Chapter 15: Multiple Integrals (March 28)

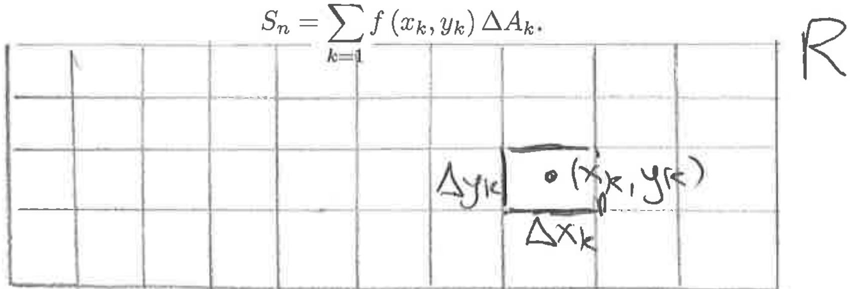
2 Section 15.1: Double and Iterated Integrals over Rectangles

Consider a rectangle

$$R : a \leq x \leq b,\ c \leq y \leq d,$$

and a function $f(x, y)$ defined on R. We can define Riemann sums over the rectangle R in a way analogous to that we did for an interval. Subdivide R into small rectangles using a network of lines parallel to the x and y axes. These rectangles form a **partition** P of R. A small rectangle of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. Suppose that there are n small rectangles with areas ΔA_1, ΔA_2, ..., ΔA_n. To form a **Riemann sum** over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at that point by the area ΔA_k, and then add over all rectangles:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$



Next, we shrink the sizes of the rectangles to 0, and take limits. Let's make this more precise. The **norm** of a partition P, denoted $\|P\|$, is defined to be the largest width or height of any rectangle in the partition. Thus if $\|P\| \leq 0.001$, then every rectangle in the partition has width and height at most 0.001. We take limits of the Riemann sum as $\|P\| \to 0$, and write this as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$

Since the number of rectangles approaches 0 as $n \to \infty$, this also can be written as

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$

Of course, different subdivisions and choices of the points lead to different Riemann sums. However, when the limit of the sums exists, giving the same answer no matter what choices are made, then the function f is said to be

integrable and the limit is called the **double integral** of f over R, written as

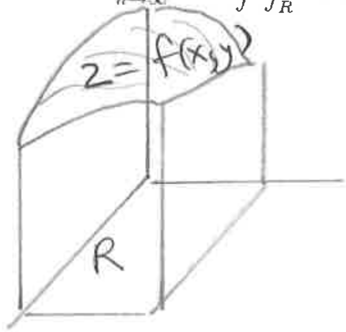
$$\int\int_R f(x,y)\,dA \text{ or } \int\int_R f(x,y)\,dx\,dy.$$

It can be shown that if f is continuous on R, then it is integrable (but this is only a sufficient, not a necessary condition).

Double Integrals as Volumes

Suppose that $f(x,y)$ is a positive function over a rectangular region R in the xy−plane. Then we can interpret $\int\int_R f(x,y)\,dA$ as the volume of the solid region in 3 dimensions bounded below by R and bounded above by the surface $z = f(x,y)$. Each term $f(x_k,y_k)\,\Delta A_k$ in the Riemann sum S_n is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus is an approximation to the total volume of the solid. Define the volume to be

$$\text{Volume} = \lim_{n\to\infty} S_n = \int\int_R f(x,y)\,dA.$$



Fubini's Theorem for Calculating Double Integrals

Fubini's theorem enables us to express a double integral as an "iterated" or "repeated" integral. One way is to use "slices" rather than little rectangles. Let's look at the idea for integrating

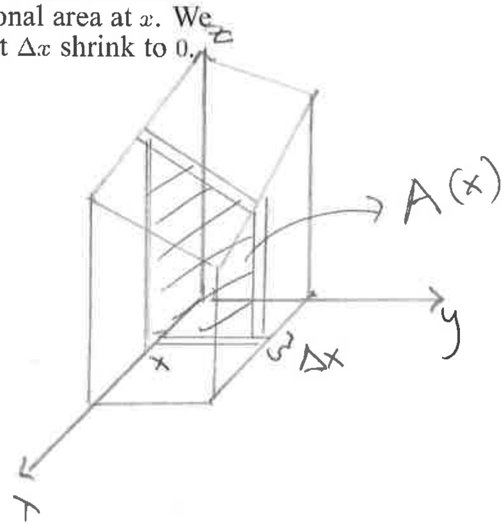
$$z = f(x,y) = 4 - x - y$$

over the rectangle

$$R : 0 \le x \le 2, 0 \le y \le 1$$

(It was discussed in Chapter 6 of Thomas). The idea of slices is to use thin slices perpendicular to, say, the x-axis, with very small width Δx in the x−direction. The volume of one of the slices with width Δx centered at a given x would be $(\Delta x)\,A(x)$, where $A(x)$ is the cross-sectional area at x. We add these much as we would a Riemann-sum, and then let Δx shrink to 0. The volume is then effectively

$$\text{Volume} = \int_{x=0}^{x=2} A(x)\,dx,$$



where $A(x)$ is the cross-sectional area, given by

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)dy.$$

Note that in calculating $A(x)$ for a given x, we keep x fixed when we work out the integral. So,

$$A(x) = \left[4y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1} = \left[\left(4 - x - \frac{1}{2}\right) - 0\right] = \frac{7}{2} - x.$$

The volume is then

$$\begin{aligned} \text{Volume} &= \int_{x=0}^{x=2} A(x)\,dx \\ &= \int_{x=0}^{x=2} \left[\frac{7}{2} - x\right] dx \\ &= \left[\frac{7}{2}x - \frac{x^2}{2}\right]_{x=0}^{x=2} = [7-2] - [0] = 5. \end{aligned}$$

We can write this as a repeated integral:

$$\text{Volume} = \int_0^2 \left[\int_0^1 (4 - x - y)dy\right] dx$$

or just

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)dy\,dx. \tag{1}$$

Notice that in the last integral, we first work out the inner integral by integrating with respect to y and then we integrate with respect to x.

We could also have taken slices with planes perpendicular to the $y-$axis. In much the same way, this leads to the formula

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)dx\,dy. \tag{2}$$

Here we first work out the inner integral, integrating with respect to x while keeping y fixed, and then integrate with respect to y. Both the repeated integrals in (1) and (2) give a specific way of computing the double integral

$$\int\int_R (4 - x - y)dA.$$

It is a general rule, due to the Italian mathematician Guido Fubini, that double integrals of continuous functions over rectangles can be expressed as repeated integrals:

Theorem 1 (Fubini's Theorem, first form)
Let $f(x, y)$ be continuous throughout the rectangular region $R: a \leq x \leq b, c \leq$

$y \leq d$. Then

$$\int\int_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Remarks

Thus we can integrate either first with respect to x, or first with respect to y. Of course we must take care that we are placing x and y in the correct intervals.

Example 1

Calculate $\int\int_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$ and $R: 0 \leq x \leq 2, -1 \leq y \leq 1$.

Solution

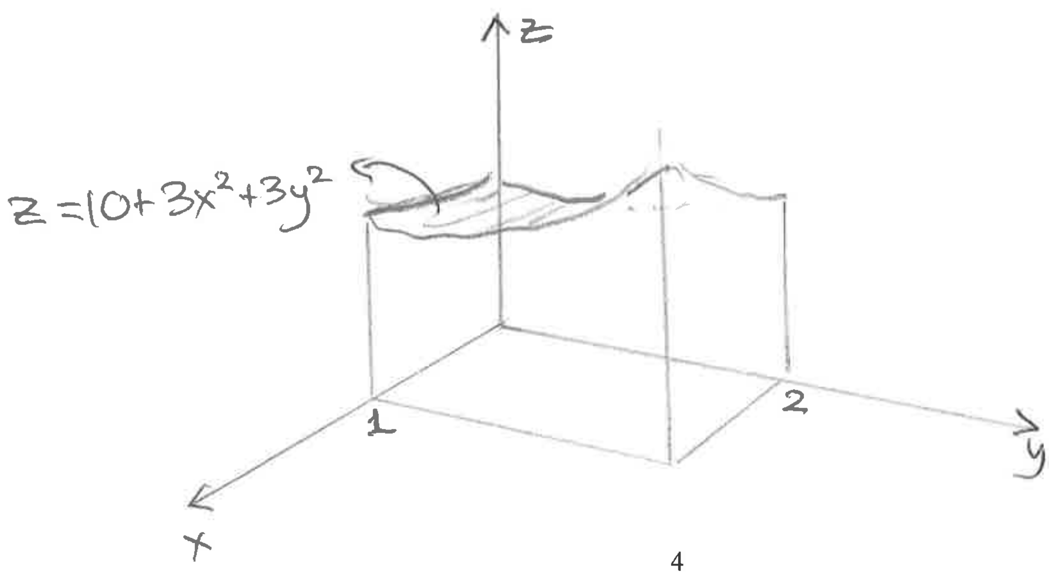
$$\begin{aligned}\int\int_R f(x,y)\,dA &= \int_{-1}^1 \int_0^2 \left(100 - 6x^2y\right) dx\,dy \\ &= \int_{-1}^1 \left[100x - 2x^3y\right]_{x=0}^{x=2} dy \\ &= \int_{-1}^1 [200 - 16y]\,dy \\ &= \left[200y - 8y^2\right]_{y=-1}^{y=1} \\ &= [200 - 8] - [-200 - 8] = 400.\end{aligned}$$

Example 2

Find the volume of the region bounded above by the (elliptical paraboloid) $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \leq x \leq 1, 0 \leq y \leq 2$.

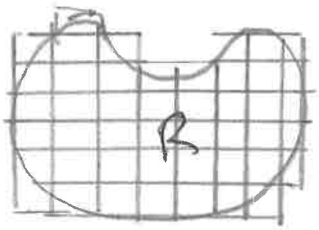
Solution

Note that $z = 10 + 3x^2 + y^2 \geq 0$ in R, so the volume is given by the double



integral

$$\begin{aligned}
V &= \int\int_R \left(10 + x^2 + 3y^2\right) dA \\
&= \int_0^1 \int_0^2 \left(10 + x^2 + 3y^2\right) dy\, dx \\
&= \int_0^1 \left[10y + x^2y + y^3\right]_{y=0}^{y=2} dx \\
&= \int_0^1 \left[20 + 2x^2 + 8\right] dx \\
&= \int_0^1 \left[28 + 2x^2\right] dx \\
&= \left[28x + \frac{2}{3}x^3\right]_{x=0}^{x=1} \\
&= \left[28 + \frac{2}{3}\right] - 0 = 28\frac{2}{3}.
\end{aligned}$$



Section 15.2: Double Integrals over General Regions

In this section, we consider double integrals over regions other than rectangles. If R is such a region, we can define the double integral as follows: we again cover R with a grid of small rectangles, whose union covers all of R. When we form a partition, we sum only over the small rectangles that lie inside R. Thus we exclude any small rectangle that lies partially or totally outside R. For "nice" regions, more and more of R is included as the norm of a partition approaches 0.

Once we have a partition of R, we number the rectangles in some order from 1 to n, and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle, and from (as before) a Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$

Again, as the norm $\|P\| \to 0$, the width and height of each rectangle approaches 0, and their number n goes to ∞. If $f(x, y)$ is a function continuous on R, then these Riemann sums converge to a limiting value, not depending on the particular partitions. This limit is the **double integral** of f over R:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = \int\int_R f(x, y)\, dA.$$

To describe this as a repeated integral, we observe that quite a lot of regions R can be described in the form

$$a \le x \le b,\, g_1(x) \le y \le g_2(x)$$

with g_1 and g_2 continuous on $[a, b]$, or its analogue

$$c \le y \le d,\, h_1(y) \le x \le h_2(y).$$

Example

If R is the unit disk, $\{(x, y) : x^2 + y^2 \le 1\}$, we see that x can take any value in the interval $[-1, 1]$, while for a given x, from $x^2 + y^2 \le 1$, we see that $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$. Thus R is the region

$$-1 \le x \le 1,\; -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}.$$

In terms of y, it can instead be described as

$$-1 \le y \le 1,\; -\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}.$$

Theorem 2 - Fubini's Theorem (Stronger/ More General Form)

Let $f(x, , y)$ be continuous on a region R.

(1) If R can be described in the form

$$a \le x \le b,\, g_1(x) \le y \le g_2(x)$$

with g_1 and g_2 continuous on $[a,b]$, then

$$\int\int_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(2) If R can be described in the form

$$c \le y \le d,\ h_1(y) \le x \le h_2(y)$$

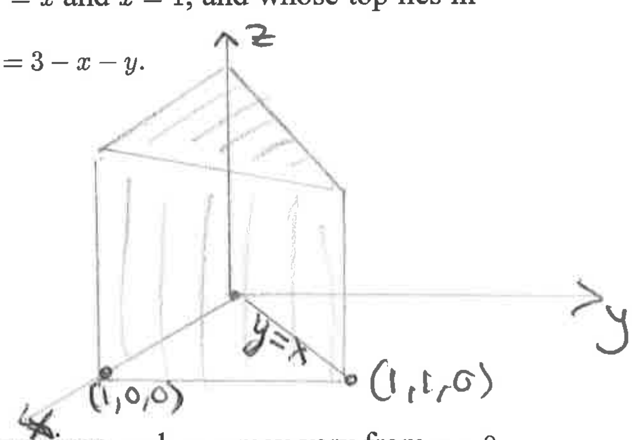
with h_1 and h_2 continuous on $[c,d]$, then

$$\int\int_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example 1

Find the volume of the prism whose base is the triangle in the xy–plane bounded by the x–axis and the lines $y = x$ and $x = 1$, and whose top lies in the plane

$$z = f(x,y) = 3 - x - y.$$



Solution

We see that x varies from 0 to 1. For any given such x, y may vary from $y = 0$ to $y = x$. So our R has form $R: 0 \le x \le 1, 0 \le y \le x$. Thus

$$\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y)\,dy\,dx \\ &= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left[\left(3x - x^2 - \frac{x^2}{2}\right) - 0\right] dx \\ &= \int_0^1 \left[3x - \frac{3}{2}x^2\right] dx \\ &= \left[\frac{3}{2}x^2 - \frac{1}{2}x^3\right]_{x=0}^{x=1} = \frac{3}{2} - \frac{1}{2} = 1. \end{aligned}$$

We could also do this using y as our "outer variable". Here y varies from 0 to 1. For a given y we see that x varies from y to 1, that is $y \le x \le 1$.

