Chapter 13: Vector Valued Functions and Motion in Space

Section 13.1: Curves in Space and Their Tangents

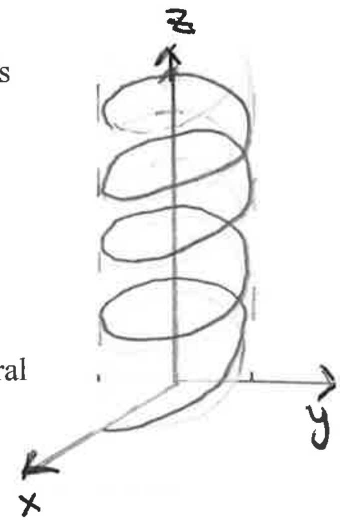
The position of a particle moving through space during a time interval I, is defined by the three **component functions**

$$x = f(t);\ y = g(t);\ z = h(t),\ t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t))$ make up the curve in space, which is called the particle's path. In vector form, we describe this as

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} = (f(t), g(t), h(t)), \text{ for } t \in I.$$

We call $\mathbf{r}(t)$ a vector-valued function (or vector function).



Example 1
The vector function

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\mathbf{k} = (\cos t, \sin t, t), \text{ for } t \in (-\infty, \infty)$$

describes a helix. It winds around a circular cylinder of radius 1 like a spiral spring.

Example 2

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\mathbf{k} = \left(\cos t, \sin t, \frac{1}{2}t\right), \text{ for } t \in (-\infty, \infty)$$

is also a helix. The vertical component increases more slowly so the spirals in this second example are closer together.

Limits and Continuity
Just as scalar valued functions have limits, so also do vector valued functions:

Definition: Limit of Vector Valued Function
Let

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} = (f(t), g(t), h(t)), \text{ for } t \in D. \tag{1}$$

(D is the domain). Let $t_0 \in D$. We say that $\mathbf{r}(t)$ has **limit** L as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L} \tag{2}$$

iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that for $t \in D$, with $0 < |t - t_0| < \delta$, we have

$$|\mathbf{r}(t) - \mathbf{L}| < \varepsilon.$$

Remark
The limit exists iff the corresponding limits exist for the component

functions. Thus if $\mathbf{L} = (L_1, L_2, L_3)$, (2) holds iff

$$\lim_{t \to t_0} f(t) = L_1 \text{ and } \lim_{t \to t_0} g(t) = L_2 \text{ and } \lim_{t \to t_0} h(t) = L_3.$$

Definition: Continuity of Vector Valued Functions
Let the function $\mathbf{r}(t)$ be given by (1). We say that $\mathbf{r}(t)$ is **continuous at** $t_0 \in D$ if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

We say $\mathbf{r}(t)$ is **continuous** if it is continuous for all $t \in D$.

Remark
$\mathbf{r}(t)$ is continuous iff all of the component functions f, g, h are continuous.

Example

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\mathbf{k} = (\cos t, \sin t, t), \text{ for } t \in (-\infty, \infty)$$

is continuous since the component functions $\cos t, \sin t, t$ are.

Derivatives and Motion
We know that

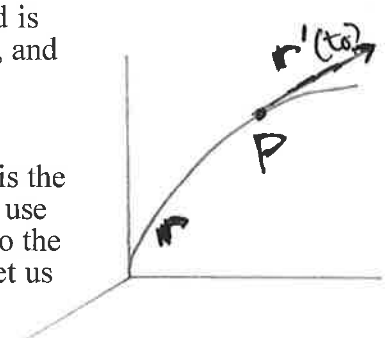
$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

In the same way we can define the derivative of a vector valued function:

Definition: Derivative of a Vector Valued Function
Let the function $\mathbf{r}(t)$ be given by (1).
(a) We say that $\mathbf{r}(t)$ is **differentiable at** $t \in D$ (or has a derivative at t) if f, g, h are differentiable at t. The derivative is

$$\mathbf{r}'(t) = \frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

(b) We say $\mathbf{r}(t)$ is **differentiable** if it is differentiable for all $t \in D$.
(c) We say the curve traced by $\mathbf{r}(t)$ is **smooth** if $\frac{d\mathbf{r}(t)}{dt}$ is continuous and is never the $\mathbf{0}$ vector. We say it is **piecewise smooth** if $\mathbf{r}(t)$ is continuous, and we can divide the curve into a finite number of smooth curves.



Important Remark
There is a geometric interpretation for $\mathbf{r}'(t)$. Let $P = \mathbf{r}(t_0)$. Then $\mathbf{r}'(t_0)$ is the tangent vector, at the point P, to the curve traced by $\mathbf{r}(t)$. We can also use this to get a formula for the tangent line to the curve: the tangent line to the curve at P is the line passing through P that is parallel to $\mathbf{r}(t_0)$. Now let us give a physical interpretation:

Definitions
If $\mathbf{r}$ is the position of a particle moving along a smooth curve in space, then

(1) $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ is the particle's **velocity vector**;
(2) The **speed** is the magnitude of the velocity: speed = $|\mathbf{v}|$;
(3) $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2}$ is the particle's **acceleration vector** (if it exists);
(4) The unit vector $\frac{\mathbf{v}}{|\mathbf{v}|}$ is the **direction of motion.**

Example
Let

$$\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + \left(5\cos^2 t\right)\mathbf{k}$$

The velocity vector is

$$\begin{aligned}\mathbf{v}(t) &= (-2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} + (-10\cos t\sin t)\,\mathbf{k} \\ &= (-2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} + (-5\sin 2t)\,\mathbf{k}.\end{aligned}$$

The acceleration vector is

$$\mathbf{a}(t) = (-2\cos t)\,\mathbf{i} + (-2\sin t)\,\mathbf{j} + (-10\cos 2t)\,\mathbf{k}.$$

The speed is

$$\begin{aligned}|\mathbf{v}(t)| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} \\ &= \sqrt{4 + 25\sin^2 2t}.\end{aligned}$$

The direction of motion at time t is

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-2\sin t, 2\cos t, -5\sin 2t)}{\sqrt{4 + 25\sin^2 2t}}.$$

