CALCULUS III, MATH 2551

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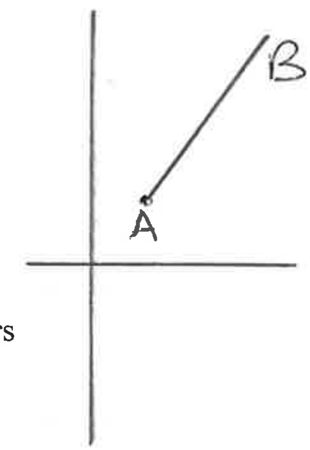
Abstract

These typed notes are used to prepare my lectures for Math 2551. They are not intended as a replacement for the textbook, Thomas' Calculus, Thirteenth Edition. Moreover, this is the first time these notes have been used, so they will need modification.

Section 12.2: Vectors



We start with a somewhat loose definition:

Definition
Let A, B be points in either two dimensional space or in three dimensional space. The vector represented by the directed line segment $\overrightarrow{AB}$ has initial point A and terminal point B and its length is denoted by $\left|\overrightarrow{AB}\right|$. Two vectors are equal if they have the same length and direction.

Definition
(i) If $\mathbf{v}$ is a two-dimensional vector in the plane equal to the vector with initial point at the origin $0 = (0, 0)$ and terminal point (v_1, v_2), then the component form of $\mathbf{v}$ is

$$\mathbf{v} = (v_1, v_2).$$

Its **magnitude** or **length** is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}.$$

(ii) If $\mathbf{v}$ is a three-dimensional vector in the plane equal to the vector with initial point at the origin $0 = (0, 0, 0)$ and terminal point (v_1, v_2, v_3), then the component form of $\mathbf{v}$ is

$$\mathbf{v} = (v_1, v_2, v_3).$$

Its **magnitude** or **length** is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Remarks about notation
(a) In the typed notes, as in the book, we shall use boldface (for example $\mathbf{v}$)

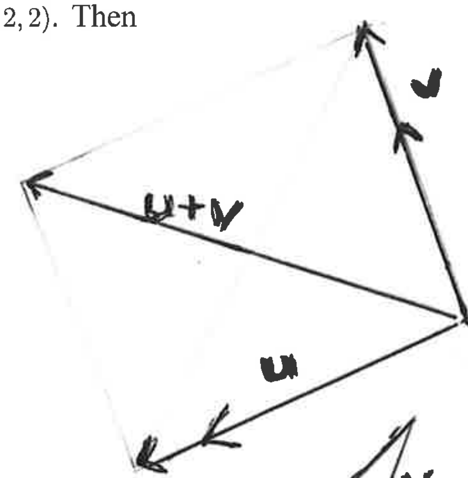
for a vector. As we cannot use bold when writing notes in class, we shall use $\overrightarrow{v}$ there.
(b) If P and Q are given points, then $\overrightarrow{PQ}$ is the vector from P to Q.
(b) Also in the book, they use $< v_1, v_2, v_3 >$ rather than (v_1, v_2, v_3).

Example 1 Vectors with given initial and terminal points
Let $\mathbf{v} = \overrightarrow{PQ}$ be the vector with initial point $P(-3, 4, 1)$ and $Q(-5, 2, 2)$. Then the component form of v is

$$\begin{aligned} \mathbf{v} &= (-5-(-3), 2-4, 2-1) \\ &= (-2, -2, 1). \end{aligned}$$

The magnitude or length of v is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3.$$



Vector Algebra

Definition
Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ and k be a scalar.
(i) We define **addition** of $\mathbf{u}, \mathbf{v}$ by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3);$$

(ii) We define **scalar multiplication**

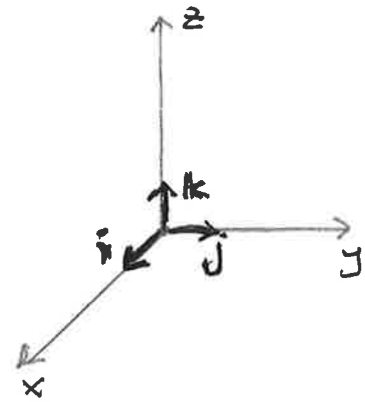
$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

(iii) We define the **difference of** $\mathbf{u}, \mathbf{v}$ by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, u_3 - v_3).$$

Properties of Vector Operations
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars.
(1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)
(2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)
(3) $\mathbf{u} + \mathbf{0} = \mathbf{u}$
(4) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
(5) $0\mathbf{u} = \mathbf{0}$
(6) $1\mathbf{u} = \mathbf{u}$
(7) $a(b\mathbf{u}) = (ab)\mathbf{u}$
(8) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
(9) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors
It is often useful to have vectors in a given direction with length 1.