So

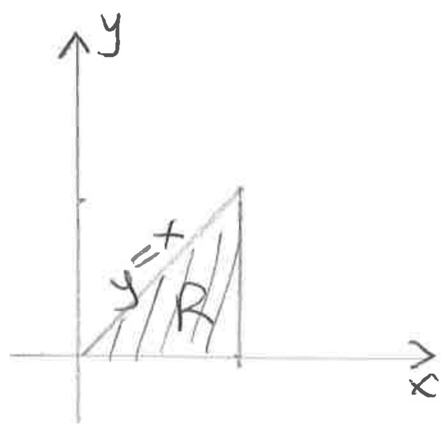
$$\begin{aligned}
V &= \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy \\
&= \int_0^1 \left[3x - \frac{x^2}{2} - yx\right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(\left[3 - \frac{1}{2} - y\right] - \left[3y - \frac{y^2}{2} - y^2\right]\right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy \\
&= \left[\frac{5}{2}y - 2y^2 + \frac{1}{2}y^3\right]_{y=0}^{y=1} = \left[\frac{5}{2} - 2 + \frac{1}{2}\right] - 0 = 1.
\end{aligned}$$

Example 2
Calculate

$$\int\int_R \frac{\sin x}{x} dA,$$

where R is the region in the $xy-$plane bounded by the $x-$axis, the line $y = x$, and the line $x = 1$.
Solution
We can describe R as $0 \leq x \leq 1$, $0 \leq y \leq x$. So

$$\begin{aligned}
&\int\int_R \frac{\sin x}{x} dA \\
&= \int_0^1 \int_0^x \frac{\sin x}{x} dy\, dx \\
&= \int_0^1 \left[\frac{\sin x}{x} y\right]_{y=0}^{y=x} dx \\
&= \int_0^1 \left[\frac{\sin x}{x} x - 0\right] dx \\
&= \int_0^1 \sin x\, dx \\
&= [-\cos x]_{x=0}^{x=1} = -\cos 1 + \cos 0 = 1 - \cos 1.
\end{aligned}$$



If we try to reverse the order of integration, we find we cannot evaluate the integral. We see that R is $0 \leq y \leq 1$, $y \leq x \leq 1$. So

$$\begin{aligned}
&\int\int_R \frac{\sin x}{x} dA \\
&= \int_0^1 \int_y^1 \frac{\sin x}{x} dx\, dy.
\end{aligned}$$

The problem now is that we cannot evaluate explicitly $\int_y^1 \frac{\sin x}{x} dx$. So the order in which we work out the integral can make a difference.

Finding Limits of Integration
We have seen that we can describe the unit disk with limits either as functions of x, or functions of y. Let's discuss doing this for

$$\int\int_{R} f(x,y)\,dA,$$

where R is a general region.

Using Vertical Cross Sections
Here the inner integral involves integration with respect to y. The steps are:
(1) Sketch the region R and label the bounding curves.
(2) Work out the interval $[a,b]$ in which x lies.
(2) For each $x \in [a,b]$, find the y−limits of integration, say $g_1(x) \leq y \leq g_2(x)$. Think of a vertical line L that cuts the horizontal axis at x. Mark the y−values where L enters and leaves R. Then

$$\int\int_{R} f(x,y)\,dA = \int_{a}^{b}\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Using Horizontal Cross Sections
Here the inner integral involves integration with respect to y. The steps are:
(1) Sketch the region R and label the bounding curves.
(2) Work out the interval $[c,d]$ in which y lies.
(2) For each $y \in [c,d]$, find the x−limits of integration, say $h_1(y) \leq x \leq h_2(y)$. Think of a horizontal line L that cuts the vertical axis at y. Mark the y−values where L enters and leaves R. Then

$$\int\int_{R} f(x,y)\,dA = \int_{c}^{d}\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

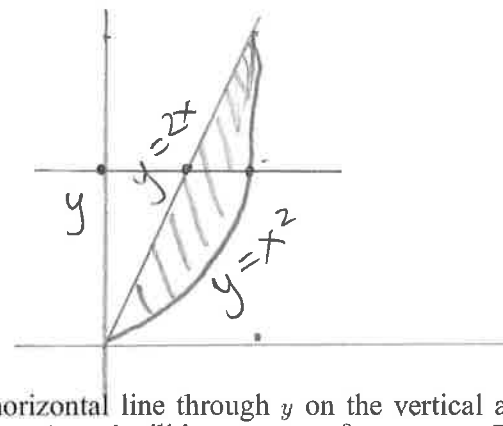
Example 3
Consider

$$\int_{0}^{2}\int_{x^2}^{2x} f(x,y)\,dy\,dx.$$

Sketch the region R of integration and write the integral with the order of integration reversed.
Solution
So R is

$$0 \leq x \leq 2,\ x^2 \leq y \leq 2x.$$

We see the bounding curves are $y = x^2$ and $y = 2x$. See the sketch.



For a given y, a horizontal line through y on the vertical axis, will enter at $y = 2x$, or $x = \frac{y}{2}$, and will leave at $y = x^2$ or $x = \sqrt{y}$. From the picture, we see that y runs from $y = 0$ to $y = 4$. Thus our integral can be written as

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f(x, y)\, dx\, dy.$$

Properties of Double Integrals

(a)

$$\int\int_R (f(x,y) \pm g(x,y))\, dA = \int\int_R f(x,y)\, dA \pm \int\int_R g(x,y)\, dA.$$

(b) If c is a number,

$$\int\int_R (cf(x,y))\, dA = c\int\int_R f(x,y)\, dA.$$

(c) If R is the union of non-overlapping regions R_1 and R_2,

$$\int\int_R f(x,y)\, dA = \int\int_{R_1} f(x,y)\, dA + \int\int_{R_2} f(x,y)\, dA.$$

Example 4

Find the volume of the wedgeable solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x–axis.

Solution

Observe that the curves $y = 2\sqrt{x}$ and $y = 4x - 2$ intersect where

$$\begin{aligned} 2\sqrt{x} &= 4x - 2 \\ &\Rightarrow \sqrt{x} = 2x - 1 \\ &\Rightarrow x = (2x-1)^2 = 4x^2 - 4x + 1 \\ &\Rightarrow 4x^2 - 5x + 1 = 0 \\ &\Rightarrow (4x-1)(x-1) = 0 \\ &\Rightarrow x = \frac{1}{4} \text{ or } x = 1. \end{aligned}$$

For $x = \frac{1}{4}$, we have $y = -1$, which is not part of R. So we consider only $x = 1$ where $y = 2$. We sketch the region R of integration, and see that y varies from 0 to 2. For a given y, we see that the horizontal line enters R where

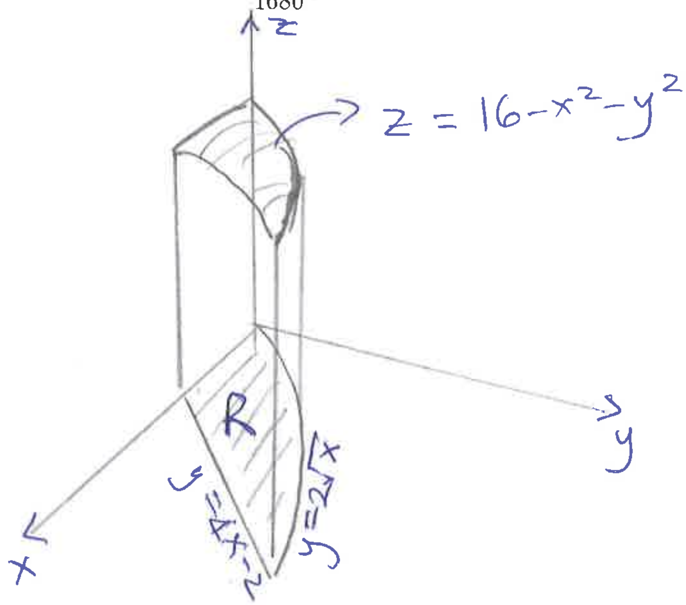
$$y = 2\sqrt{x} \Longleftrightarrow \frac{y^2}{4} = x$$

and leaves where

$$y = 4x - 2 \Longleftrightarrow \frac{y+2}{4} = x.$$

So the volume is the iterated integral

$$\begin{aligned} & \int\int_R \left(16 - x^2 - y^2\right) dA \\ = & \int_0^2 \int_{y^2/4}^{(y+2)/4} \left(16 - x^2 - y^2\right) dx\, dy \\ = & \int_0^2 \left[16x - \frac{x^3}{3} - y^2 x\right]_{x=y^2/4}^{x=(y+2)/4} dy \\ = & \int_0^2 \left(\left[4(y+2) - \frac{1}{3}(y+2)^3 - y^2(y+2)/4\right] - \left[4y^2 - \frac{y^6}{192} - \frac{y^4}{4}\right]\right) dy \\ = & \frac{20803}{1680}. \end{aligned}$$



Section 15.3: Area by Double Integration

We see how to use double integrals to calculate areas. If R is a region in the $xy-$plane, then integrating $f(x, y) = 1$ over R gives the area:

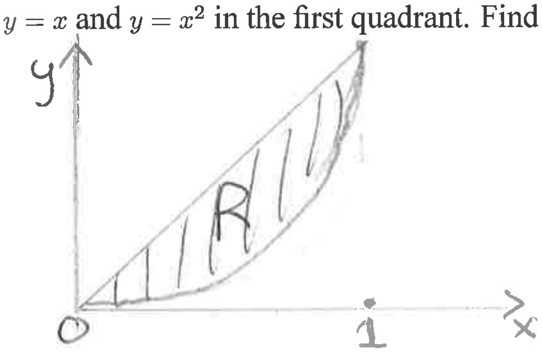
Definition
The **area** of a closed bounded plane region R is

$$A = \int\int_R 1 \, dA.$$

Remark
Sometimes we just drop the 1 and write

$$A = \int\int_R dA.$$

Example 1
Let R be the region bounded by $y = x$ and $y = x^2$ in the first quadrant. Find its area.
Solution
First, we sketch the region.

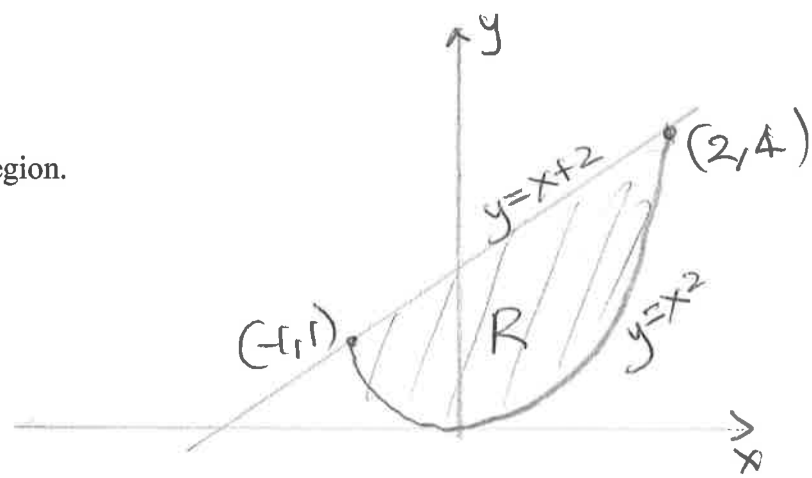


We see that $x = x^2$ at $x = 0$ and $x = 1$. Thus from the sketch $0 \leq x \leq 1$. For a given such x, we see that the vertical line L at x, enters R where $y = x^2$ and leaves where $y = x$. So $x^2 \leq y \leq x$. Thus

$$\begin{aligned} A &= \int_0^1 \int_{x^2}^{x} 1 \, dy \, dx \\ &= \int_0^1 [y]_{y=x^2}^{y=x} \, dx \\ &= \int_0^1 \left(x - x^2\right) dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

Example 2
Let R be the region bounded by $y = x + 2$ and $y = x^2$. Find its area.
Solution

First, we sketch the region.



Note that

$$\begin{aligned} x^2 &= x+2 \\ &\Longleftrightarrow x^2 - x - 2 = 0 \\ &\Longleftrightarrow (x-2)(x+1) = 0 \\ &\Longleftrightarrow x = -1 \text{ or } x = 2. \end{aligned}$$

So the curves intersect at $(-1,1)$ and $(2,4)$. If we use x as our "outer" variable, we see that R is

$$R : -1 \le x \le 2,\ x^2 \le y \le x+2.$$

So

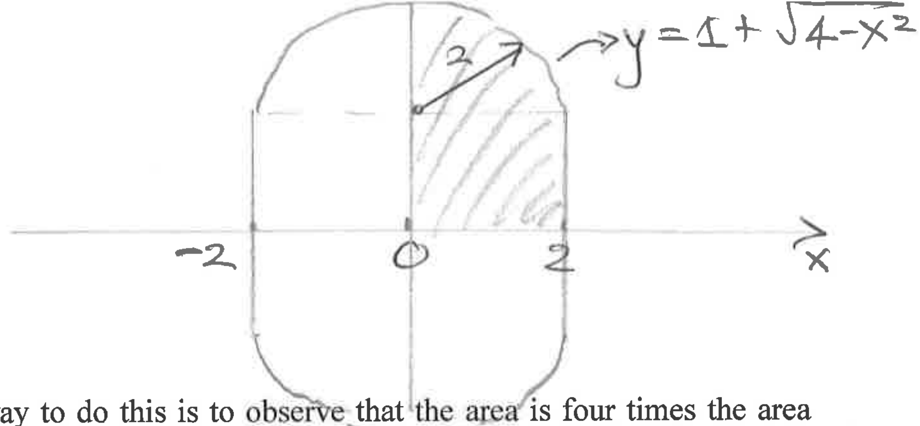
$$\begin{aligned} A &= \int_{-1}^{2} \int_{x^2}^{x+2} 1\, dy\, dx \\ &= \int_{-1}^{2} [y]_{y=x^2}^{y=x+2}\, dx \\ &= \int_{-1}^{2} \left[x + 2 - x^2\right] dx \\ &= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{x=-1}^{x=2} = \left[2 + 4 - \frac{8}{3}\right] - \left[\frac{1}{2} - 2 + \frac{1}{3}\right] = 4\frac{1}{2}. \end{aligned}$$

(If instead we use y as our outer variable, we have to divide R into two regions.)