For example,

$$\begin{aligned}\mathbf{v}\left(\frac{\pi}{2}\right) &= (-2, 0, 0) = -2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}; \\ \mathbf{a}\left(\frac{\pi}{2}\right) &= (0, -2, 10)) = 0\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}; \\ \text{The speed at time } \frac{\pi}{2} \text{ is } |\mathbf{v}(t)| &= 2.\end{aligned}$$

Differentiation Rules for Vector Functions
Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable functions of t, let C be a constant vector, c a scalar, and f be any differentiable function.
(1) Derivative of a constant vector:

$$\frac{d}{dt}\mathbf{C} = \mathbf{0}.$$

(2) Scalar multiple rules:

$$\frac{d}{dt}(c\mathbf{u}(t)) = c\mathbf{u}'(t);$$

$$\frac{d}{dt}(f(t)\,\mathbf{u}(t)) = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t).$$

(3) Sums:

$$\frac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t))=\mathbf{u}'(t)+\mathbf{v}'(t).$$

(4) Differences:

$$\frac{d}{dt}(\mathbf{u}(t)-\mathbf{v}(t))=\mathbf{u}'(t)-\mathbf{v}'(t).$$

(5) Dot Products:

$$\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t))=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t).$$

(6) Cross Products:

$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t))=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t).$$

(7) Chain Rule

$$\frac{d}{dt}\mathbf{u}(f(t))=\mathbf{u}'(f(t))f'(t).$$

Proof of the Rule for Dot Products

Write

$$\mathbf{u}(t)=(u_1(t),u_2(t),u_3(t))\text{; and }\mathbf{v}(t)=(v_1(t),v_2(t),v_3(t)).$$

Then

$$\mathbf{u}(t)\cdot\mathbf{v}(t)=u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t).$$

The usual product rule for scalar functions gives

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= u_1'(t)v_1(t)+u_1(t)v_1'(t)\\ &\quad +u_2'(t)v_2(t)+u_2(t)v_2'(t)\\ &\quad +u_3'(t)v_3(t)+u_3(t)v_3'(t)\\ &= (u_1'(t)v_1(t)+u_2'(t)v_2(t)+u_3'(t)v_3(t))\\ &\quad +(u_1(t)v_1'(t)+u_2(t)v_2'(t)+u_3(t)v_3'(t))\\ &= (u_1'(t),u_2'(t),u_3'(t))\cdot(v_1(t),v_2(t),v_3(t))\\ &\quad +(u_1(t),u_2(t),u_3(t))\cdot(v_1'(t),v_2'(t),v_3'(t))\\ &= \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t).\end{aligned}$$

■

Section 13.2: Integrals of Vector Functions; Projectile Motion

We learn about integrals of vector functions.

Integrals of Vector Functions
A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval I if at each point of I,

$$\frac{d\mathbf{R}}{dt} = \mathbf{r}(t).$$

Note that if $\mathbf{C}$ is any constant vector, $\mathbf{R}(t) + \mathbf{C}$ is also an antiderivative.

Definition: Indefinite Integral
The indefinite integral of $\mathbf{r}$ with respect to t is the set of all anti-derivatives of $\mathbf{r}$, and is denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is an anti-derivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + C.$$

Remark
If $\mathbf{r}(t) = (f(t), g(t), h(t))$ and f, g, h are integrable over $[a, b]$, then for any constant vector $\mathbf{C}$,

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \left(\int f(t)\,dt\right)\mathbf{i} + \left(\int g(t)\,dt\right)\mathbf{j} + \left(\int h(t)\,dt\right)\mathbf{k} + \mathbf{C} \\ &= \left(\int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt\right) + \mathbf{C}.\end{aligned}$$

Definition
If $\mathbf{r}(t) = (f(t), g(t), h(t))$ and f, g, h are integrable over $[a, b]$, then so is $\mathbf{r}$, and the definite integral of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Example
Suppose we know the acceleration of a hang glider, and its initial position (position at time 0) and initial velocity (velocity at time 0). Then we can find the path of the hang glider. As an example, suppose the acceleration at time t is

$$\mathbf{a}(t) = -(3\cos t)\,\mathbf{i} - (3\sin t)\,\mathbf{j} + 2\mathbf{k} = (-3\cos t, -3\sin t, 2).$$

Suppose the initial position is

$$\mathbf{r}(0) = 4\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = (4, 0, 0)$$

and initial velocity is

$$\mathbf{v}(0) = 0\mathbf{i} + 3\mathbf{j} + 0\mathbf{k} = (0,3,0).$$

We see by integrating the component functions that

$$\mathbf{v}(t) = (-3\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j} + (2t)\,\mathbf{k} + \mathbf{C}_1 = (-3\sin t, 3\cos t, 2t) + \mathbf{C}_1.$$

We use $\mathbf{v}(0)$ to find $\mathbf{C}_1$:

$$(0,3,0) = \mathbf{v}(0) = (0,3,0) + \mathbf{C}_1.$$

So

$$\mathbf{C}_1 = (0,0,0).$$

So the glider's velocity at time t is

$$\mathbf{v}(t) = (-3\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j} + (2t)\,\mathbf{k} = (-3\sin t, 3\cos t, 2t).$$

We integrate again to find $\mathbf{r}$:

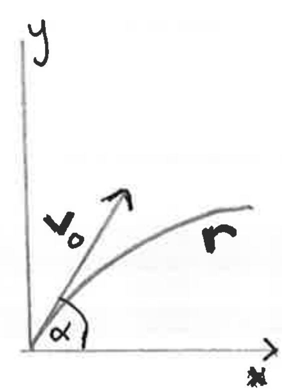
$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + \left(t^2\right)\mathbf{k} + \mathbf{C}_2 = \left(3\cos t, 3\sin t, t^2\right) + \mathbf{C}_2$$

We find $\mathbf{C}_2$ using $r(0) = (4,0,0)$:

$$(4,0,0) = \mathbf{r}(0) = (3,0,0) + \mathbf{C}_2$$
$$\Rightarrow \mathbf{C}_2 = (1,0,0).$$

So the glider's position at time t is

$$\begin{aligned}\mathbf{r}(t) &= (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + \left(t^2\right)\mathbf{k} + (\mathbf{1},\mathbf{0},\mathbf{0}) \\ &= (1 + 3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + \left(t^2\right)\mathbf{k} \\ &= \left(1 + 3\cos t, 3\sin t, t^2\right).\end{aligned}$$



The Vector and Parametric Equation for Ideal Projectile Motion

Suppose we fire a projectile, and ignore friction and the earth's rotation. Thus we are assuming that there is an initial velocity and the only force acting on the projectile is gravity. It acts straight down.