Definition
(i) A vector $\mathbf{v}$ of length 1 is called a **unit vector**.
(ii) The **standard unit vectors** are

$$\mathbf{i} = (1,0,0)\,;\mathbf{j} = (0,1,0)\,;\mathbf{k} = (0,0,1)\,.$$

Remarks
(a) Any three dimensional vector $\mathbf{v} = (v_1, v_2, v_3)$ can be expressed as a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

(b) If $\mathbf{v} \neq 0$, then

$$\frac{\mathbf{v}}{|\mathbf{v}|}$$

is a unit vector having the same length as $\mathbf{v}$. We say $\frac{\mathbf{v}}{|\mathbf{v}|}$ is the **direction** of v.

Example
Let $\mathbf{v} = \mathbf{3i} - \mathbf{4j}$ be a velocity vector. Express $\mathbf{v}$ as a product of its speed times its direction ●
Solution
Recall that speed is the magnitude of velocity. So the speed is

$$|\mathbf{v}| = |3\mathbf{i} - 4\mathbf{j}| = \sqrt{3^2 + (-4)^2} = 5.$$

The direction is

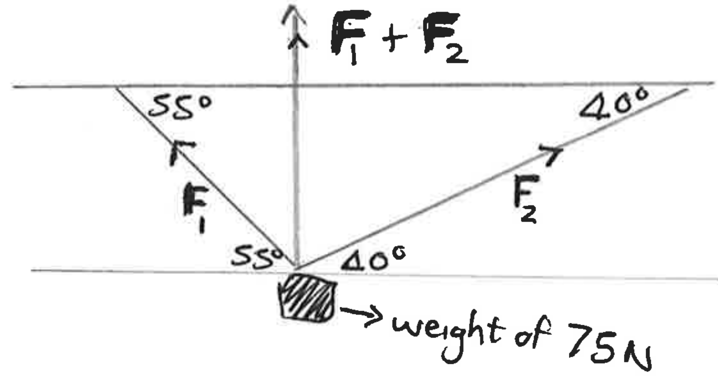
$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

Thus

$$\begin{aligned} \mathbf{v} &= \text{(speed)(direction)} \\ &= (5)\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right). \end{aligned}$$

The midpoint of a line segment
The **midpoint** M of the line segment joining $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$



Application: Suspended Weights

Example
A weight of 75 Newtons is suspended by two wires as in the figure. Find the coordinates of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$.
Solution
The trick in this type of problem is to realize that gravity acts vertically downwards, it has no horizontal component. Thus the equal and opposite force $\mathbf{F}_1+\mathbf{F}_2$ must act vertically upwards to balance the weight of 75 Newtons, so has

$$\mathbf{F}_1+\mathbf{F}_2 = (0, 75)\,.$$

The rest is trigonometry and simultaneous equations. Recall that $\mathbf{F}_1$ has magnitude $|\mathbf{F}_1|$. So

$$\mathbf{F}_1 = (-\,|\mathbf{F}_1|\cos 55^\circ, |\mathbf{F}_1|\sin 55^\circ)\,.$$

Similarly,

$$\mathbf{F}_2 = (|\mathbf{F}_2|\cos 40^\circ, |\mathbf{F}_2|\sin 40^\circ)\,.$$

Then adding these two forces, we have

$$(-\,|\mathbf{F}_1|\cos 55^\circ, |\mathbf{F}_1|\sin 55^\circ) + (|\mathbf{F}_2|\cos 40^\circ, |\mathbf{F}_2|\sin 40^\circ) = (0, 75)\,,$$

which leads to

$$\begin{aligned} -\,|\mathbf{F}_1|\cos 55^\circ + |\mathbf{F}_2|\cos 40^\circ &= 0; \\ |\mathbf{F}_1|\sin 55^\circ + |\mathbf{F}_2|\sin 40^\circ &= 75. \end{aligned}$$

From this,

$$|\mathbf{F}_2| = |\mathbf{F}_1|\,\frac{\cos 55^\circ}{\cos 40^\circ}$$

and hence

$$|\mathbf{F}_1|\sin 55^\circ + \left(|\mathbf{F}_1|\,\frac{\cos 55^\circ}{\cos 40^\circ}\right)\sin 40^\circ = 75,$$

so

$$|\mathbf{F}_1| = \frac{75}{\sin 55^\circ + \left(\frac{\cos 55^\circ}{\cos 40^\circ}\right)\sin 40^\circ} \approx 57.67N.$$