Example 3
Find the area of the playing field described by

$$R : -2 \le x \le 2,\ -1 - \sqrt{4 - x^2} \le y \le 1 + \sqrt{4 - x^2}.$$

Solution
First sketch the area.



One way to do this is to observe that the area is four times the area of that part of the region in the first quadrant. We can describe that part of the region in the quadrant as

$$R_1 : 0 \le x \le 2, 0 \le y \le 1 + \sqrt{4 - x^2},$$

so the area in the first quadrant is

$$\begin{aligned}
\int\int_{R_1} dA &= \int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\, dx \\
&= \int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx \\
&= \int_0^2 dx + \int_0^2 \sqrt{4 - x^2} dx \\
&= 2 + \int_0^2 \sqrt{4 - x^2} dx.
\end{aligned}$$

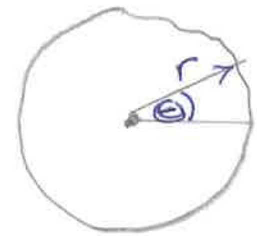
Let's make the substitution $x = 2\cos\theta$ in the second integral. (The book does it differently). When $x = 0$, we need $\theta = \frac{\pi}{2}$ and when $x = 2$, we need $\theta = 0$. So

$$\begin{aligned}
&\int_0^2 \sqrt{4 - x^2} dx \\
&= \int_{\frac{\pi}{2}}^0 \sqrt{4 - 4\cos^2\theta} \frac{dx}{d\theta} d\theta \\
&= \int_{\frac{\pi}{2}}^0 (2\sin\theta)(-2\sin\theta)\, d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \frac{1 - \cos(2\theta)}{2} d\theta \\
&= 4\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} = 4\left[\frac{\pi}{4} - 0\right] - 0 = \pi.
\end{aligned}$$

So the total area in the first quadrant is $2 + \pi$. Then the area of the original playing field is

$$4(2 + \pi) = 8 + 4\pi.$$

Section 15.4: Double Integrals in Polar Form

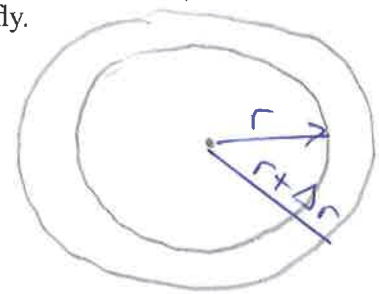


So far we have described regions using "rectangular coordinates" x, y. Sometimes it is easier to use polar coordinates (r, θ). Thus we express

$$(x, y) = (r\cos\theta, r\sin\theta).$$

For example the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$ may also be expressed in polar coordinates as $\{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$.

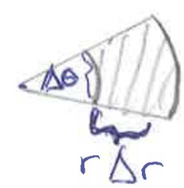
When we express integrals in polar coordinates, there is a factor of r that arises, and let us motivate this briefly.



The area of the annulus center 0, with inner radius r and outer radius $r + \Delta r$, may be calculated by subtracting the area of the circle center 0, radius r from the area of the concentric circle radius $r + \Delta r$:

$$\begin{aligned} & \pi(r + \Delta r)^2 - \pi r^2 \\ = \ & \pi r^2 + 2\pi r\Delta r + \pi(\Delta r)^2 - \pi r^2 \\ = \ & (2\pi\Delta r)\left(r + \frac{\Delta r}{2}\right) \approx (2\pi\Delta r)\, r. \end{aligned}$$

When integrating, the Δr becomes dr, and there is the factor r. Thus the further we are from the origin, that is the larger is r, the bigger effect we have in increasing the area of the annulus. When instead we consider a small angular wedge, with angular opening $\Delta\theta$ and r varying from r to $r + \Delta r$, its area is approximately $r\ \Delta r\,(\Delta\theta)$.



Suppose now that we describe a region R in polar coordinates as

$$R : \alpha \leq \theta \leq \beta,\ g_1(\theta) \leq r \leq g_2(\theta).$$

Then

$$\int\int_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

or just

$$\int\int_R f(r, \theta)\, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

(Don't forget that factor of r).

Finding Limits of Integration for a region R

(1) Sketch the region and label the bounding curves.

(2) Find the r–limits of integration. Thus draw a line L from the origin cutting the region R. Mark the r–values where L enters and leaves R. These will usually depend on the angle θ that L makes with the x–axis.
(3) Find the θ–limits of integration.

Area in Polar Coordinates
When we integrate 1 over R, we obtain the area:

$$\text{Area} = \int\int_R r\ dr\ d\theta.$$

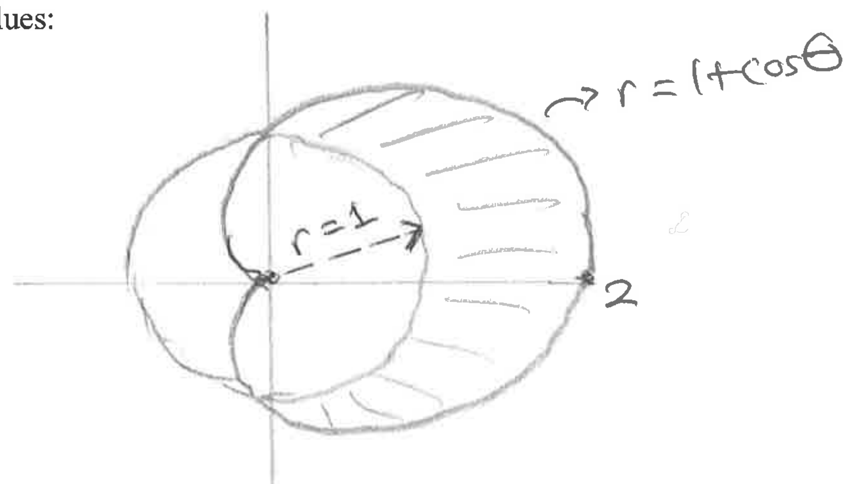
Example 1
The quarter circle $R: 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2$. Here the area is

$$\int_0^{\frac{\pi}{2}} \int_0^2 r\ dr\ d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\ d\theta = \pi.$$

Example 2
Find the limits of integration for the region R that lies inside over the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.
Solution
(1) Let's start with a table of values:

θ	r
0	2
$\frac{\pi}{4}$	$1 + \frac{1}{\sqrt{2}}$
$\frac{\pi}{2}$	1
π	0

Next, sketch



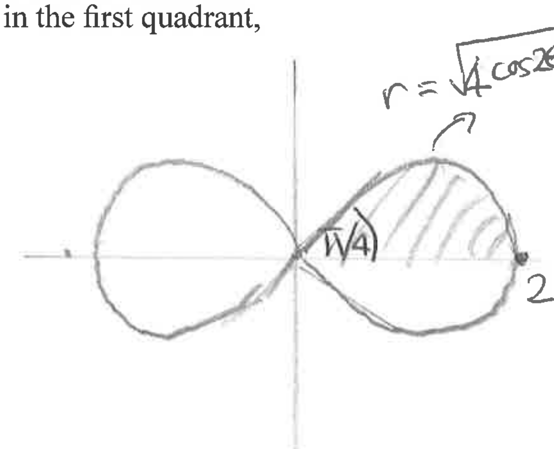
(2) Find r–limits of integration.
For any given θ, we see that a ray from the origin making an angle θ with the

$x-$axis enters at $r = 1$ and leaves at $r = 1 + \cos\theta$.
(3) Find the $\theta-$limits of integration.
We see that θ varies from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. So

$$\int\int_R f(r,\theta)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta.$$

Example 3
Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.
Solution
Warning: this problem is not so clearly stated. You have to know a priori what a lemniscate looks like. We use symmetry and consider only that part of the lemniscate in the first quadrant. The total area will be four times that in the first quadrant. We draw the part of the lemniscate in the first quadrant, using a table:

θ	r
0	$\sqrt{4} = 2$
$\frac{\pi}{8}$	$\sqrt{4\cos\frac{\pi}{4}} = 2/\sqrt[4]{2}$
$\frac{\pi}{4}$	0



We see that

$$0 \le r \le \sqrt{4\cos 2\theta}$$

and

$$0 \le \theta \le \frac{\pi}{4}.$$

So the area in the first quadrant is

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta \\ =& \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta \\ =& \int_0^{\frac{\pi}{4}} \frac{4\cos 2\theta}{2} d\theta \\ =& \left[\sin 2\theta\right]_{\theta=0}^{\theta=\frac{\pi}{4}} = \sin\frac{\pi}{2} = 1. \end{aligned}$$

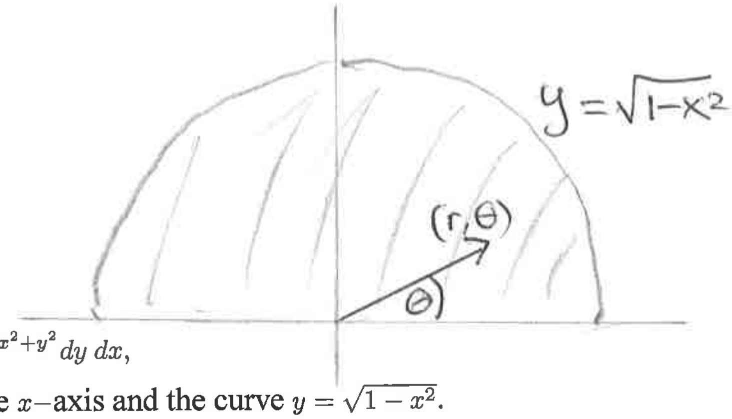
The total area is then 4.

From Cartesian/ Rectangular to Polar Coordinates
Suppose we want to integrate $f(x,y)$ over a region R, described in Cartesian or x,y coordinates. Suppose that we now describe R in polar coordinates, and call this G. Then

$$\int\int_R f(x,y)\, dx\, dy = \int\int_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example 4

Evaluate



$$\int\int_R e^{x^2+y^2}\,dy\,dx,$$

where R is the region bounded by the x–axis and the curve $y = \sqrt{1-x^2}$.

Solution

Let us use polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$. From the sketch we see that $0 \le r \le 1$ and $0 \le \theta \le \pi$. Note that

$$e^{x^2+y^2} = e^{r^2\cos^2\theta + r^2\sin^2\theta} = e^{r^2}.$$

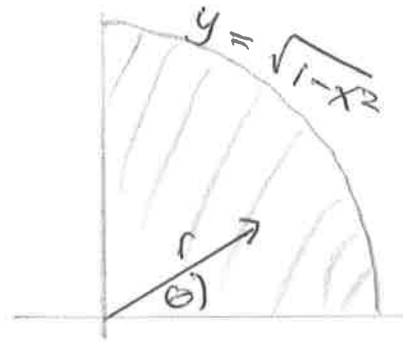
Then

$$\begin{aligned}
&\int\int_R e^{x^2+y^2}\,dy\,dx \\
=\;& \int_0^{\pi}\int_0^1 e^{r^2} r\,dr\,d\theta \\
=\;& \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_{r=0}^{r=1} d\theta \\
=\;& \int_0^{\pi}\left[\frac{1}{2}e^1 - \frac{1}{2}\right] d\theta \\
=\;& \pi\frac{1}{2}(e-1) = \frac{\pi}{2}(e-1).
\end{aligned}$$

Example 5

Evaluate

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$



Solution

So our R is

$$R:\ 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}.$$

From the sketch, we see that in polar coordinates, this is

$$G:\ 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}.$$

So

$$\begin{aligned}
&\int\int_R (x^2+y^2)\,dA \\
=\;& \int\int_G r^2\,r\,dr\,d\theta \\
=\;& \int_0^{\frac{\pi}{2}}\int_0^1 r^3\,dr\,d\theta \\
=\;& \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta \\
=\;& \int_0^{\frac{\pi}{2}} \frac{1}{4}d\theta = \frac{\pi}{8}.
\end{aligned}$$

Example 6
Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the $xy-$ plane.
Solution
So our R is

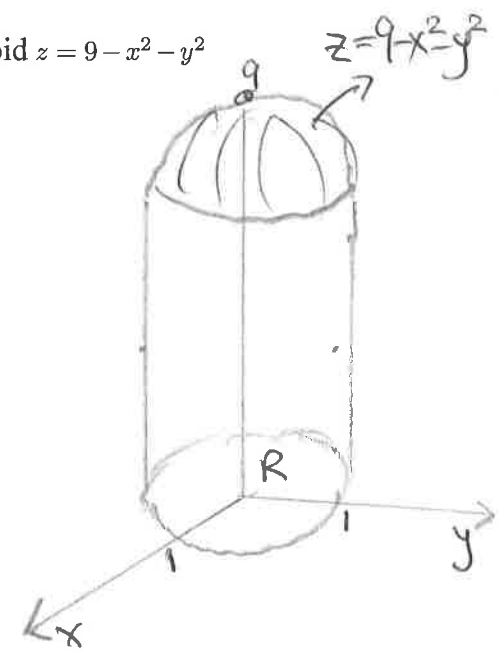
$$R: \ -1 \le x \le 1, \ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$

In polar coordinates, this is

$$G: \ 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

So the volume is

$$\begin{aligned} & \int\int_R (9 - x^2 - y^2)\, dA \\ = & \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta \\ = & \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta \\ = & \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta \\ = & \int_0^{2\pi} \left[\frac{9}{2} - \frac{1}{4} - 0 \right] d\theta = (2\pi)\frac{17}{4} = \frac{17\pi}{2}. \end{aligned}$$