We assume that the initial velocity (at time $t = 0$) is $\mathbf{v}_0$, and it acts at an angle α to the horizontal. If we denote the speed by $v_0 = |\mathbf{v}_0|$, then

$$\mathbf{v}_0 = (v_0\cos\alpha)\,\mathbf{i} + (v_0\sin\alpha)\,\mathbf{j}. \qquad (1)$$

The initial position of the projectile is

$$\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}. \qquad (2)$$

We let $\mathbf{r} = \mathbf{r}(t)$ denote its position at time t. Newton's second law of motion says that the force acting on the projectile equals its mass $m \times$ its acceleration, that is $m\frac{d^2\mathbf{r}}{dt^2}$. If the force is only gravitational force $-mg\mathbf{j}$, then

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j},$$

so

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}.$$

This is our differential equation. The initial conditions are (1) and (2). Integrating once with respect to t gives

$$\frac{d\mathbf{r}}{dt} = -gt\mathbf{j} + \mathbf{v}_0.$$

Integrating again gives

$$\begin{aligned}\mathbf{r} &= -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0 \\ &= -\frac{1}{2}gt^2\mathbf{j} + \left(\left(v_0 \cos\alpha\right)\mathbf{i} + \left(v_0 \sin\alpha\right)\mathbf{j}\right)t + \mathbf{0}.\end{aligned}$$

In summary:
The Equation of Ideal Projectile Motion is

$$\mathbf{r} = \left(v_0 \cos\alpha\right) t\mathbf{i} + \left(\left(v_0 \sin\alpha\right) t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

Example
A projectile is fired over horizontal ground at an initial speed of 500m/sec, and at a launch angle of $60°$. What is the equation of its motion? Where will it be after 10 sec?
Solution
The initial speed is $v_0 = 500$, $\alpha = 60°$, and $g = 9.8$. So

$$\begin{aligned}\mathbf{r} &= \left(500 \cos 60°\right) t\mathbf{i} + \left(\left(500 \sin 60°\right) t - 4.9t^2\right)\mathbf{j} \\ &= 250t\mathbf{i} + \left(250\sqrt{3}t - 4.9t^2\right)\mathbf{j}.\end{aligned}$$

At $t = 10$, we have

$$\begin{aligned}\mathbf{r} &= 2500\mathbf{i} + \left(2500\sqrt{3} - 490\right)\mathbf{j} \\ &\simeq 2500\mathbf{i} + 3840\mathbf{j}.\end{aligned}$$

Path, Height, Flight Time of an Ideal Projectile
Next, we analyze the path of an ideal projectile in terms of its horizontal component x and vertical component y. Write

$$\begin{aligned}\mathbf{r} &= \left(v_0 \cos\alpha\right) t\mathbf{i} + \left(\left(v_0 \sin\alpha\right) t - \frac{1}{2}gt^2\right)\mathbf{j} \\ &= x\mathbf{i} + y\mathbf{j}.\end{aligned}$$

Then

$$\begin{aligned}x &= \left(v_0 \cos\alpha\right) t \\ &\Rightarrow t = x / \left(v_0 \cos\alpha\right).\end{aligned}$$

So

$$\begin{aligned} y &= (v_0 \sin \alpha)\, t - \frac{1}{2} g t^2 \\ &= \left(\frac{v_0 \sin \alpha}{v_0 \cos \alpha} \right) x - \frac{1}{2} g \left(\frac{x}{v_0 \cos \alpha} \right)^2 . \end{aligned}$$

Or,

$$y = -\frac{g}{2 v_0^2 \cos^2 \alpha} x^2 + (\tan \alpha)\, x.$$

Thus is the equation of a parabola in x. From this we can find the **maximum height** (the highest point of the projectile), the **range** (the distance from start to when it again hits the ground) and the **flight time** (the total time it is in the air).

The maximum height occurs at the point where $\frac{dy}{dx} = 0$:

$$\frac{dy}{dx} = -\left(\frac{g}{2 v_0^2 \cos^2 \alpha} \right) 2x + (\tan \alpha)$$

so

$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{\tan \alpha}{\left(\frac{g}{v_0^2 \cos^2 \alpha} \right)} = \frac{v_0^2 \tan \alpha \cos^2 \alpha}{g} = \frac{v_0^2 \sin \alpha \cos \alpha}{g}.$$

Then

$$\begin{aligned} y &= -\frac{g}{2 v_0^2 \cos^2 \alpha} \left(\frac{v_0^2 \sin \alpha \cos \alpha}{g} \right)^2 + (\tan \alpha) \left(\frac{v_0^2 \sin \alpha \cos \alpha}{g} \right) \\ &= \frac{(v_0 \sin \alpha)^2}{2g}. \end{aligned}$$

The flight time is that t for which again $y = 0$, thus

$$y = -\frac{g}{2 v_0^2 \cos^2 \alpha} x^2 + (\tan \alpha)\, x = 0,$$

and the corresponding value of x is the range:

$$\begin{aligned} x &= \frac{\tan \alpha}{\left(\frac{g}{2 v_0^2 \cos^2 \alpha} \right)} = \frac{2 v_0^2}{g} \tan \alpha \cos^2 \alpha \\ &= \frac{2 v_0^2}{g} \sin \alpha \cos \alpha = \frac{v_0^2}{g} \sin 2\alpha. \end{aligned}$$

Then the flight time is

$$t = \frac{x}{v_0 \cos \alpha} = \frac{2 v_0^2 \sin \alpha \cos \alpha}{g v_0 \cos \alpha} = \frac{2 v_0}{g} \sin \alpha.$$

Summary

Maximum height:

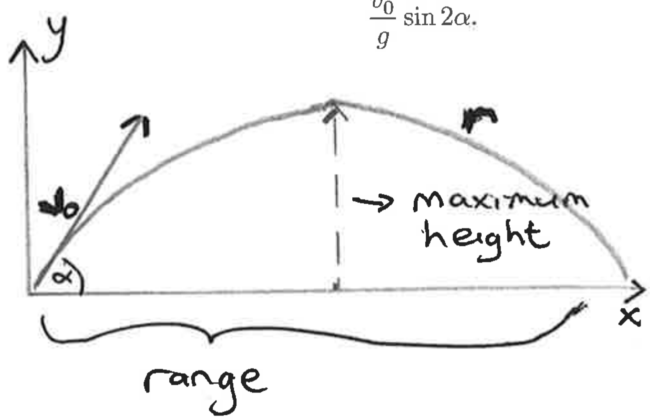
$$y = \frac{(v_0 \sin \alpha)^2}{2g}.$$

Flight Time

$$t = \frac{2v_0}{g} \sin \alpha.$$

Range

$$\frac{v_0^2}{g} \sin 2\alpha.$$



Section 13.3: Arc Length in Space

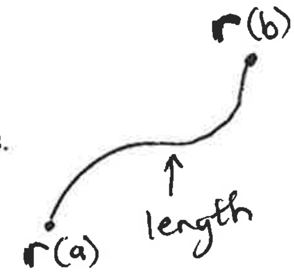
There is a nice formula for the length of a "smooth" curve in three dimensions:

Definition
Let a curve be parametrized (described) by the formula

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},\ t \in [a,b],$$

and assume that x, y, z have continuous (or piecewise continuous) derivatives. The **length** of the curve as t increases from a to b is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$



Remark
If $\mathbf{r}(t)$ is thought of as describing the position of a particle at time t, then its velocity is

$$\mathbf{v}(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}$$

so the speed is

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

Thus we can also write

$$L = \int_a^b |\mathbf{v}(t)|\,dt.$$

Example 1
A glider is soaring upwards on the helix

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\mathbf{k},\ t \in [0, 2\pi].$$

Then we see that

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k},$$

so

$$|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2},$$

so

$$L = \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2}.$$

We can also use **arc length**, or length along the curve from some starting point to a given point, to measure how far we have walked along the curve. This is also useful for other purposes such as measuring curvature, twists and turns, ... So we give it a name:

Definition
Assume that the curve C is described by

$$P(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} = (x(t), y(t), z(t)),\ t \in [a,b].$$

Choose a base point $P(t_0)$ on curve C. The **arc length parameter** with base point $P(t_0)$ is

$$s(t) = \int_{t_0}^{t} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\,d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\,d\tau. \tag{2}$$

Remark
Sometimes we can use this to reparametrize the curve in terms of s. For this we need to be able to solve for t as a function of s.

Example 2
Consider the helix from Example 1,

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\mathbf{k},\ t \in [0, 2\pi]. \tag{1}$$

for which

$$|\mathbf{v}(t)| = \sqrt{2}.$$

Suppose we choose $t_0 = 0$ as our base point. Then the arc length parameter is

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\,d\tau = \int_{0}^{t} \sqrt{2}\,d\tau = \sqrt{2}t.$$

Thus

$$t = \frac{s}{\sqrt{2}}.$$

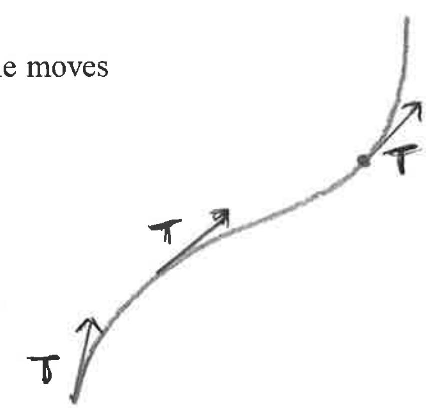
Substituting this into (1) gives a parametrization of the curve in terms of arc length:

$$\mathbf{r}(t(s)) = \left(\cos\frac{s}{\sqrt{2}}\right)\mathbf{i} + \left(\sin\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}.$$

Speed on a Smooth Curve
The fundamental theorem of calculus and (2) gives

$$\frac{ds}{dt} = |\mathbf{v}(t)|.$$

This says something we know: the speed with which the particle moves along the curve is the magnitude of $\mathbf{v}$.



Unit Tangent Vector
We have seen in earlier sections that the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

is tangent to the curve $\mathbf{r}(t)$. The **unit tangent vector** is therefore

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Example 3
Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (1 + 3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + t^2\mathbf{k}.$$

Solution
Then

$$\mathbf{v} = \mathbf{r}'(t) = (-3\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j} + 2t\mathbf{k},$$

so

$$|\mathbf{v}| = \sqrt{9+4t^2}$$

and

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(-\frac{3\sin t}{\sqrt{9+4t^2}}\right)\mathbf{i} + \left(\frac{3\cos t}{\sqrt{9+4t^2}}\right)\mathbf{j} + \frac{2t}{\sqrt{9+4t^2}}\mathbf{k}.$$

Remark
For general smooth curves, we can use $\mathbf{T}$ to describe the rate of change of the position vector with respect to time t. By the chain rule:

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{d\mathbf{r}}{dt}/\frac{ds}{dt} = \mathbf{v}/\left|\mathbf{v}\right| = \mathbf{T}.$$

Section 13.4: Curvature and Normal Vectors of a Curve

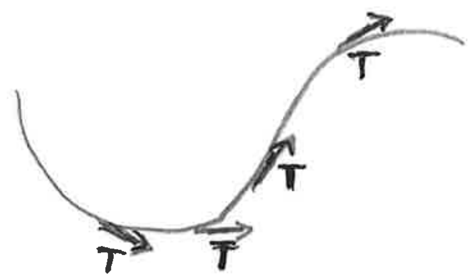
We shall study how a curve turns or bends.

Curvature of a Plane Curve
As a particle moves along a smooth curve in the plane, the unit tangent vector to the curve $\mathbf{T} = \frac{d\mathbf{r}}{ds}$ turns as the curve bends. Since $|\mathbf{T}| = 1$, its magnitude remains constant and only its direction changes. The rate at which $\mathbf{T}$ turns per unit length along the curve is called the **curvature**. It is denoted by κ (Greek kappa).