Then

$$|\mathbf{F}_2| = |\mathbf{F}_1|\,\frac{\cos 55^\circ}{\cos 40^\circ} \approx 43.18N.$$

So we obtain

$$\mathbf{F}_1 \approx (-33.08, 47.24)\,.$$
$$\mathbf{F}_2 \approx (33.08, 27.76)\,.$$

Section 12.3: The Dot Product

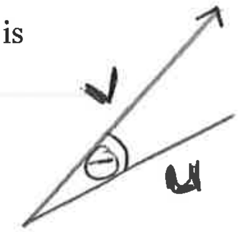
Definition: The Dot Product, Angles, Orthogonal Vectors
(a) The **dot product** (or scalar product) of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

(b) If $\mathbf{u}, \mathbf{v}$ are non-zero vectors, the **angle between them** is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

(c) We say that $\mathbf{u}, \mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.



Remark
Note that if $\mathbf{u}, \mathbf{v}$ are non-zero vectors, and are orthogonal, then the angle between them is

$$\theta = \cos^{-1}(0) = \frac{\pi}{2},$$

that is the vectors are perpendicular.

Example
Find the angle between

$$\mathbf{u} = (1, -2, -2) \text{ and } \mathbf{v} = (6, 3, 2).$$

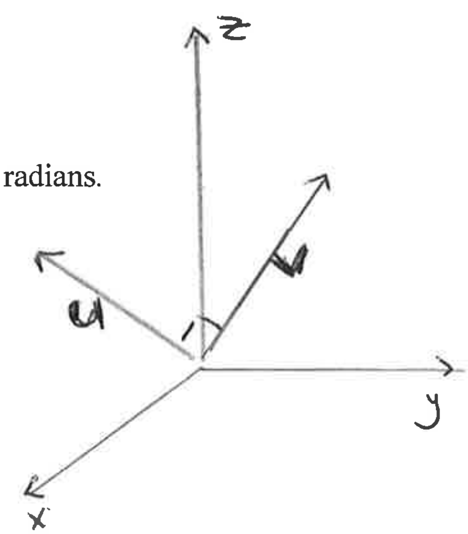
Solution

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 1(6) + (-2)(3) + (-2)2 = -4; \\ |\mathbf{u}| &= \sqrt{1+4+4} = 3; \\ |\mathbf{v}| &= \sqrt{36+9+4} = 7; \end{aligned}$$

So

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{3(7)}\right) = \cos^{-1}\left(-\frac{4}{21}\right) = 1.76\ldots \text{ radians.}$$

These vectors are not orthogonal.



Example
Show that

$$\mathbf{u} = (3, -2, 1) \text{ and } \mathbf{v} = (0, 2, 4)$$

are orthogonal.
Solution

$$\mathbf{u} \cdot \mathbf{v} = 3(0) + (-2)2 + (1)4 = 0,$$

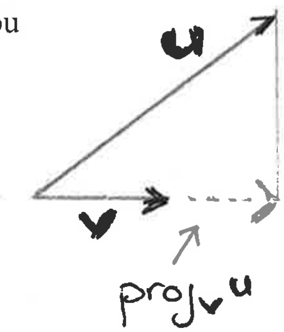
so these vectors are orthogonal.

Properties of the Dot Product
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and c be a scalar.
(1) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity)
(2) $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

(3) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributivity)
(4) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
(5) $\mathbf{0} \cdot \mathbf{u} = 0$

Projections
Let $\mathbf{u}, \mathbf{v}$ be non-zero vectors. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is obtained by dropping a perpendicular from $\mathbf{u}$ onto $\mathbf{v}$. In your linear algebra course, you probably saw that the formula for this projection is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$



The **scalar component** of $\mathbf{u}$ in the direction of $\mathbf{v}$ is $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$.