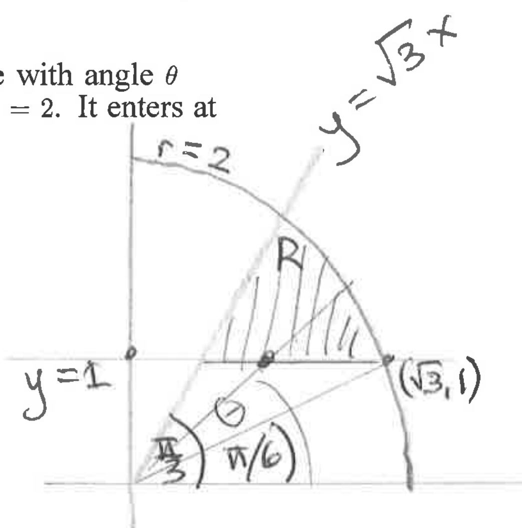


Example 7
Find the area of the region R in the $xy-$plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$ and below the line $y = \sqrt{3}x$.
Solution
We sketch the region. Let us try to describe it in polar coordinates.
(I) First, the line $y = \sqrt{3}x$ makes an angle $\frac{\pi}{3}$ to the $x-$axis. (To see this, observe e.g. that the point $(1, \sqrt{3})$ lies on this line, and use elementary trigonometry)
(II) Next the line $y = 1$ intersects the circle $x^2 + y^2 = 4$, where $x^2 = 4 - 1 = 3$, so $(x, y) = (\sqrt{3}, 1)$. The line from the origin to $(\sqrt{3}, 1)$ makes an angle of $\frac{\pi}{6}$ with the $x-$axis.
So the angle θ in R varies from $\frac{\pi}{6}$ to $\frac{\pi}{3}$.
(III) Now for a given such θ, let us see where the line with angle θ enters and leaves R. The easy part is that it leaves at $r = 2$. It enters at $(x, y) = (r\cos\theta, r\sin\theta)$, where

$$y = r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta} = \csc\theta.$$

Thus in polar coordinates, our region is

$$G: \frac{\pi}{6} \le \theta \le \frac{\pi}{3}, \ \csc\theta \le r \le 2.$$



Then the area is

$$
\begin{aligned}
\int\int_R dA &= \int\int_G r\, dr\, d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc\theta}^{2} r\, dr\, d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\left[4-(\csc\theta)^2\right] d\theta \\
&= \frac{1}{2}[4\theta+\cot\theta]_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{3}} \\
&= \frac{1}{2}\left[4\frac{\pi}{3}+\frac{1}{\sqrt{3}}\right]-\frac{1}{2}\left[4\frac{\pi}{6}+\sqrt{3}\right] \\
&= \frac{\pi}{3}+\frac{1}{2}\left(\frac{\sqrt{3}}{3}-\sqrt{3}\right) \\
&= \frac{\pi}{3}-\frac{\sqrt{3}}{3}.
\end{aligned}
$$

Section 15.5: Triple Integrals in Rectangular Coordinates

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region D in space, we can define the integral of f over D using Riemann sums much as we did for double integrals. We partition a rectangular boxlike region containing D into small rectangular boxes, with kth box having volume $\Delta V = (\Delta x_k)(\Delta y_k)(\Delta z_k)$, where the sides have lengths $\Delta x_k, \Delta y_k, \Delta z_k$. We choose a point (x_k, y_k, z_k) in the kth box, and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k$$

Let the norm $\|P\|$ of the partition be the largest of any of the sides $\Delta x_k, \Delta y_k, \Delta z_k$. As $\|P\| \to 0$, the number n of cells $\to \infty$, and the sums S_n hopefully approach a limit. We call this limit the triple integral of F over D and write

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\, dV$$

or

$$\lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\, dx\, dy\, dz.$$

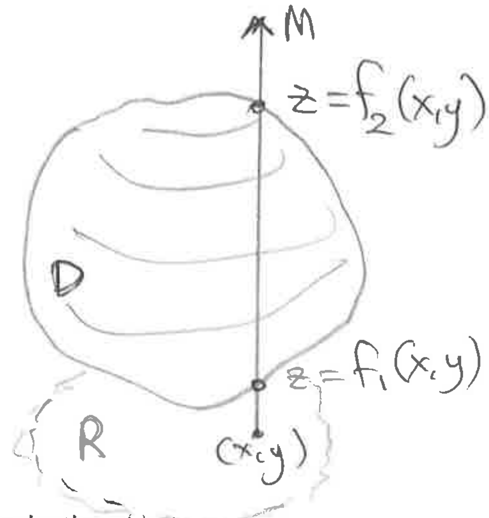
Volume of a Region in Space

If we integrate $F = 1$ over D, we obtain the volume of D:

Definition

The **volume** of a closed bounded region D in space is

$$V = \iiint_D 1\, dV, \text{ or just } \iiint dV.$$

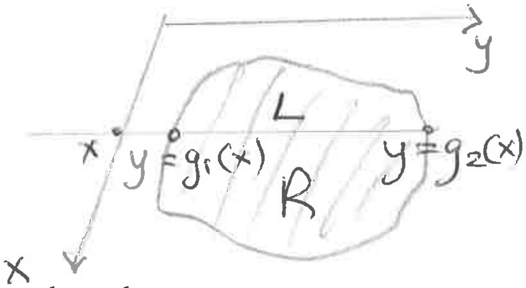


Finding Limits of Integration in the Order $dz\, dy\, dx$.

To evaluate

$$\iiint_D F(x, y, z)\; dV$$

(1) Sketch the region D along with its shadow, or vertical projection R, in the xy-plane. Label the upper and lower bounding curves of R.



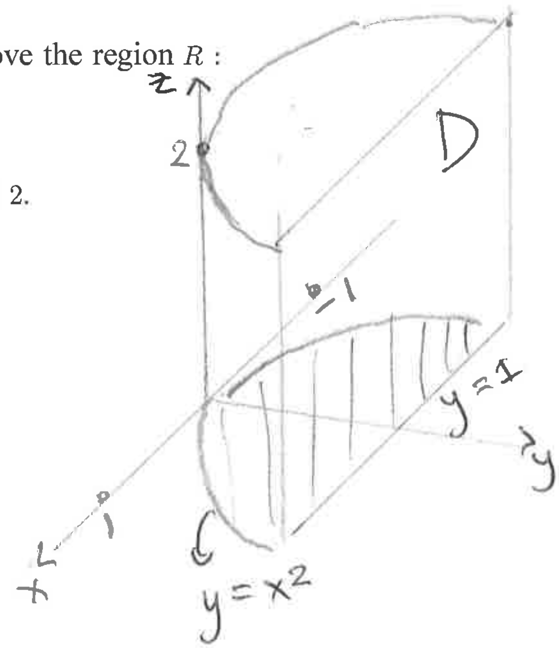
(2) Find the $z-$limits of integration. Draw a line M passing through a typical point (x, y) in R parallel to the $z-$axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the $z-$limits of integration.
(3) Find the $y-$limits of integration. Draw a line L passing through (x, y) parallel to the $y-$axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the $y-$limits of integration.
(4) Find the $x-$limits of integration. Choose $x-$limits that include all lines through R parallel to the $y-$axis, giving $a \leq x \leq b$. The integral is then

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z-f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) dz\, dy\, dx.$$

Remark
This formulation requires D to be bounded above and below by a surface. Sometimes we shall need to have a different outer variable than x.

Example 1
Let D be the "cylinder" between $z = 0$ and $z = 2$, above the region R : $-1 \leq x \leq 1, x^2 \leq y < 1$. Find $\int\int\int_D yz\, dV$.
Solution
First, sketch the region: we see that D is given by

$$D: \ -1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 2.$$



So

$$\begin{aligned}
& \int\int\int_D yz dV \\
= & \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{2} yz\, dz\, dy\, dx \\
= & \int_{-1}^{1}\int_{x^2}^{1} \left[y\frac{z^2}{2}\right]_{z=0}^{z=2} dy\, dx \\
= & \int_{-1}^{1}\int_{x^2}^{1} [2y]\, dy\, dx \\
= & \int_{-1}^{1} \left[y^2\right]_{y=x^2}^{y=1} dx \\
= & \int_{-1}^{1} \left[1 - x^4\right] dx = \left[x - \frac{x^5}{5}\right]_{x=-1}^{x=1} = 2\left[1 - \frac{1}{5}\right] = 1\frac{3}{5}.
\end{aligned}$$

Example 2
Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.
Solution
We want to calculate

$$V = \int\int\int_D dz\, dy\, dx.$$

The two surfaces intersect where

$$x^2 + 3y^2 = 8 - x^2 - y^2$$

or

$$2x^2 + 4y^2 = 8 \Rightarrow x^2 + 2y^2 = 4.$$

Note that we also need $z > 0$. When we project D onto the xy–axis, we obtain the boundary curve of R, namely the ellipse

$$x^2 + 2y^2 = 4.$$

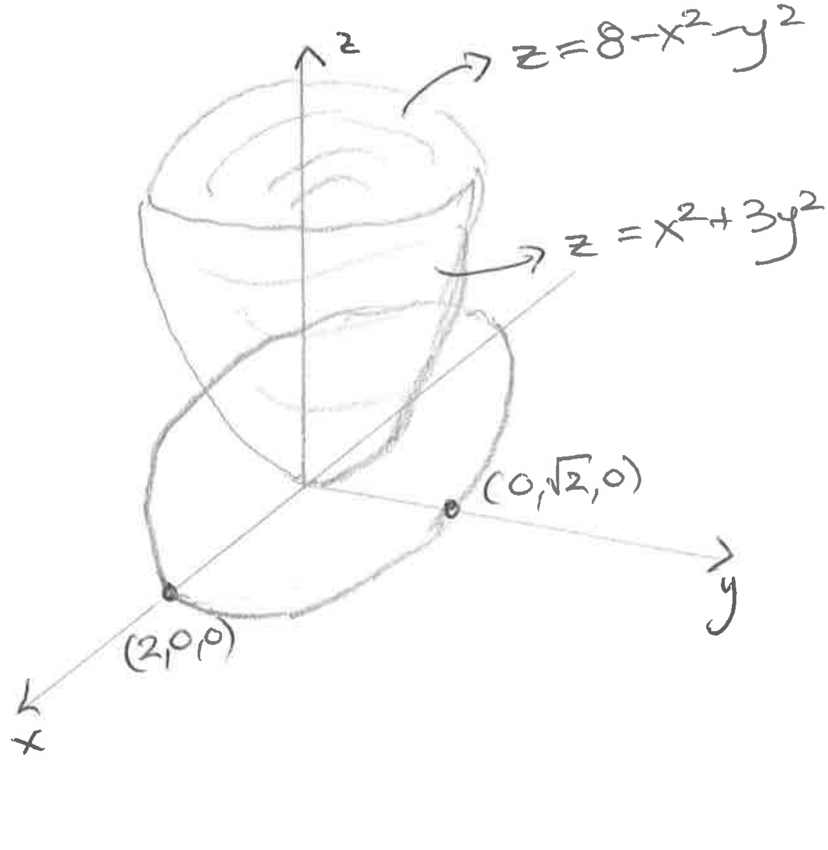
We see that inside R, x can vary from -2 to 2, while for a given x,

$$-\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}.$$

(1) Now we find the z-limits of integration. The line M parallel to the z–axis passing through a typical point (x, y) in R enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

(2) Next we find the y–limits of integration. The line L through (x, y) parallel to the y–axis enters R at $y = -\sqrt{\frac{4-x^2}{2}}$ and leaves at $y = \sqrt{\frac{4-x^2}{2}}$.

(3) Finally, we have already seen that the x–limits of integration are $-2 \leq x \leq 2$.



So the volume is

$$
\begin{aligned}
V &= \iiint_D dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left((8-x^2-y^2)-(x^2+3y^2)\right) dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left(8-2x^2-4y^2\right) dy\,dx \\
&= \int_{-2}^{2}\left[\left(8-2x^2\right)y-\frac{4}{3}y^3\right]_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=\sqrt{\frac{4-x^2}{2}}} dx \\
&= \int_{-2}^{2}\left[\left(8-2x^2\right)2\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\sqrt{\frac{4-x^2}{2}}\right)^3\right] dx \\
&= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\sqrt{\frac{4-x^2}{2}}\right)^3\right] dx \\
&= \int_{-2}^{2}\left[8\left(\sqrt{\frac{4-x^2}{2}}\right)^3-\frac{8}{3}\left(\sqrt{\frac{4-x^2}{2}}\right)^3\right] dx \\
&= \left(8-\frac{8}{3}\right)\left(\frac{1}{\sqrt{2}}\right)^3\int_{-2}^{2}\left(\sqrt{4-x^2}\right)^{3/2} dx \\
&= \frac{8}{3\sqrt{2}}\int_{-2}^{2}\left(\sqrt{4-x^2}\right)^{3/2} dx.
\end{aligned}
$$

Now we make the substitution $x = 2\sin u$. When $x = \pm 2$, we see that $u = \pm\frac{\pi}{2}$:

$$\begin{aligned}
&\int_{-2}^{2} \left(\sqrt{4-x^2}\right)^{3/2} dx \\
= \; & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sqrt{4\cos^2 u}\right)^{3/2} 2\cos u \, du \\
= \; & 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 u \, du \\
= \; & 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2 u\right)\left(1-\sin^2 u\right) du \\
= \; & 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2 u - \cos^2 u \sin^2 u\right) \\
= \; & 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2u}{2} - \left(\frac{\sin 2u}{2}\right)^2\right) du \\
= \; & 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2u}{2} - \frac{1-\cos 4u}{8}\right) du \\
= \; & 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{8} + \frac{1}{2}\cos 2u + \frac{1}{8}\cos 4u\right) du \\
= \; & 16\left[\frac{3}{8}u + \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right]_{u=-\frac{\pi}{2}}^{u=\frac{\pi}{2}} \\
= \; & 16\frac{3}{8}\pi = 6\pi.
\end{aligned}$$

Then the volume is

$$V = \frac{8}{3\sqrt{2}}(6\pi) = \frac{16}{\sqrt{2}}\pi = 8\sqrt{2}\pi.$$

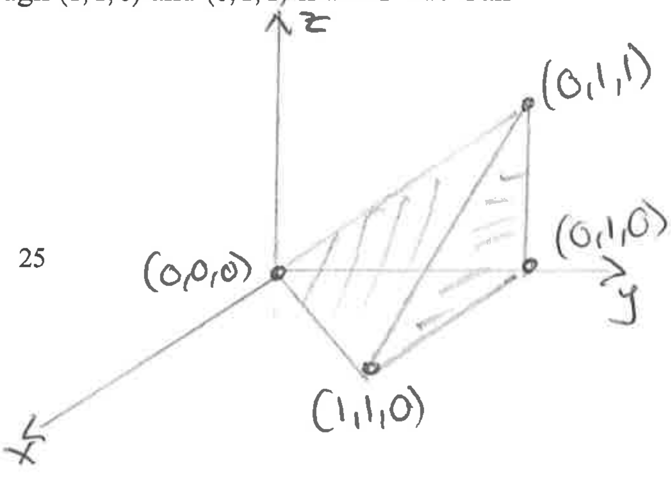
Example 3

Let D be the tetrahedron with vertices $(0,0,0,)$, $(1,1,0)$, $(0,1,0)$, and $(0,1,1)$. Use the order of integration $dy\,dz\,dx$ to set up the limits of integration for

$$\int\int\int_D F(x,y,z)\,dy\,dz\,dx.$$

Solution

First we sketch D. This we can do by just plotting the four given points and then drawing lines between them. We also look at the equations of these lines. For example that passing through $(1,1,0)$ and $(0,1,1)$ is $z = 1 - x$. Can you see this?