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

If $|\frac{d\mathbf{T}}{ds}|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large. If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t, (other than arc length s), we can calculate the curvature as

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\ &= \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| \\ &= \left|\frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt}\right| \\ &= \left|\frac{d\mathbf{T}}{dt}\right|\frac{1}{|\mathbf{v}|}.\end{aligned}$$



Let us summarize:

Formula for Calculating Curvature
If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is the unit tangent vector.
Remark
We emphasize that κ is a function of t.

Example 2
Let us find the curvature of a circle in the plane (two dimensions). A circle center 0, of radius $a > 0$, is parametrized by

$$\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j},\ t \in [0, 2\pi].$$

Then

$$\mathbf{v}(t) = (-a\sin t)\,\mathbf{i} + (a\cos t)\,\mathbf{j},\ t \in [0, 2\pi].$$

so

$$|\mathbf{v}(t)| = \sqrt{a^2\sin^2 t + a^2\cos t} = a.$$

Hence

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j},\ t \in [0, 2\pi].$$

Then

$$\frac{d\mathbf{T}}{dt} = (-\cos t)\,\mathbf{i} + (-\sin t)\,\mathbf{j},$$

and

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1.$$

Then the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}1 = \frac{1}{a}.$$

So the curvature of the circle is the reciprocal of its radius. Of course, it is not surprising that a circle has the same curvature at all of its points.

While the tangent vector $\mathbf{T}$ is tangent to the curve, it is also useful to have a normal to the curve, showing the direction in which the curve is turning. Since $\mathbf{T}$ has constant length one, $\mathbf{T} \cdot \mathbf{T} = 1$, and then we see (as before),

$$\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0,$$

so if we divide $\frac{d\mathbf{T}}{ds}$ by its length $\kappa = |\frac{d\mathbf{T}}{ds}|$, we obtain a unit vector orthogonal to $\mathbf{T}$:

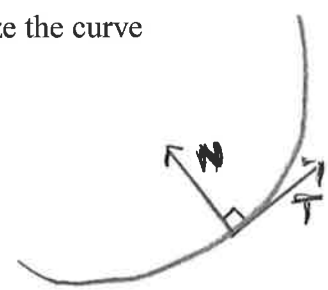
Definition

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Remarks

(a) Note that $\frac{d\mathbf{T}}{ds}$ points in the direction in which $\mathbf{T}$ turns as we go along the curve. If we face in the direction of increasing arc length, the vector $\frac{d\mathbf{T}}{ds}$ points towards the right if $\mathbf{T}$ turns clockwise; and to the left if $\mathbf{T}$ turns counterclockwise. Geometrically, this means that $\mathbf{N}$ **points towards the concave side of the curve**.



(b) It is also possible to give a formula for $\mathbf{N}$ when we parametrize the curve in terms of t rather than s: using the chain rule,

$$\begin{aligned}\mathbf{N} &= \frac{1}{|\frac{d\mathbf{T}}{ds}|}\frac{d\mathbf{T}}{ds} \\ &= \frac{1}{|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}|}\frac{d\mathbf{T}}{dt}\frac{dt}{ds} \\ &= \frac{1}{|\frac{d\mathbf{T}}{dt}|}\frac{d\mathbf{T}}{dt},\end{aligned}$$

since $\frac{dt}{ds} > 0$, as t increases as s does. Let us summarize this as:

Formula for Calculating $\mathbf{N}$

If $\mathbf{r}(t)$ is a smooth curve, then the **principal unit normal** is

$$\mathbf{N} = \frac{1}{\left|\frac{d\mathbf{T}}{dt}\right|}\frac{d\mathbf{T}}{dt}.$$

Example
Find $\mathbf{T}$ and $\mathbf{N}$ for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\,\mathbf{i} + (\sin 2t)\,\mathbf{j}.$$

Solution
Now

$$\mathbf{v}(t) = (-2\sin 2t)\,\mathbf{i} + (2\cos 2t)\,\mathbf{j},$$

so

$$|\mathbf{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2,$$

so

$$T = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\sin 2t)\,\mathbf{i} + (\cos 2t)\,\mathbf{j}.$$

Then

$$\frac{d\mathbf{T}}{dt} = (-2\cos 2t)\,\mathbf{i} + (-2\sin 2t)\,\mathbf{j},$$

so

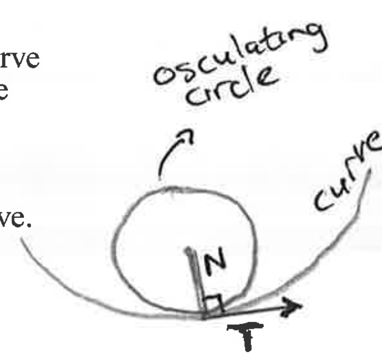
$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2,$$

so

$$\mathbf{N} = \frac{1}{\left|\frac{d\mathbf{T}}{dt}\right|}\frac{d\mathbf{T}}{dt} = (-\cos 2t)\,\mathbf{i} + (-\sin 2t)\,\mathbf{j}.$$

(You can check that $\mathbf{N}\cdot\mathbf{T} = 0$).

Circle of Curvature for Plane Curves
The **circle of curvature** or **osculating circle** at a point P on a plane curve (for which $\kappa \neq 0$) is a circle in the same plane with the following three properties:
(1) The circle has the same tangent line at P as does the curve;
(2) The circle has the same curvature at P as does the curve;
(3) The circle has its center towards the concave or inner side of the curve.



The **radius of curvature** of the curve at P is $\frac{1}{\kappa}$, the reciprocal of the curvature. (It is the radius of the osculating circle).