Work
Suppose that a force $\mathbf{F}$ moves an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$. Then the work done in this is

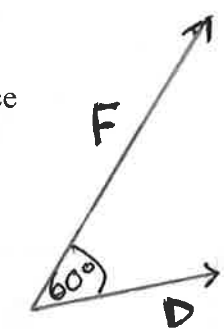
$$\begin{aligned} \text{Work} &= (\text{Scalar component of } \mathbf{F} \text{ in direction } \mathbf{D})(\text{length of } \mathbf{D}) \\ &= \left(\mathbf{F} \cdot \frac{\mathbf{D}}{|\mathbf{D}|} \right) |\mathbf{D}| = \mathbf{F} \cdot \mathbf{D}. \end{aligned}$$

If θ is the angle between $\mathbf{F}$ and $\mathbf{D}$, this is also

$$|\mathbf{F}|\,|\mathbf{D}| \cos \theta.$$

Example
Suppose that the magnitude of the force $\mathbf{F}$ is $40N$ (Newtons). Suppose also that the magnitude of the displacement is $3m$ and the angle between the force vector and the direction of displacement is $60°$. Then the work done in this displacement is

$$\begin{aligned} & |\mathbf{F}|\,|\mathbf{D}| \cos \theta \\ = \; & (40N)\,(3m) \cos 60 = (40)\,(3) \left(\frac{1}{2} \right) Nm = 60\text{Joules}. \end{aligned}$$

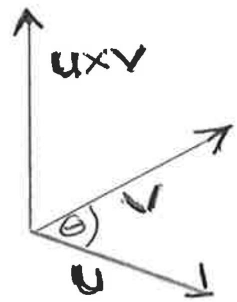


Section 12.4: The Cross Product

Suppose that we are given vectors $\mathbf{u}, \mathbf{v}$ in three dimensional space that are not parallel. They both then lie in a certain plane. It is often useful to be able to form a vector that is perpendicular to this plane. This can be achieved with the aid of the **cross product**. We shall first give a definition in terms of angles and then later a definition in terms of the components of $\mathbf{u}, \mathbf{v}$.

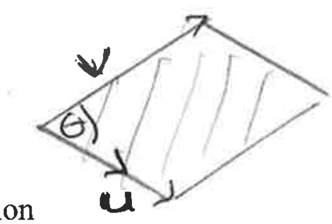
Definition: Cross Product
Let $\mathbf{u}, \mathbf{v}$ be two vectors in (three dimensional) space. Assume they are not parallel, so that they determine a plane. Choose $\mathbf{n}$ to be a unit vector perpendicular to this plane, by the **right-hand rule**. That is, we choose $\mathbf{n}$ to be the unit normal vector that points the way your right thumb does when your fingers curl through the angle θ from $\mathbf{u}$ to $\mathbf{v}$. Then define the **cross product** (or vector product)

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}| \sin\theta)\, \mathbf{n}.$$



Remarks
(a) While $\mathbf{u} \cdot \mathbf{v}$ is a scalar, $\mathbf{u} \times \mathbf{v}$ is a vector.
(b) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ iff $\mathbf{u}, \mathbf{v}$ are parallel (in that case $\sin\theta = 0$).

Properties of the Cross Product
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and r, s be scalars.
(1) $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)\, \mathbf{u} \times \mathbf{v}$
(2) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
(3) $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$
(4) $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
(5) $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
(6) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}$



Remarks
(a) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i}$; $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and of course $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.
(b) Because the unit normal $\mathbf{n}$ is a unit vector, we see that (with the notation above),

$$\boxed{|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta.}$$

Remember this formula! It is also the area of the parallelogram defined by the vectors $\mathbf{u}, \mathbf{v}$.

A determinant formula for $\mathbf{u} \times \mathbf{v}$
If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$,

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Example 1
Let $\mathbf{u} = (2,1,1)$ and $\mathbf{v} = (-4,3,1)$, then

$$\mathbf{u}\times\mathbf{v} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{bmatrix}.$$

Using cofactor expansion by the first row,

$$\begin{aligned} & \mathbf{u}\times\mathbf{v} \\ = \; & \mathbf{i}\det\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} - \mathbf{j}\det\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix} + \mathbf{k}\det\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} \\ = \; & \mathbf{i}\,(1\,(1) - 3\,(1)) - \mathbf{j}\,(2\,(1) - (-4)\,1) + \mathbf{k}\,(2\,(3) - (-4)\,1) \\ = \; & -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} = (-2,-6,10)\,. \end{aligned}$$