The bounding lines in the $xy-$ plane (or $z = 0$ plane) are $y = 1, y = x, x = 0$. As y is our innermost variable, we have to get limits for y in terms of (x, z), so we look for the shadow of D in the $xz-$plane. The upper (or right-hand) bounding surface lies in the plane $y = 1$. The left-side of the tetrahedron has bounding lines

$$y = x \text{ in the plane } z = 0;$$

$$z = y \text{ in the plane } x = 0;$$
$$z = 1 - x \text{ in the plane } y = 1.$$

Note that on all three of these lines,

$$z + x = y,$$

and this is the equation of the plane defining the left-hand side of the tetrahedron. (We could also derive this using vectors and normals, in the way we did in Section 12.5).
(1) First we find the $y-$limits of integration. We see that the line through a point (x, z) parallel to the y-axis enters D where $y = x + z$ and leaves when $y = 1$.
(2) Second, we see that the line L through (x, z) parallel to the $z-$axis enters at $z = 0$ and leaves at $z = 1 - x$.
(3) We see that $0 \leq x \leq 1$.
So the integral is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) \; dy \, dz \, dx.$$

Example 4
Find the volume of the tetrahedron in Example 2.
Solution

This is

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x} \int_{x+z}^1 1 \, dy \, dz \, dx \\
&= \int_0^1 \int_0^{1-x} (1-(x+z)) \, dz \, dx \\
&= \int_0^1 \left[(1-x) z - \frac{z^2}{2} \right]_{z=0}^{z=1-x} dx \\
&= \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2} \right] dx \\
&= \frac{1}{2} \int_0^1 (1-x)^2 \, dx \\
&= \left[-\frac{1}{2(3)} (1-x)^3 \right]_{x=0}^{x=1} = \frac{1}{6}.
\end{aligned}
$$

Note that we could also do this in $dz\ dy\ dx$ or $dy\ dx$ directions.

Average Value of a Function

The average value of a function F over a region D in space is defined by

$$\text{Average Value of } F \text{ over } D = \frac{1}{\text{Volume of } D} \iiint_D F \, dV.$$

Thus if $F(x,y,z)$ is the temperature at a point (x,y,z) in D, then this would give the average temperature over D.

Properties of Triple Integrals

(a)

$$\iiint_D (F(x,y,z) \pm G(x,y,z)) \, dV = \iiint_D F(x,y,z) \, dV \pm \iiint_D G(x,y,z) \, dV.$$

(b) If c is a number,

$$\iiint_D (cF(x,y,z)) \, dV = c \iiint_D F(x,y,z) \, dV.$$

(c) If D is the union of non-overlapping regions D_1 and D_2,

$$\iiint_D F(x,y,z) \, dV = \iiint_{D_1} F(x,y,z) \, dV + \iiint_{D_2} F(x,y,z) \, dV.$$

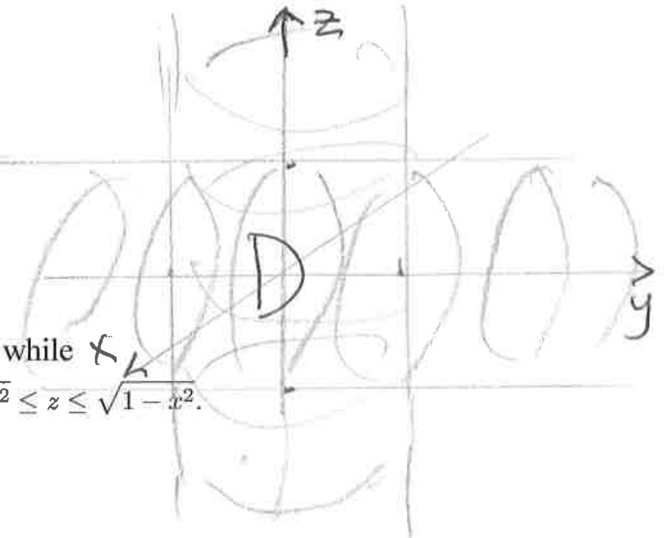
Example 5

(a) Find the volume of the intersection D of the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

(b) Assume the temperature at a point (x,y,z) in D is $T(x,y,z) = x^2$ (so is independent of y, z). Find the average temperature over D.

Solution

(a) Our region D is the set of all points (x,y,z) with both $x^2 + y^2 \leq 1$ and



$x^2 + z^2 \leq 1$. So we see that x ranges from -1 to 1, while

$$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \text{ and } -\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}.$$

So

$$\begin{aligned}
V &= \iiint_D dV \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 dz\, dy\, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(2\sqrt{1-x^2}\right) dy\, dx \\
&= \int_{-1}^{1} \left[2\sqrt{1-x^2y}\right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} \left[2\sqrt{1-x^2}\right]\left[2\sqrt{1-x^2}\right] dx \\
&= 4\int_{-1}^{1} \left(1-x^2\right) dx \\
&= 4\left[x - \frac{x^3}{3}\right]_{x=-1}^{x=1} = 4\left(2 - \frac{2}{3}\right) = \frac{16}{3}.
\end{aligned}$$

(b) Here we want to compute

$$\begin{aligned}
&\frac{1}{V}\iiint_D T(x,y,z)\, dV \\
= &\frac{3}{16}\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 dz\, dy\, dx.
\end{aligned}$$

Proceeding as above, we see this is

$$\begin{aligned}
&\frac{3}{16}\int_{-1}^{1} x^2\left(4\left(1-x^2\right)\right) dx \\
= &\frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{x=-1}^{x=1} = \frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{1}{5}.
\end{aligned}$$

Section 15.6: Moments and Centers of Mass

We learn how to calculate masses and centers of mass of objects in space.

Masses and First Moments
Suppose that $\delta(x,y,z)$ is the density (mass per unit volume) at the point (x,y,z) of an object filling a region D in space. The (total) mass of the object is obtained by integrating $\delta(x,y,z)$ over D. To see this, suppose we partition D into n small pieces. Say the kth one has volume ΔV_k and (x_k,y_k,z_k) is a point in the kth one, and the piece is so small that the density is approximately $\delta(x_k,y_k,z_k)$. Then its mass is approximately $\Delta m_k = \delta(x_k,y_k,z_k)\Delta V_k$. To obtain the total mass, we add the masses of the small pieces, giving

$$M = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta m_k = \lim_{n\to\infty}\sum_{k=1}^{n}\delta(x_k,y_k,z_k)\Delta V_k = \int\int\int_D \delta(x,y,z)\,dV.$$

The **first moment** of a solid region D about a coordinate plane is defined as the triple integral over D of the distance from the plane to a point (x,y,z) in D, multiplied by the density of the solid at that point.

For example, the **first moment** about the yz–plane is the integral

$$M_{yz} = \int\int\int_D x\delta(x,y,z)\,dV.$$

Similarly,

$$M_{xz} = \int\int\int_D y\delta(x,y,z)\,dV.$$
$$M_{xy} = \int\int\int_D z\delta(x,y,z)\,dV.$$

The **center of mass** is defined as

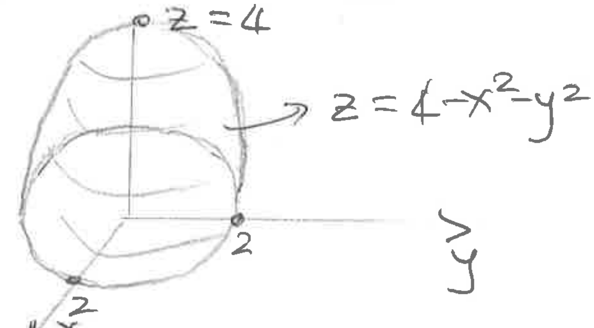
$$(\bar{x},\bar{y},\bar{z})$$

where

$$\begin{aligned}\bar{x} &= M_{yz}/M;\\ \bar{y} &= M_{xz}/M;\\ \bar{z} &= M_{yz}/M.\end{aligned}$$

We can often just use symmetry to "guess" at least some of $\bar{x},\bar{y},\bar{z}$. For example, the center of mass of a solid box of equal sides and constant density is fairly clearly the center of the box. Its coordinates are the midpoints of each of the sides.

Example 1
Find the center of mass of a solid of constant density δ, that is bounded below in the xy-plane by the disk $R: x^2+y^2 \le 4$, and above by the paraboloid $z = 4 - x^2 - y^2$.
Solution



Looking at the sketch, we see from symmetry that $\bar{x} = \bar{y} = 0$. Let us find

$$
\begin{aligned}
\bar{z} &= M_{yz}/M \\
&= \int\int\int_D z\delta dV / \int\int\int_D \delta dV \\
&= \int\int\int_D z dV / \int\int\int_D dV.
\end{aligned}
$$

We need the limits to do this. We see that in D

$$0 \le z \le 4 - x^2 - y^2.$$

We can also describe R in rectangular coordinates, and work out the integral. However in this example, it is easier to first integrate with respect to z, and then to express R in polar coordinates

$$R : 0 \le \theta \le 2\pi, 0 \le r \le 2.$$

Thus

$$
\begin{aligned}
M_{yz} &= \int\int\int_D \delta z dV \\
&= \delta \int\int_R \int_{z=0}^{z=4-x^2-y^2} z\, dz\, dy\, dx \\
&= \delta \int\int_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} dy\, dx \\
&= \delta \int\int_R \frac{1}{2}\left(4 - x^2 - y^2\right)^2 dy\, dx \\
&= \delta \int_0^{2\pi}\int_0^2 \frac{1}{2}\left(4 - r^2\right)^2 r dr\, d\theta \\
&= \frac{\delta}{4}\int_0^{2\pi}\int_0^2 \left(4 - r^2\right)^2 (2r)\, dr\, d\theta \\
&= \frac{\delta}{4}\int_0^{2\pi}\left[-\frac{1}{3}\left(4 - r^2\right)^3\right]_{r=0}^{r=2} d\theta \\
&= \frac{\delta}{4}\int_0^{2\pi}\left[\frac{1}{3}4^3\right] d\theta \\
&= \frac{\delta}{3}4^2(2\pi) = \frac{32}{3}\delta\pi.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
M &= \int\int\int_D \delta dV \\
&= \int\int_R \int_{z=0}^{z=4-x^2-y^2} \delta \, dz \, dy \, dx \\
&= \delta \int\int_R \left(4 - x^2 - y^2\right) dy \, dx \\
&= \delta \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r dr \, d\theta \\
&= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 \left(4 - r^2\right)(2r) \, dr \, d\theta \\
&= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{2} \left(4 - r^2\right)^2 \right]_{r=0}^{r=2} d\theta \\
&= \frac{\delta}{2} \int_0^{2\pi} \left[\frac{1}{2} 4^2 \right] d\theta \\
&= 8\delta\pi.
\end{aligned}$$

So

$$\bar{z} = \left(\frac{32}{3}\delta\pi\right) / (8\delta\pi) = \frac{4}{3},$$

and the center of mass is

$$\left(0, 0, \frac{4}{3}\right).$$

Remark
When the density of a body is constant, the centre of mass is also called the **centroid**. Thus in the example above, the centroid is $(0, 0, \frac{4}{3})$.

Center of Mass/ Centroid of a Two Dimensional Plate
When we have a plate in two dimensions rather than 3, we can still define mass and center of mass. If R is a region in two dimensions, and the density at the point (x, y) is $\delta(x, y)$, then its mass is

$$M = \int\int_R \delta(x, y) \, dA.$$

We can define the moments

$$M_y = \int\int_R x\delta(x, y) \, dA \text{ and } M_x = \int\int_R y\delta(x, y) \, dA$$

and the **center of mass**

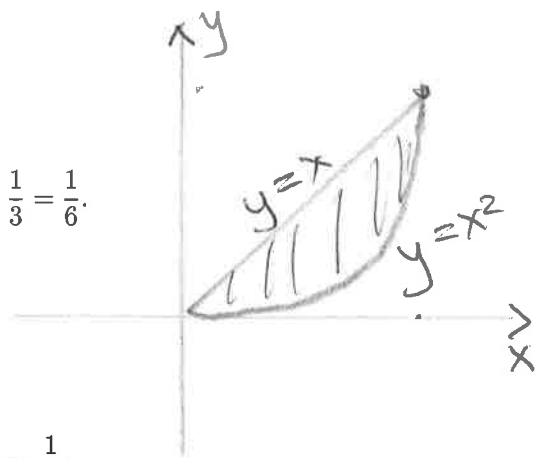
$$(\bar{x}, \bar{y})$$

where

$$\begin{aligned}
\bar{x} &= M_y / M; \\
\bar{y} &= M_x / M.
\end{aligned}$$

If the density is constant, we also call this the **centroid**.(We can take $\delta = 1$).