Example
Find the osculating circle of the parabola $y = x^2$ at the origin.
Solution
First let us parametrize this in terms of a parameter t. If we set $x = t$, then $y = t^2$, so the parabola is given by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ \text{all real } t.$$

Then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j},$$

and

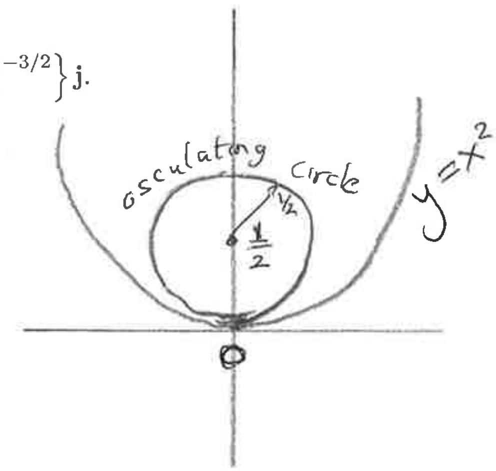
$$|\mathbf{v}| = \sqrt{1+4t^2},$$

so

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\mathbf{j}.$$

Then

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= \left(-\frac{1}{2}\left(1+4t^2\right)^{-3/2}(8t)\right)\mathbf{i} + \\ &\quad \left\{\frac{2\left(\sqrt{1+4t^2}\right) - 2t\left(\frac{8t}{2\sqrt{1+4t^2}}\right)}{1+4t^2}\right\}\mathbf{j} \\ &= -4t\left(1+4t^2\right)^{-3/2}\mathbf{i} + \left\{2/\sqrt{1+4t^2} - 8t^2\left(1+4t^2\right)^{-3/2}\right\}\mathbf{j}.\end{aligned}$$



At $t = 0$, we see that

$$\begin{aligned}|\mathbf{v}| &= \sqrt{1+0} = 1; \\ \frac{d\mathbf{T}}{dt} &= 0\mathbf{i} - 2\mathbf{j},\end{aligned}$$

so the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{1}\sqrt{0^2+2^2} = 2.$$

Then the radius of curvature is $\frac{1}{\kappa} = \frac{1}{2}$. Also,

$$\mathbf{T} = \mathbf{i} + 0\mathbf{j} = \mathbf{i},$$

so the unit normal must be

$$\mathbf{N} = \mathbf{j}.$$

(One could also do this by calculation). So we want a circle of radius $\frac{1}{2}$, with center in the concave part of the parabola (namely above the x-axis), with tangent at $(0,0)$ being the horizontal vector $\mathbf{i}$. This is the circle

$$(x-0)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

Curvature and Normal Vectors for Space Curves

All the above was in two dimensions. However, the same formulae apply for curves in space (three dimensions). If $\mathbf{r}(t)$ describes the position of the particle at time t, then still

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt};$$

The **unit tangent vector** is still

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

The **curvature** is still

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

The **principal unit normal** is still

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{1}{\left|\frac{d\mathbf{T}}{dt}\right|}\frac{d\mathbf{T}}{dt}.$$

Example

(a) Let $a, b > 0$. Find the curvature of the helix

$$r(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j} + bt\mathbf{k}$$

Solution

Here

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-a\sin t)\,\mathbf{i} + (a\cos t)\,\mathbf{j} + b\mathbf{k}$$

so

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

and

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(-\frac{a\sin t}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \left(\frac{a\cos t}{\sqrt{a^2+b^2}}\right)\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}.$$

Then

$$\frac{d\mathbf{T}}{dt} = \left(-\frac{a\cos t}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \left(-\frac{a\sin t}{\sqrt{a^2+b^2}}\right)\mathbf{j} + 0\mathbf{k},$$

so

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{a^2}{a^2+b^2}\cos^2 t + \frac{a^2}{a^2+b^2}\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}},$$

and

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}.$$

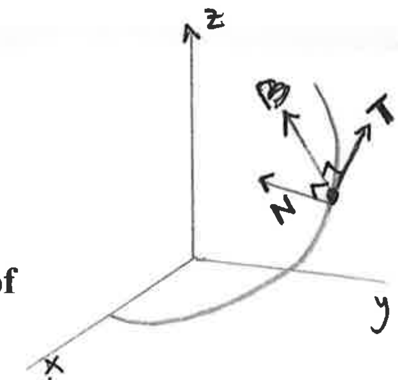
(b) Find the principal unit normal at any point on the helix

Solution

We know that

$$\begin{aligned}\mathbf{N} &= \frac{1}{\left|\frac{d\mathbf{T}}{dt}\right|}\frac{d\mathbf{T}}{dt} \\ &= \frac{1}{\left(\frac{a}{\sqrt{a^2+b^2}}\right)}\left(-\frac{a\cos t}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \left(-\frac{a\sin t}{\sqrt{a^2+b^2}}\right)\mathbf{j} \\ &= (-\cos t)\,\mathbf{i} + (-\sin t)\,\mathbf{j}.\end{aligned}$$

Note that $\mathbf{N}$ always points towards the $z-$axis and is always parallel to the horizontal $xy-$plane.



Section 13.5: Tangential and Normal Components of Acceleration

When travelling along a curve in space, the tangent $\mathbf{T}$ gives the direction in which you are moving forward, and the principal unit normal $\mathbf{N}$ describes the direction in which you are turning. Another important vector is the unit binormal vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

which gives the tendency of your motion to twist out of the plane defined by $\mathbf{T}$ and $\mathbf{N}$, into a perpendicular direction.

Together these three vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ define a frame of mutually orthogonal vectors called the $\mathbf{TNB}$ frame, or **Frenet frame**. They move along the curve, and are a **right-handed frame**. In this section, we learn how to describe acceleration in terms of this frame.

When analyzing the motion of an object that is accelerating, it is useful to know how much of the acceleration acts in the direction of motion, that is in the direction of the tangent $\mathbf{T}$. One can use the chain rule to calculate this:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}.$$

Now differentiate both sides again to get the acceleration:

$$\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d\mathbf{T}}{dt}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\frac{d\mathbf{T}}{ds} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\kappa\mathbf{N},
\end{aligned}$$

recall that $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$. This expresses the acceleration as a linear combination of $\mathbf{T}$ and $\mathbf{N}$.

Definition
If the acceleration vector is written as

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}, \tag{1}$$

then

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \tag{2a}$$

is the **tangential scalar component of acceleration** and

$$a_{\mathbf{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa\,|\mathbf{v}|^2 \tag{2b}$$

is the **normal scalar component of acceleration**.

Remark
(a) The binormal vector $\mathbf{B}$ plays no role in this last equation. So the acceleration is always in the plane defined by $\mathbf{T}$ and $\mathbf{N}$.
(b) Observe that if we double the speed $|\mathbf{v}|$, then we multiply the normal component of acceleration by 4. This explains why when a car goes around a sharp curve at high speed (large κ, high $|\mathbf{v}|$), then you need to hold on.