Note that then

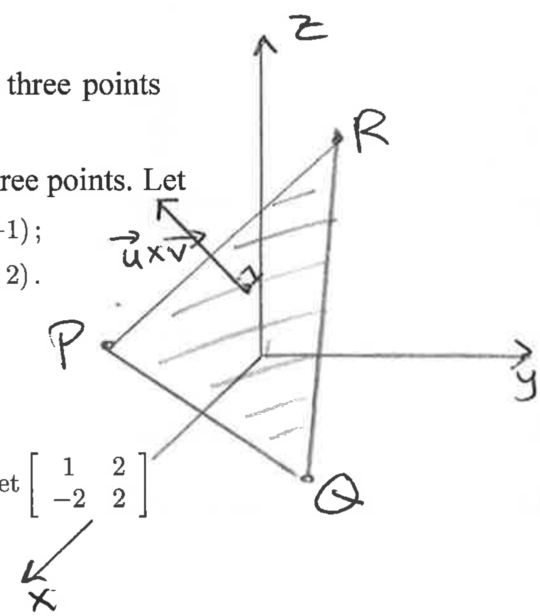
$$\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} = (2,6,-10)\,.$$

Example 2
Find a vector perpendicular to the plane containing the three points $P\,(1,-1,0)$; $Q\,(2,1,-1)$; $R\,(-1,1,2)$.
Solution
Let us first find two vectors in the plane containing the three points. Let

$$\begin{aligned} \mathbf{u} &= \overrightarrow{PQ} = (2,1,-1) - (1,-1,0) = (1,2,-1)\,; \\ \mathbf{v} &= \overrightarrow{PR} = (-1,1,2) - (1,-1,0) = (-2,2,2)\,. \end{aligned}$$

Then the desired perpendicular vector is

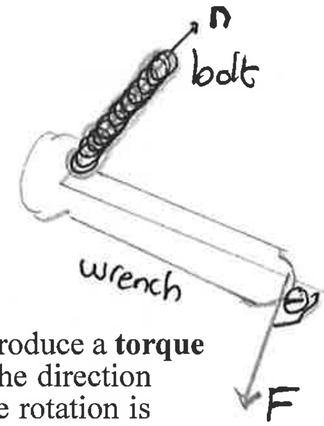
$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{bmatrix} \\ &= \mathbf{i}\det\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} - \mathbf{j}\det\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} + \mathbf{k}\det\begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} \\ &= \mathbf{i}\,(6) - \mathbf{j}\,(0) + \mathbf{k}\,(6) = (6,0,6)\,. \end{aligned}$$



Example 3
Find the area of the triangle with vertices $P\,(-1,1,0)$, $Q\,(2,1,-1)$, and $R\,(-1,1,2)$.
Solution
We know that the area of the parallelogram defined by the vectors $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$ is

$$|\mathbf{u}\times\mathbf{v}| = |(6,0,6)| = \sqrt{36+0+36} = 6\sqrt{2}.$$

The triangle has half the area of the parallegram, so this will be $3\sqrt{2}$.



Torque
When we turn a bolt by applying a force **F** to a wrench, we produce a **torque** that causes the bolt to rotate. The **torque vector** points in the direction of the axis of the bolt according to the right-hand rule. (The rotation is counterclockwise when viewed from the tip of the vector). The size or (magnitude) of the torque increases when it is applied further away along the wrench. Let **r** be the vector describing the lever arm of the wrench. Then **the magnitude of the torque** is

$$|\mathbf{r} \times \mathbf{F}|$$

and the torque vector is

$$\mathbf{r} \times \mathbf{F}.$$

Note that it points in the direction of the axis of the bolt. Note that if θ is the angle between **r** and **F**, then also the magnitude of the torque is

$$|\mathbf{r}|\,|\mathbf{F}|\sin\theta$$

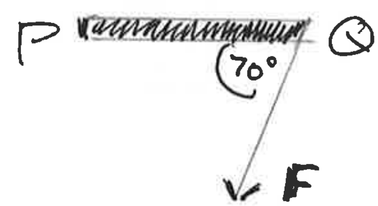
and the **torque vector** is also,

$$|\mathbf{r}|\,|\mathbf{F}|\,(\sin\theta)\,\mathbf{n}$$

where **n** is a unit vector normal (in the direction of the bolt) to the plane containing **r** and **F**.