Example
Find the centroid of the region R in the first quadrant bounded above by the line $y = x$ and below by the parabola $y = x^2$.
Solution
From the sketch, we see that we can describe R by



$$R : 0 \le x \le 1,\ x^2 \le y \le x.$$

Then

$$M = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \left(x - x^2\right) dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Also

$$\begin{aligned} M_x &= \int_0^1 \int_{x^2}^{x} y\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{2}y^2\right]_{y=x^2}^{y=x} dx \\ &= \frac{1}{2}\int_0^1 \left(x^2 - x^4\right) dx = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}. \end{aligned}$$

$$\begin{aligned} M_y &= \int_0^1 \int_{x^2}^{x} x\, dy\, dx \\ &= \int_0^1 x\left(x - x^2\right) dx \\ &= \int_0^1 \left(x^2 - x^3\right) dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \end{aligned}$$

So

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{(1/12)}{(1/6)} = \frac{1}{2}; \\ \bar{y} &= \frac{M_x}{M} = \frac{(1/15)}{(1/6)} = \frac{2}{5}. \end{aligned}$$

So the centroid is

$$\left(\frac{1}{2}, \frac{2}{5}\right).$$

Moments of Inertia
Suppose we have a cylindrical shaft rotating about the z axis, with constant angular velocity $\omega = \frac{d\theta}{dt}$ radians per second. If we consider a small block in this cylinder of mass Δm_k, at a distance r_k from the centre of the shaft, then

the small block's center of mass will move at a linear speed of

$$v_k = \frac{d}{dt}(r_k\theta) = r_k\frac{d\theta}{dt} = r_k\omega.$$

The kinetic energy of the block is

$$\frac{1}{2}(\Delta m_k)\,v_k^2 = \frac{1}{2}(\Delta m_k)\,r_k^2\omega^2.$$

If we add over all the small blocks, say n in all, we obtain the total kinetic energy of approximately

$$\sum_{k=1}^{n}\frac{1}{2}r_k^2\omega^2(\Delta m_k).$$

As the shaft is partitioned into smaller and smaller blocks, and we take limits, we obtain the kinetic energy of the shaft:

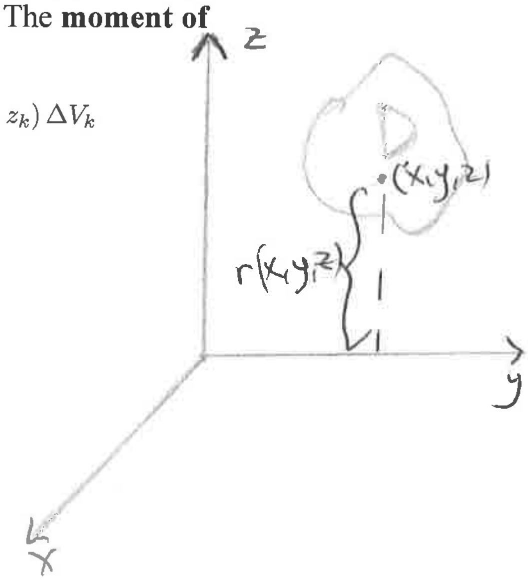
$$\mathrm{KE}_{shaft} = \int\frac{1}{2}r^2\omega^2dm = \frac{1}{2}\omega^2\int r^2dm.$$

The **moment of inertia** of the shaft is

$$I = \int r^2dm$$

The bigger the moment of inertia, the more effort/energy it takes to stop the shaft from turning, or to start it turning.

One can generalize this from a cylindrical shaft to an object in space. Let D be a region on space and L be a line. Let $r(x,y,z)$ be the distance from the point (x,y,z) in D to L. Consider a small block of D of volume ΔV_k and density $\delta(x_k,y_k,z_k)$, so that its mass is $\Delta m_k = \delta(x_k,y_k,z_k)\,\Delta V_k$. Its moment of inertia about L is approximately $\Delta I_k = r^2(x_k,y_k,z_k)\,\Delta m_k$. The **moment of inertia about** L of the entire object is



$$\begin{aligned} I_L &= \lim_{n\to\infty}\sum_{k=1}^{n}\Delta I_k = \lim_{n\to\infty}\sum_{k=1}^{n}r^2(x_k,y_k,z_k)\,\delta(x_k,y_k,z_k)\,\Delta V_k \\ &= \iiint_D r^2(x,y,z)\,\delta(x,y,z)\,dV. \end{aligned}$$

If L is the $x-$axis, then $r^2(x,y,z) = y^2+z^2$, so

$$I_x = \iiint_D (y^2+z^2)\,\delta(x,y,z)\,dV.$$

Similarly, if L is the $y-$axis, or $z-$axis, then

$$I_y = \iiint_D (x^2+z^2)\,\delta(x,y,z)\,dV,$$

or

$$I_z = \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV,$$

Example

Let D be a rectangular solid of sides a, b, c with center at the origin and axes parallel to the x, y, z axes. Assume it has constant density δ. Find I_x, I_y, I_z.

Solution

We see that D can be described as

$$D: -\frac{a}{2} \leq x \leq \frac{a}{2}; \ -\frac{b}{2} \leq y \leq \frac{b}{2}; \ -\frac{c}{2} \leq z \leq \frac{c}{2}.$$

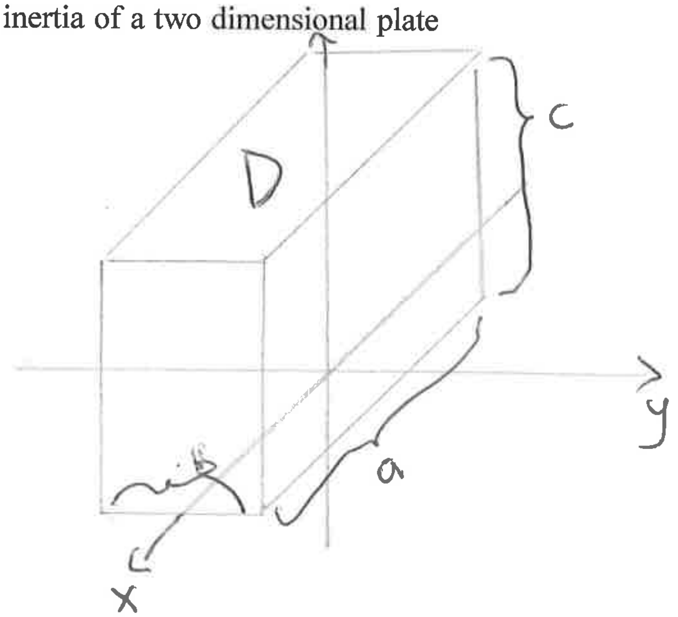
Then

$$\begin{aligned}
I_x &= \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \delta \left(y^2 + z^2\right) dx\, dy\, dz \\
&= \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} a\delta \left(y^2 + z^2\right) dy\, dz \\
&= \int_{-\frac{c}{2}}^{\frac{c}{2}} a\delta \left(\frac{2}{3}\left(\frac{b}{2}\right)^3 + z^2 b\right) dz \\
&= a\delta \frac{2}{3}\left(\frac{b}{2}\right)^3 (c) + a\delta b \frac{2}{3}\left(\frac{c}{2}\right)^3 \\
&= a\delta \frac{1}{12}\left(b^3 c + b c^3\right) \\
&= \frac{abc\delta}{12}\left(b^2 + c^2\right) = \frac{M}{12}\left(b^2 + c^2\right),
\end{aligned}$$

where $M = abc\delta$ is the mass of D. Similarly,

$$I_y = \frac{M}{12}\left(a^2 + c^2\right) \text{ and } I_z = \frac{M}{12}\left(a^2 + b^2\right).$$

One can similarly define moments of inertia of a two dimensional plate about a line.



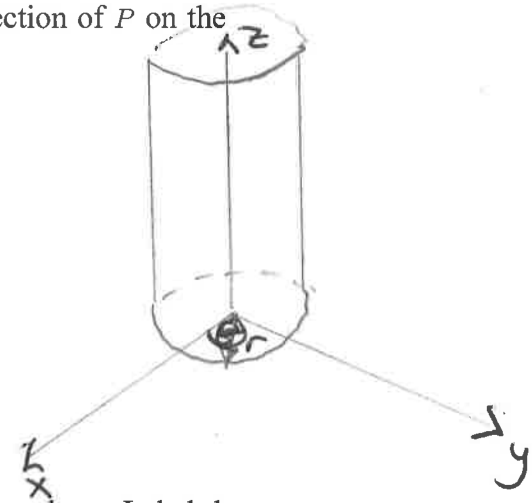
Section 15.7: Triple Integrals in Cylindrical and Spherical Coordinates

We have seen how polar coordinates help us to evaluate some double integrals. Now we do analogous things for triple integrals.

Integration in Cylindrical Coordinates

Here we combine polar coordinates in the xy-plane with the usual z–axis. We already used this in an example of moments of inertia.

Definition
Cylindrical coordinates represent a P in space by ordered triples (r, θ, z) in which $r \geq 0$.
(1) r and θ are polar coordinates for the vertical projection of P on the xy–plane.
(2) z is the rectangular vertical coordinate.



Remarks
In relating (r, θ, z) to (x, y, z), we see that

$$x = r\cos\theta;\ y = r\sin\theta;\ z = z.$$

In the other direction

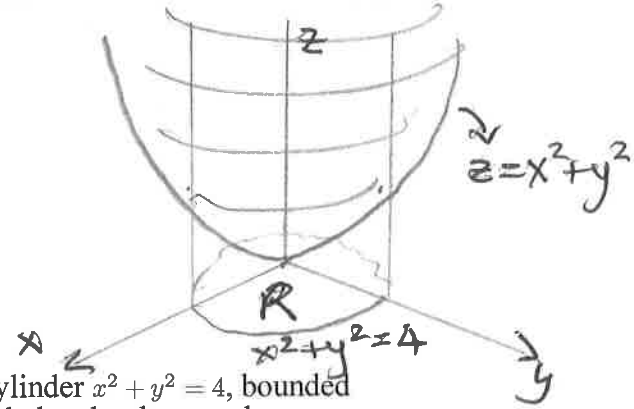
$$r^2 = x^2 + y^2;\ \ \tan\theta = y/x.$$

Integrating in Cylindrical Coordinates
To evaluate

$$\int\int\int_D f(r, \theta, z)\, dV,$$

(1) Sketch the region D with its projection R on the xy–plane. Label the surfaces and curves that bound D and R.
(2) Find the z–limits of integration. Draw a line M parallel to the z–axis through a typical point (r, θ) of R. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves D at $g_2(r, \theta)$. So $g_1(r, \theta) \leq z \leq g_2(r, \theta)$.
(3) Find the r–limits of integration. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. So $h_1(\theta) \leq r \leq h_2(\theta)$.
(4) Find the θ–limits of integration. Find the range of θ as L above sweeps through R, say $\alpha \leq \theta \leq \beta$.
Then

$$\int\int\int_D f(r, 0, z)\, dz\, dr\, d0 = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, 0, z)\, dz\, r\, dr\, d0.$$



Example
Find the centroid of the solid D enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy–plane.
Solution
Recall that for the centroid, we assume there is a constant density $\delta = 1$. We see that the base R of the cylinder in the xy–plane can be expressed as

$$R : 0 \le r \le 2, 0 \le \theta \le 2\pi.$$

The paraboloid $z = x^2 + y^2$ can be expressed as $z = r^2$, so

$$0 \le z \le r^2.$$

Next, the region is symmetric about the z–axis, so its centroid should lie on the z–axis, and consequently

$$\bar{x} = \bar{y} = 0.$$

Now we calculate

$$\begin{aligned} \bar{z} &= M_{xy}/M \\ &= \iiint_D z\, dV \Big/ \iiint_D dV. \end{aligned}$$

First,

$$\begin{aligned} M_{xy} &= \iiint_D z\, dV \\ &= \int_0^{2\pi}\int_0^2\int_0^{r^2} z\, dz\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2\right]_{z=0}^{z=r^2} r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^5 dr\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left[\frac{1}{6}r^6\right]_{r=0}^{r=2} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1}{6}2^6\, d\theta \\ &= \frac{1}{2}\left(\frac{1}{6}2^6\right)2\pi = \frac{32\pi}{3}. \end{aligned}$$

Next,

$$
\begin{aligned}
M &= \iiint_D dV \\
&= \int_0^{2\pi} \int_0^2 \int_0^{r^2} dz\ r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^2 r^3\ dr\ d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_{r=0}^{r=2} d\theta \\
&= \int_0^{2\pi} \frac{1}{4} 2^4\ d\theta \\
&= (4)\, 2\pi = 8\pi.
\end{aligned}
$$

Thus

$$z = \left(\frac{32}{3} \pi \right) / \left(8\pi \right) = \frac{4}{3}.$$

The centroid is

$$\left(0, 0, \frac{4}{3} \right).$$

It is interesting in this case that the centroid actually lies outside the region D (this can only happen for non-convex regions).

Spherical Coordinates and Integration

While cylindrical coordinates use two distances (z, r) and one angle θ, spherical coordinates use two angles and one distance:

Definition

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

(1) ρ is the distance from P to the origin $(\rho \geq 0)$.

(2) ϕ is the angle $\overrightarrow{OP}$ makes with the positive $z-$axis $(0 \leq \phi \leq \pi)$.

(3) θ is the angle from cylindrical coordinates.

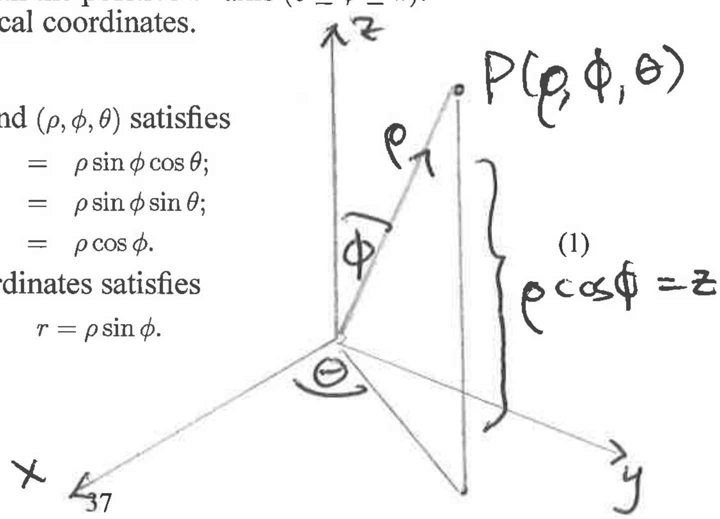
Remarks

The relation between (x, y, z) and (ρ, ϕ, θ) satisfies

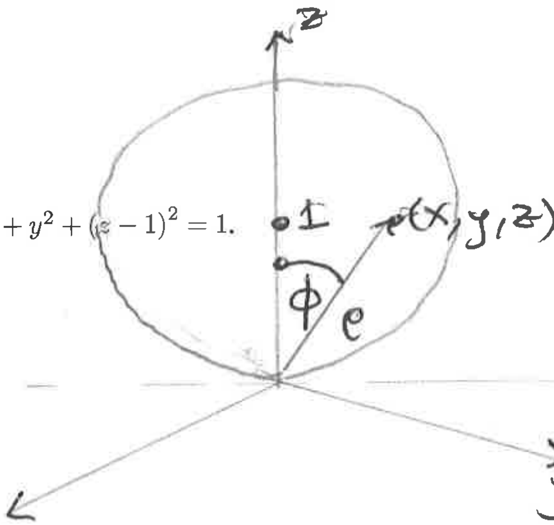
$$
\begin{aligned}
x &= \rho \sin\phi \cos\theta; \\
y &= \rho \sin\phi \sin\theta; \\
z &= \rho \cos\phi.
\end{aligned}
$$

Note that the r from polar coordinates satisfies

$$r = \rho \sin\phi.$$

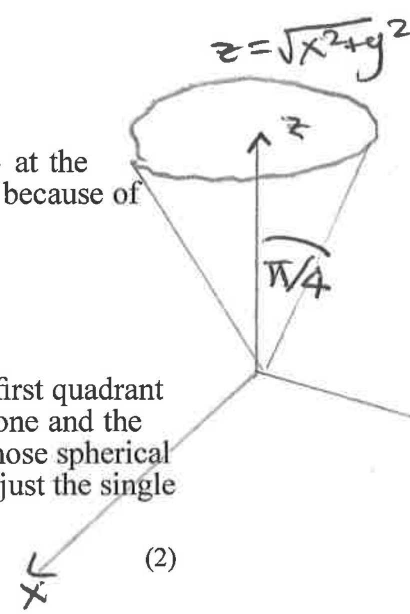


Example 3
Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.
Solution



We use the formulae for x, y, z in (1) above:

$$\begin{aligned} &(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + (\rho\cos\phi - 1)^2 = 1 \\ \Rightarrow\ &(\rho\sin\phi)^2\left(\cos^2\theta + \sin^2\theta\right) + (\rho\cos\phi)^2 - 2\rho\cos\phi + 1 = 1 \\ \Rightarrow\ &\rho^2\left(\sin^2\phi + \cos^2\phi\right) - 2\rho\cos\phi = 0 \\ \Rightarrow\ &\rho^2 = 2\rho\cos\phi \\ \Rightarrow\ &\rho = 2\cos\phi. \end{aligned}$$



Note that ϕ varies from 0 at the north pole of the sphere to $\frac{\pi}{2}$ at the south pole. The angle θ does not appear in the identity above, because of symmetry; it varies from 0 to 2π.