Calculating $a_{\mathbf{N}}$
We use the fact that $\mathbf{N}\cdot\mathbf{T}=0$ and that $\mathbf{T},\mathbf{N}$ are unit vectors:

$$\begin{aligned}
|\mathbf{a}|^2 &= \mathbf{a}\cdot\mathbf{a} \\
&= (a_{\mathbf{T}}\mathbf{T}+a_{\mathbf{N}}\mathbf{N})\cdot(a_{\mathbf{T}}\mathbf{T}+a_{\mathbf{N}}\mathbf{N}) \\
&= (a_{\mathbf{T}}\mathbf{T})\cdot(a_{\mathbf{T}}\mathbf{T})+2\,(a_{\mathbf{T}}\mathbf{T})\cdot(a_{\mathbf{N}}\mathbf{N})+(a_{\mathbf{N}}\mathbf{N})\cdot(a_{\mathbf{N}}\mathbf{N}) \\
&= (a_{\mathbf{T}})^2\,\mathbf{T}\cdot\mathbf{T}+0+(a_{\mathbf{N}})^2\,\mathbf{N}\cdot\mathbf{N} \\
&= (a_{\mathbf{T}})^2\,|\mathbf{T}|^2+(a_{\mathbf{N}})^2\,|\mathbf{N}|^2 \\
&= (a_{\mathbf{T}})^2+(a_{\mathbf{N}})^2,
\end{aligned}$$

This gives:

Formula for Calculating the Normal Component of Acceleration

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2-(a_{\mathbf{T}})^2}.$$

Summary

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T}+a_{\mathbf{N}}\mathbf{N} \tag{1}$$

where

$$a_{\mathbf{T}} = \frac{d}{dt}\,|\mathbf{v}| \tag{2}$$

and

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2-(a_{\mathbf{T}})^2}. \tag{3}$$

Example 1
Consider the motion given by

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j},\ t>0.$$

Without finding the $\mathbf{T}$ and $\mathbf{N}$, write the acceleration in the form $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.

Solution

First we compute

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\,\mathbf{i} + (\cos t - \cos t + t\sin t)\,\mathbf{j} \\ &= (t\cos t)\,\mathbf{i} + (t\sin t)\,\mathbf{j}\end{aligned}$$

so

$$|\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t.$$

Then

$$a_{\mathbf{T}} = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1.$$

Next,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j}.$$

Then

$$\begin{aligned}|\mathbf{a}|^2 &= (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 \\ &= \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t \\ &\quad + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t \\ &= 1 + t^2.\end{aligned}$$

We then use (3):

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2} = \sqrt{1 + t^2 - 1} = t.$$

Thus

$$\begin{aligned}\mathbf{a} &= a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} \\ &= \mathbf{T} + t\mathbf{N}.\end{aligned}$$

Torsion

Let us calculate

$$\begin{aligned}\frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) \\ &= \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds}.\end{aligned}$$

Here $\mathbf{N}$ is the direction of $\frac{d\mathbf{T}}{ds}$, so their cross product is $\mathbf{0}$, and

$$\frac{d\mathbf{B}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}.$$

Thus $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$. As $\mathbf{B}$ has constant unit length, $\frac{d\mathbf{B}}{ds}$ is also orthogonal to $\mathbf{B}$. Thus it has to be parallel to $\mathbf{N}$. So for some scalar τ,

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

We call τ the **torsion** along the curve. We can compute τ by taking the dot product with $\mathbf{N}$:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau \mathbf{N} \cdot \mathbf{N} = -\tau.$$

Let us summarize:

Definition

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion function** of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

Remark

(a) Here is one interpretation of torsion. Suppose a train is climbing up a curved track. The rate at which the headlight turns is the curvature of the track. The rate at which the engine tends to twist out of the plane defined by $\mathbf{N}$ and $\mathbf{T}$ is the torsion.

(b) It can be shown that a smooth curve in 3 dimensional space is a helix iff it has constant non-0 curvature and constant non-0 torsion.

Formulas for Computing Curvature and Torsion

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt}\mathbf{T}\right) \times [a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}] \\ &= \left(\frac{ds}{dt}a_{\mathbf{T}}\right)\mathbf{T} \times \mathbf{T} + \left(\frac{ds}{dt}a_{\mathbf{N}}\right)\mathbf{T} \times \mathbf{N} \\ &= \mathbf{0} + \left(\frac{ds}{dt}a_{\mathbf{N}}\right)\mathbf{T} \times \mathbf{N} \\ &= \left(\frac{ds}{dt}a_{\mathbf{N}}\right)\mathbf{T} \times \mathbf{N} \\ &= \kappa\left(\frac{ds}{dt}\right)^3 \mathbf{T} \times \mathbf{N} \\ &= \kappa\left(\frac{ds}{dt}\right)^3 \mathbf{B}. \end{aligned}$$

Then as $\mathbf{B}$ is a unit vector,

$$|\mathbf{v} \times \mathbf{a}| = \kappa\left(\frac{ds}{dt}\right)^3 = \kappa\,|\mathbf{v}|^3.$$

Let us summarize:

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

One can also prove:

Formula for Torsion:
Suppose $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$, etc. Then

$$\tau = \frac{\det \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{bmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}.$$

Example
Use these last two formulas to find the curvature and torsion of the helix

$$\mathbf{r}(t) = (a \cos t)\,\mathbf{i} + (a \sin t)\,\mathbf{j} + bt\mathbf{k}.$$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-a \sin t)\,\mathbf{i} + (a \cos t)\,\mathbf{j} + b\mathbf{k};$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-a \cos t)\,\mathbf{i} + (-a \sin t)\,\mathbf{j}.$$

Then

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

and

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{bmatrix} \\ &= \mathbf{i}\,[ba \sin t] - \mathbf{j}\,[ba \cos t] + \mathbf{k}\left[a^2 \sin^2 t + a^2 \cos^2 t\right] \\ &= (ab \sin t)\,\mathbf{i} - (ab \cos t)\,\mathbf{j} + a^2\mathbf{k}. \end{aligned}$$

so

$$|\mathbf{v} \times \mathbf{a}| = \sqrt{(ab)^2 + a^4} = a\sqrt{b^2 + a^2}.$$

Then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{a\sqrt{b^2 + a^2}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a}{a^2 + b^2}.$$