Example 4
Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.
Solution 1 using geometry
The cone is symmetric with respect to the z–axis and cuts the first quadrant of the yz–plane along the line $z = y$. The angle between the cone and the positive z–axis is $\frac{\pi}{4}$ radians. The cone consists of the points whose spherical coordinates have angle $\phi = \frac{\pi}{4}$. So the description of the cone is just the single equation

$$\phi = \frac{\pi}{4}. \tag{2}$$

Solution 1 using algebra
Our equations (1) for spherical coordinates give

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \Rightarrow\ \rho\cos\phi &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} = \sqrt{\rho^2\sin^2\phi} = \rho\left|\sin\phi\right|. \end{aligned}$$

Cancelling $\rho > 0$, and using that $\phi \in [0, \pi]$ so $\sin\phi \geq 0$, we have

$$\cos\phi = \sin\phi,$$

which again gives (2).

Example 5

The sphere $x^2 + y^2 + z^2 = 4$ has the very simple spherical description $\rho^2 = 4$, or $\rho = 2$ (while $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$).

Integrating in Spherical Coordinates
When calculating volume using spherical coordinates, we base our calculation on the volume of a small spherical wedge containing a point $(\rho_k, \phi_k, \theta_k)$. One edge is a circular arc of length $\rho_k(\Delta\phi_k)$; another edge is a circular arc of length $\rho_k(\sin\phi_k)\Delta\theta_k$, and the thickness in the ρ direction is $\Delta\rho_k$. So the volume of the small spherical wedge is

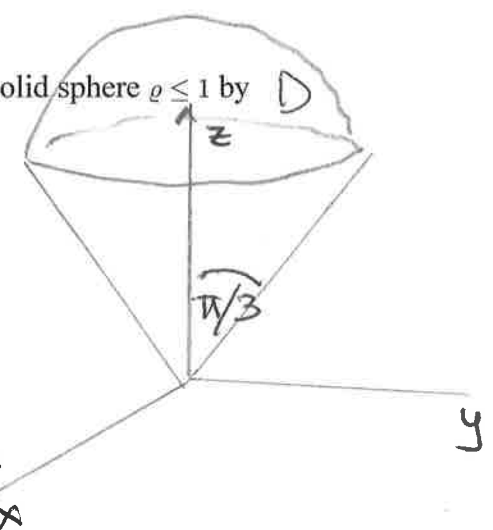
$$\begin{aligned}\Delta V_k &= (\Delta\rho_k)(\rho_k(\Delta\phi_k))(\rho_k(\sin\phi_k)\Delta\theta_k) \\ &= \rho_k^2(\sin\phi_k)(\Delta\rho_k)(\Delta\phi_k)(\Delta\theta_k).\end{aligned}$$

When we add up over all such small wedges, we obtain a Riemann sum. The we take limits, and this gives the formula for a triple integral in spherical coordinates:

$$\int\int\int_D f(\rho,\phi,\theta)\,dV = \int\int\int_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

To evaluate integrals using spherical coordinates, we proceed much as we have before: first we sketch, then find limits of integration for ρ, then for ϕ and finally for θ.

Example 6
Find the volume of the "ice cream cone" D cut from the solid sphere $\varrho \leq 1$ by the cone $\phi = \frac{\pi}{3}$.
Solution



We know the volume is

$$\int\int\int_D 1\,dV = \int\int\int_D \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Now let us describe D. See the sketch.
(1) Draw a ray M from the origin through D, making an angle ϕ with the positive $z-$axis. We see that M enters D at $\rho = 0$ and leaves at $\rho = 1$.
(2) We see that ϕ runs from $\phi = 0$ to $\phi = \frac{\pi}{3}$.

(3) Because of the symmetry, we see that θ runs from 0 to 2π. So

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta. \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left[\frac{\rho^3}{3} \sin\phi \right]_{\rho=0}^{\rho=1} d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \frac{1}{3} \sin\phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3} \cos\phi \right]_{\phi=0}^{\phi=\frac{\pi}{3}} d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3} \cos\frac{\pi}{3} + \frac{1}{3} \right] d\theta \\
&= (2\pi) \frac{1}{3} \left[-\frac{1}{2} + 1 \right] = \frac{\pi}{3}.
\end{aligned}$$

Example 7

A solid of constant density $\delta = 1$ occupies the region D in the previous example. Find the solid's moment of inertia about the z-axis.

Solution

In rectangular coordinates, the moment of inertia is

$$I_z = \int\int\int_D (x^2 + y^2) \, dV.$$

In spherical coordinates,

$$x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi,$$

so

$$\begin{aligned}
I_z &= \int\int\int_D \left(\rho^2 \sin^2 \phi\right) \left(\rho^2 \sin \phi\right) d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left[\frac{1}{5}\rho^5 \sin^3 \phi\right]_{\rho=0}^{\rho=1} d\rho \, d\phi \, d\theta \\
&= \frac{1}{5} \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin^3 \phi \, d\phi \, d\theta \\
&= \frac{1}{5} \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left(1 - \cos^2 \phi\right) \sin \phi \, d\phi \, d\theta \\
\text{(substitution } t &= \cos \phi) \\
&= \frac{1}{5} \int_0^{2\pi} \int_{\cos \frac{\pi}{3}}^1 \left(1 - t^2\right) dt \, d\theta \\
&= \frac{1}{5} \int_0^{2\pi} \left[t - \frac{1}{3}t^3\right]_{t=\frac{1}{2}}^{t=1} d\theta \\
&= \frac{1}{5} \int_0^{2\pi} \left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right)\right] d\theta \\
&= \frac{1}{5} (2\pi) \left(\frac{5}{24}\right) = \frac{\pi}{12}.
\end{aligned}$$

Section 15.8: Substitutions in Multiple Integrals

We know that the substitution rule for integrals of one variable has the form

$$\int_a^b f(g(t))\, g'(t)\, dt = \int_{g(a)}^{g(b)} f(x)\, dx.$$

We now study analogue rules for double and triple integrals. The role of g' is taken by the Jacobian:

Definition
The **Jacobian determinant** or **Jacobian** of the coordinate transformation

$$x = g(u,v) \text{ and } y = h(u,v)$$

is

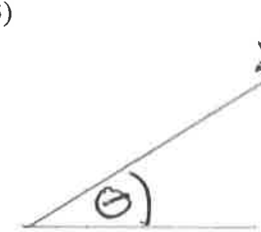
$$J(u,v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right).$$

It is also denoted

$$J(u,v) = \frac{\partial (x,y)}{\partial (u,v)}.$$

Theorem 3 - Substitution for Double Integrals
Suppose that $f(x,y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u,v)$ and $y = h(u,v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\int\int_R f(x,y)\, dx\, dy = \int\int_G f(g(u,v), h(u,v)) \left|\frac{\partial (x,y)}{\partial (u,v)}\right| du\, dv. \qquad (3)$$



Example 1
Find the Jacobian for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. Hence write down (3) for the case of polar coordinates and the region R in the first quadrant of the $xy-$ plane bounded by the circle $x^2 + y^2 = 1$.
Solution
We see from the sketch that R corresponds in the (r,θ) plane to the rectangle

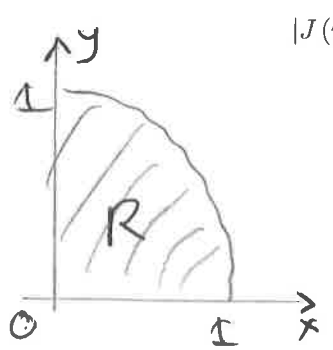
$$G: 0 \le r \le 1; 0 \le \theta \le \frac{\pi}{2}.$$

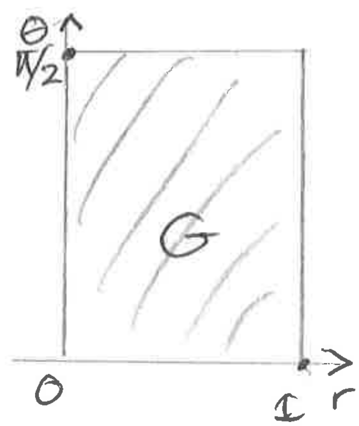
The Jacobian is

$$\begin{aligned} J(r,\theta) &= \frac{\partial (x,y)}{\partial (r,\theta)} = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r. \end{aligned}$$

Then

$$|J(r,\theta)| = r.$$





Equation (2) becomes

$$\int\int_R f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^1 f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta.$$

This is the same formula as we used in Section 15.4 for double integrals expressed in polar coordinates.

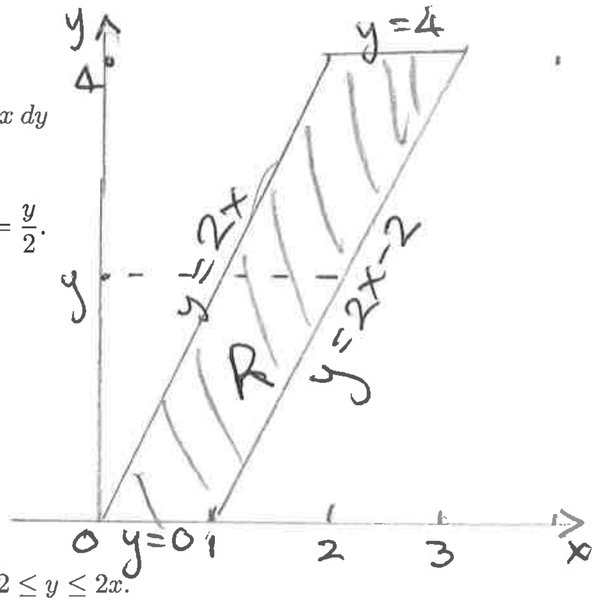
Example 2
Evaluate

$$\int_0^4\int_{\frac{y}{2}}^{\frac{y}{2}+1}\frac{2x-y}{2}dx\,dy$$

using the transformation

$$u=\frac{2x-y}{2}\text{ and }v=\frac{y}{2}.$$

Solution
Let us start by sketching R.



We see that $0\leq y\leq 4$, and

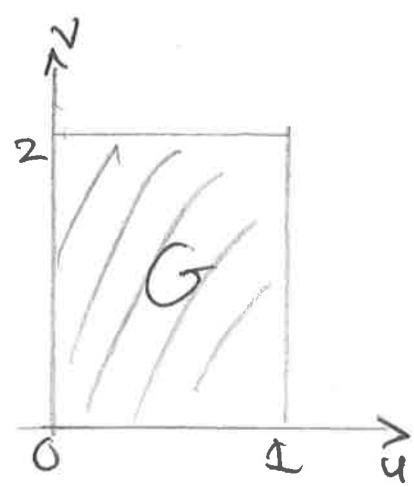
$$\frac{y}{2}\leq x\leq\frac{y}{2}+1\iff 2x-2\leq y\leq 2x.$$

So R has bounding lines $y=2x-2$ and $y=2x$, as well as $y=0$ and $y=4$. Now we need to find the region G in the uv plane. We express x,y in terms of u,v, and obtain

$$u+v=x,\ 2v=y.$$

Let us draw up a table for the boundaries of R and G :

$xy-$equation	$uv-$equation	simplified
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$y=2x-2$	$u=\frac{2x-y}{2}=1$	$u=1$
$y=2x$	$u=\frac{2x-y}{2}=0$	$u=0$



So we can describe G as

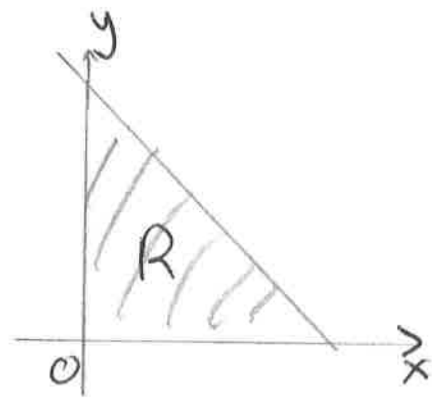
$$G:0\leq u\leq 1;0\leq v\leq 2.$$

The Jacobian is

$$\begin{aligned} J(u,v) &= \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial(u+v)}{\partial u} & \frac{\partial(u+v)}{\partial v} \\ \frac{\partial(2v)}{\partial u} & \frac{\partial(2v)}{\partial v} \end{bmatrix} \\ &= \det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = 2. \end{aligned}$$

So we can now apply our substitution rule:

$$\begin{aligned} & \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} dx\, dy \\ &= \int_0^2 \int_0^1 u\,|J(u,v)|\; du\, dv \\ &= \int_0^2 \int_0^1 (2u)\, du\, dv \\ &= \int_0^2 \left[u^2\right]_{u=0}^{u=1} dv = \int_0^2 dv = 2. \end{aligned}$$

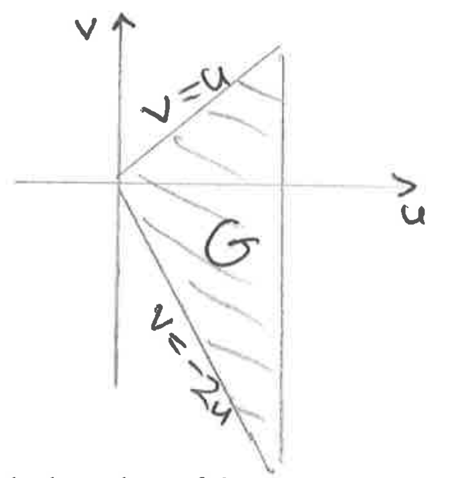


Example 3
Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy\, dx$$

Hint: use $u = x + y$ and $v = y - 2x$
Solution
First our region R is

$$R : 0 \le x \le 1, 0 \le y \le 1 - x.$$



We are told to use

$$\begin{aligned} u &= x + y; \\ v &= -2x + y. \end{aligned}$$

We solve for x, y in terms of u, v :

$$\begin{aligned} x &= \frac{u}{3} - \frac{v}{3}; \\ y &= \frac{2u}{3} + \frac{v}{3}. \end{aligned}$$

Now let us see how the boundary of R is mapped to the boundary of the new region G:

$xy-$equation	$uv-$equation	simplified
$x=0$	$\frac{u}{3}-\frac{v}{3}=0$	$v=u$
$y=0$	$\frac{2u}{3}+\frac{v}{3}=0$	$v=-2u$
$x+y=1$	$\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)=1$	$u=1$

The lines $v=u$ and $v=-2u$ intersect at $u=0$, so we see $0\leq u\leq 1$. So we can describe G as

$$G: 0\leq u\leq 1,\ -2u\leq v\leq u.$$

The Jacobian is

$$\begin{aligned} J(u,v) &= \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ &= \frac{1}{9}+\frac{2}{9}=\frac{1}{3}. \end{aligned}$$

Now we apply the change of variables formula:

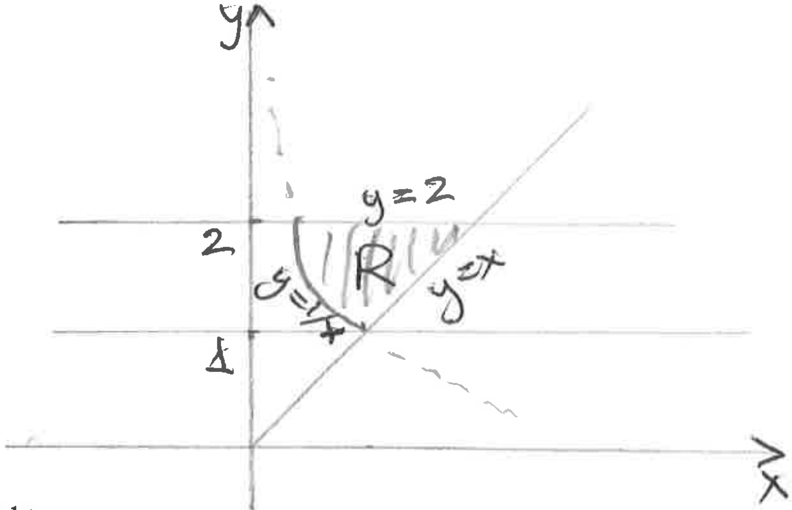
$$\begin{aligned} &\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dy\,dx \\ &= \int_0^1\int_{-2u}^{u} u^{1/2}v^2\,|J(u,v)|\,dv\,du \\ &= \int_0^1\int_{-2u}^{u} u^{1/2}v^2\frac{1}{3}dv\,du \\ &= \frac{1}{3}\int_0^1 u^{1/2}\left[\frac{v^3}{3}\right]_{v=-2u}^{v=u} du \\ &= \frac{1}{9}\int_0^1 u^{1/2}\left[u^3-(-2u)^3\right] du \\ &= \frac{1}{9}\int_0^1 u^{1/2}\left(9u^3\right) du \\ &= \int_0^1 u^{7/2}du = \frac{2}{9}\left[u^{9/2}\right]_{u=0}^{u=1} = \frac{2}{9}. \end{aligned}$$

Example 4
Evaluate

$$\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}e^{\sqrt{xy}}dx\,dy$$

Hint: use $u=\sqrt{xy}$ and $v=\sqrt{y/x}$
Solution
First our region R is

$$R: 1\leq y\leq 2,\ 1/y\leq x\leq y.$$



We are told to use

$$u = \sqrt{xy};$$
$$v = \sqrt{y/x}.$$

We solve for x, y in terms of $u, v : uv = y$ and $u/v = x$, so

$$x = \frac{u}{v};$$
$$y = uv.$$

Now let us see how the boundary of R is mapped to the boundary of the new region G:

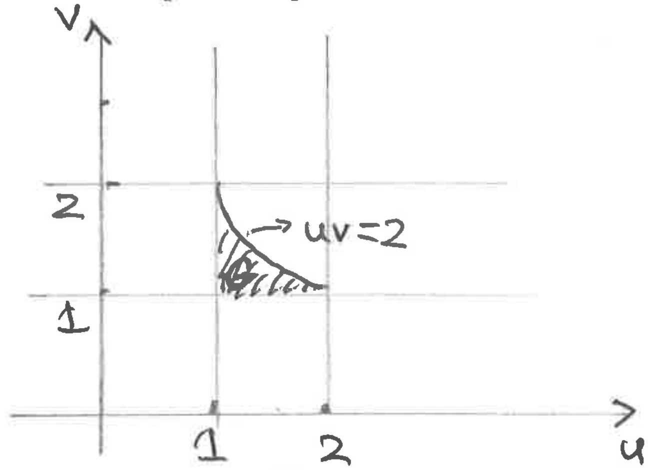
$xy-$equation	$uv-$equation	simplified
$x = \frac{1}{y}$	$\frac{u}{v} = \frac{1}{uv} \Rightarrow u^2 = 1$	$u = 1$
$x = y$	$\frac{u}{v} = uv \Rightarrow v^2 = 1$	$v = 1$
$y = 2$	$uv = 2$	$uv = 2$

The curve $v = 1$ intersects $uv = 2$ when $u = 2$. So u varies from 1 to 2. From the sketch, the transformed region is

$$G : 1 \leq u \leq 2, 1 \leq v \leq \frac{2}{u}.$$

The Jacobian is

$$J(u,v) = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial(u/v)}{\partial u} & \frac{\partial(u/v)}{\partial v} \\ \frac{\partial(uv)}{\partial u} & \frac{\partial(uv)}{\partial v} \end{bmatrix}$$
$$= \det\begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}.$$



Using our substitution rule,

$$\begin{aligned}
& \int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx\, dy \\
= & \int_1^2 \int_1^{\frac{2}{u}} v e^u \left| J(u,v) \right| dv\, du \\
= & \int_1^2 \int_1^{\frac{2}{u}} v e^u \left(\frac{2u}{v} \right) dv\, du \\
= & \; 2 \int_1^2 \int_1^{\frac{2}{u}} u e^u \, dv\, du \\
= & \; 2 \int_1^2 \left[u e^u v \right]_{v=1}^{v=\frac{2}{u}} du \\
= & \; 2 \int_1^2 \left[u e^u \frac{2}{u} - u e^u \right] du \\
= & \; 2 \int_1^2 (2-u)\, e^u du.
\end{aligned}$$

Now we integrate by parts, and continue this as

$$\begin{aligned}
= & \; 2 \left\{ (2-u)\, e^u |_{u=1}^{u=2} - \int_1^2 (-1)\, e^u du \right\} \\
= & \; 2 \left\{ 0 - 1e^1 + e^2 - e^1 \right\} \\
= & \; 2 \left(e^2 - 2e^1 \right).
\end{aligned}$$

Substitution in Triple Integrals

The cylindrical and spherical coordinates in the last section are special cases of the change of variables in three variables: suppose that the region G in uvw-space is transformed one-to-one onto the region D in $xyz-$space by

$$x = g(u,v,w)\,,\; y = h(u,v,w)\,,\; z = k(u,v,w)\,.$$

The substitution formula for triple integrals is

$$\begin{aligned}
& \int\int\int_D F(x,y,z)\, dx\, dy\, dz \\
= & \int\int\int_G F(g(u,v,w)\,, h(u,v,w)\,, k(u,v,w)) \left| J(u,v,w) \right| du\, dv\, dw, \qquad (4)
\end{aligned}$$

where again $J(u,v,w)$ is the **Jacobian determinant**

$$J(u,v,w) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \frac{\partial (x,y,z)}{\partial (u,v,w)}.$$

Example
Let us see that this is consistent with what our formula for spherical coordinates: recall

$$\begin{aligned} x &= \rho \sin\phi \cos\theta; \\ y &= \rho \sin\phi \sin\theta; \\ z &= \rho \cos\phi. \end{aligned}$$

The Jacobian is

$$\begin{aligned} J(\rho,\phi,\theta) &= \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \\ &= \det\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} \\ &= \det\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}. \end{aligned}$$

We expand by the third row, continuing this as

$$\begin{aligned} &(-1)^{3+1}\cos\phi\det\begin{bmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix} + (-1)^{4+1}(-\rho\sin\phi)\det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix} \\ =\ & \cos\phi\left\{\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right\} + \rho\sin\phi\left\{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\right\} \\ =\ & \rho^2\sin\phi\left\{\cos^2\phi\left[\cos^2\theta+\sin^2\theta\right]\right\} + \rho^2\sin\phi\left\{\sin^2\phi\left[\cos^2\theta+\sin^2\theta\right]\right\} \\ =\ & \rho^2\sin\phi\left\{\cos^2\phi+\sin^2\phi\right\} = \rho^2\sin\phi. \end{aligned}$$

So (4) becomes the familiar formula

$$\begin{aligned} &\iiint_D F(x,y,z)\,dx\,dy\,dz \\ =\ &\iiint_G F(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta. \end{aligned}$$

Example 5
Evaluate

$$\int_0^3\int_0^4\int_{y/2}^{y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

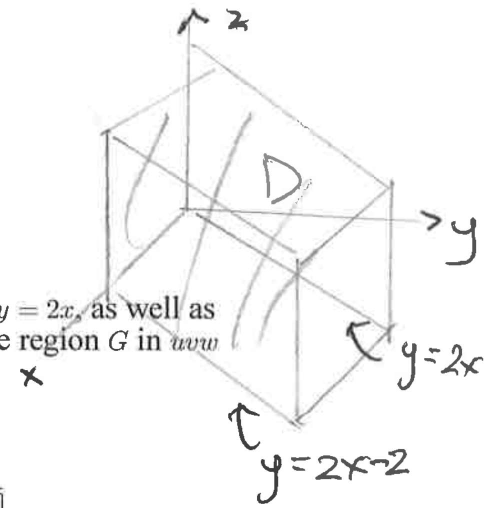
using the substitution/ transformation

$$u = \frac{2x-y}{2};\ v = \frac{y}{2};\ w = \frac{z}{3}.$$

Solution
We see that $0 \le z \le 3, 0 \le y \le 4$, and

$$\frac{y}{2} \le x \le \frac{y}{2}+1 \Longleftrightarrow 2x-2 \le y \le 2x.$$

(Using the same calculations as in Example 2 for double integrals). So our



region D in $xyz-$space has bounding planes $y = 2x - 2$ and $y = 2x$, as well as $y = 0$ and $y = 4$ and $z = 0$ and $z = 3$. Now we need to find the region G in uvw space. We express x, y, z in terms of u, v, w and obtain

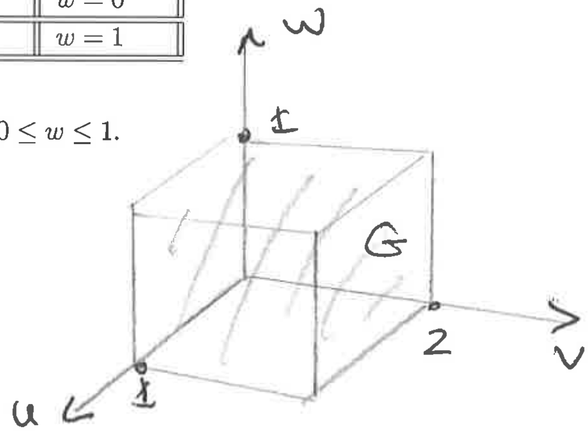
$$u + v = x, 2v = y, 3w = z.$$

Let us draw up a table for the boundaries of R and G:

$xyz-$equation	$uvw-$equation	simplified
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$y = 2x - 2$	$u = \frac{2x-y}{2} = 1$	$u = 1$
$y = 2x$	$u = \frac{2x-y}{2} = 0$	$u = 0$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

So we can describe G as

$$G : 0 \leq u \leq 1; 0 \leq v \leq 2, 0 \leq w \leq 1.$$



The Jacobian is

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 6. \end{aligned}$$

The change of variables formula gives

$$\begin{aligned}
& \int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz \\
= & \int_0^1 \int_0^2 \int_0^1 (u+w) \left| \frac{\partial (x,y,z)}{\partial (u,v,w)} \right| du\, dv\, dw \\
= & \int_0^1 \int_0^2 \int_0^1 (u+w)\, 6\, du\, dv\, dw \\
= & \ 6 \int_0^1 \int_0^2 \left[\frac{1}{2}u^2 + uw \right]_{u=0}^{u=1} dv\, dw \\
= & \ 6 \int_0^1 \int_0^2 \left[\frac{1}{2} + w \right] dv\, dw \\
= & \ 6 \int_0^1 \left[\left[\frac{1}{2} + w \right] v \right]_{v=0}^{v=2} dw \\
= & \ 6 \int_0^1 \left[\frac{1}{2} + w \right] 2\, dw \\
= & \ 6 \left[w + w^2 \right]_{w=0}^{w=1} = 6\,[2-0] = 12.
\end{aligned}$$