For τ, we also need

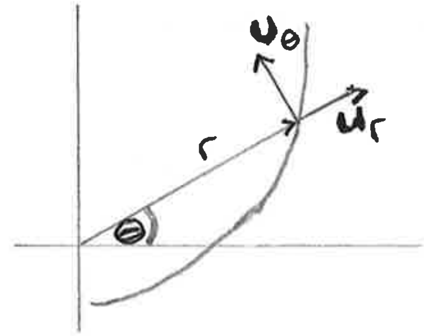
$$\frac{d\mathbf{a}}{dt} = (a \sin t)\,\mathbf{i} + (-a \cos t)\,\mathbf{j}.$$

Thus we have

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= (-a \sin t, a \cos t, b) = (\dot{x}, \dot{y}, \dot{z}); \\ \frac{d\mathbf{v}}{dt} &= (-a \cos t, -a \sin t, 0) = (\ddot{x}, \ddot{y}, \ddot{z}); \\ \frac{d\mathbf{a}}{dt} &= (a \sin t, -a \cos t, 0) = (\dddot{x}, \dddot{y}, \dddot{z}). \end{aligned}$$

So

$$\begin{aligned}
\tau &= \frac{\det \begin{bmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{bmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} \\
&= \frac{b\left(a^2\cos^2 t + a^2\sin^2 t\right)}{\left(a\sqrt{b^2+a^2}\right)^2} \\
&= \frac{ba^2}{a^2\left(b^2+a^2\right)} = \frac{b}{a^2+b^2}.
\end{aligned}$$



Section 13.6: Velocity and Acceleration in Polar Coordinates

We derive equations for velocity and acceleration in polar coordinates. As a consequence, we examine Kepler's laws of planetary motion.

Motion in Polar and Cylindrical Coordinates
First let us look at a particle moving in a plane. Let r denote distance from the origin and θ denote the angle to the origin. Let $P(r,\theta)$ denote the position of the particle. Define the unit vectors

$$\mathbf{u}_r = \frac{\mathbf{r}}{|\mathbf{r}|} = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}; \tag{1a}$$

$$\mathbf{u}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j} \tag{1b}$$

(see the picture). $\mathbf{u}_r$ points along the position vector from the origin to $P(r,\theta)$, and

$$\mathbf{r} = r\mathbf{u}_r.$$

The vector $\mathbf{u}_\theta$ is orthogonal to $\mathbf{u}_r$ and points in the direction of increasing θ. From (1),

$$\frac{d\mathbf{u}_r}{d\theta} = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j} = \mathbf{u}_\theta;$$

$$\frac{d\mathbf{u}_\theta}{d\theta} = (-\cos\theta)\,\mathbf{i} + (-\sin\theta)\,\mathbf{j} = -\mathbf{u}_r.$$

When we differentiate $\mathbf{u}_r$ and $\mathbf{u}_\theta$ with respect to t, we find out how they change with time: the chain rule gives (recall $\cdot$ means derivative with respect to time)

$$\begin{aligned} \dot{\mathbf{u}}_r &= \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} = \mathbf{u}_\theta\dot{\theta}, \\ \dot{\mathbf{u}}_\theta &= \frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt} = -\mathbf{u}_r\dot{\theta}. \end{aligned} \tag{2}$$

Then

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{u}_r) = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u}_r + r\mathbf{u}_\theta\dot{\theta}.$$

The acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d}{dt}\left(\dot{r}\mathbf{u}_r + r\mathbf{u}_\theta\dot{\theta}\right) = (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + \left(\dot{r}\mathbf{u}_\theta\dot{\theta} + r\dot{\mathbf{u}}_\theta\dot{\theta} + r\mathbf{u}_\theta\ddot{\theta}\right)$$

Now we use (2) on this:

$$\begin{aligned} \mathbf{a} &= \left(\ddot{r}\mathbf{u}_r + \dot{r}\mathbf{u}_\theta\dot{\theta}\right) + \left(\dot{r}\mathbf{u}_\theta\dot{\theta} + r\left(-\mathbf{u}_r\dot{\theta}\right)\dot{\theta} + r\mathbf{u}_\theta\ddot{\theta}\right) \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{u}_\theta. \end{aligned}$$

All these equations are for a plane. Now we extend these to motion in three dimensions: in cylindrical coordinates, we have:
Position

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$$

Velocity

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\mathbf{u}_\theta\dot{\theta} + \dot{z}\mathbf{k}$$

Acceleration

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

Note that the three vectors $\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{k}$ are orthogonal to one another and make a right-handed frame:

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k};\ \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r;\ \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta.$$

Planetary Motion

Newton's law of gravitation says that the force $\mathbf{F}$ of gravitation between the centre of a planet of mass m, and the sun of mass M, is proportional to both m and M, and inversely proportional to the square of the distance between the planet and the sun. Thus if $\mathbf{r}$ is the radius vector from the center of the sun to the centre of the planet, then

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}.$$

The number $G = 6.6378... \times 10^{-11} Nm^2/(kg)^2$ is the gravitational constant. Newton's second law also says

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

so

$$m\ddot{\mathbf{r}} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|},$$

and hence

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}.$$

This has the interpretation that the planet is accelerating towards the sun's center of mass. Notice that this also says that $\ddot{r}$ is a multiple of r. Then

$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}.$$

Then also

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}.$$

So $\mathbf{r} \times \dot{\mathbf{r}}$ does not change in time. That is, for some constant vector $\mathbf{C}$,

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C}.$$

This equations tells us that $\mathbf{r}$ and $\dot{\mathbf{r}}$ always lie in a plane perpendicular to $\mathbf{C}$. Hence the planet moves in a fixed plane through the centre of its sun. Kepler's laws describe this in a more precise way:

Kepler's First Law
The planet's path is an ellipse with the sun at one focus.

Kepler's Second Law
The radius vector from the sun sweeps out equal areas in equal times.

Kepler's Third Law
This says that the time the planet takes to go around the sun is related to the orbit's semi-major axis in a specific way.