Example
Suppose we take a bar $\overrightarrow{PQ}$ (or bolt) of length 3 feet, and apply a force of 20 pounds magnitude, with the force at an angle of $70°$ between **F** and the bar. The magnitude of the torque is

$$\begin{aligned} \left|\overrightarrow{PQ} \times \mathbf{F}\right| &= \left|\overrightarrow{PQ}\right| |\mathbf{F}| \sin 70° = (3)(20)\sin 70° \\ &= 56.4\text{...ft-lb.} \end{aligned}$$



Triple Box or Scalar Product

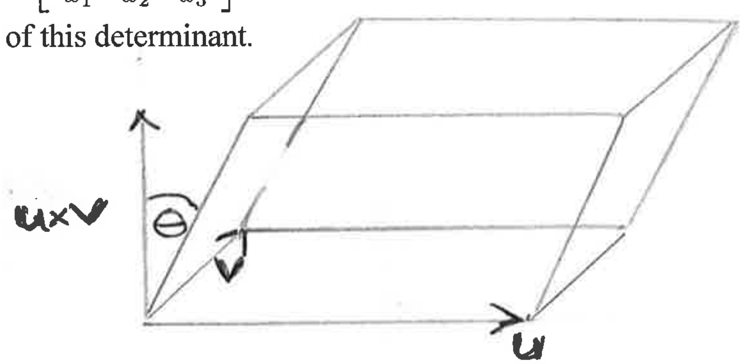
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors. We know that $|\mathbf{u} \times \mathbf{v}|$ is the area of the parallelogram, two of whose sides are $\mathbf{u}, \mathbf{v}$. If $\mathbf{w}$ makes an angle θ with the normal $\mathbf{u} \times \mathbf{v}$ to this parallelogram, then we see that the volume of the parallepiped determined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is

$$\begin{aligned} & |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}| \cos\theta \\ = & \ |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|\,. \end{aligned}$$

If we write $\mathbf{u} = (u_1, u_2, u_3)$, etc., then some algebra shows that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

So the volume is the absolute value of this determinant.



Example
Suppose $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (-2, 0, 3)$, $\mathbf{w} = (0, 7, -4)$. Find the volume of the parallelepiped formed by these three vectors.
Solution

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \det \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{bmatrix} \\ &= (1) \det \begin{bmatrix} 0 & 3 \\ 7 & -4 \end{bmatrix} - (2) \det \begin{bmatrix} -2 & 3 \\ 0 & -4 \end{bmatrix} + (-1) \det \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix} \\ &= (1)(-21) - 2(8) + (-1)(-14) = -23. \end{aligned}$$

The volume is 23.

Section 12.5: Lines and Planes in Space

Lines in Space

A line in space is determined by a point $P_0\,(x_0, y_0, z_0)$ through which it passes, and a direction vector $\mathbf{v}$. It is is the set of all points $P\,(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$. We can give its equation in two forms:

Vector equation for a line:
A vector equation for the line L through the point $P\,(x_0, y_0, z_0)$ parallel to the vector $\mathbf{v}$ is given by the formula

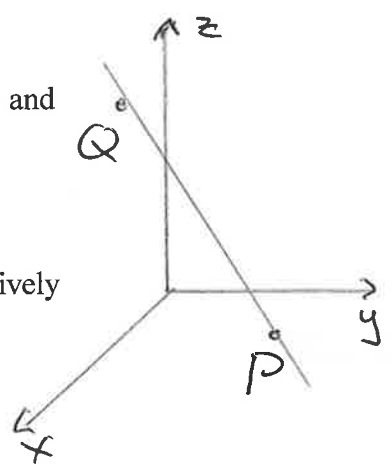
$$\mathbf{r}\,(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty.$$

Here $\mathbf{r}\,(t)$ is the position of a point $P\,(x, y, z)$ on L and

$$\mathbf{r}_0 = (x_0, y_0, z_0)\,.$$

Parametric Equation for a Line:
The standard parametrization of the line L through $P\,(x_0, y_0, z_0)$ parallel to the vector $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1;\; y = y_0 + tv_2;\; z = z_0 + tv_3, \; -\infty < t < \infty.$$



Example
Find vector and parametric equations for the line through $P\,(-3, 2, -3)$ and $Q\,(1, -1, 4)$.
Solution
We first have to find the direction vector $\mathbf{v} = \overrightarrow{PQ}$. We see that

$$\mathbf{v} = \overrightarrow{PQ} = (1, -1, 4) - (-3, 2, -3) = (4, -3, 7).$$

We can take as our point P_0 the point $P\,(-3, 2, -3)$. (We could alternatively take Q). Then the vector equation is

$$\mathbf{r}\,(t) = (-3, 2, -3) + t(4, -3, 7) = (-3 + 4t, 2 - 3t, -3 + 7t)\,.$$

The parametric equations are

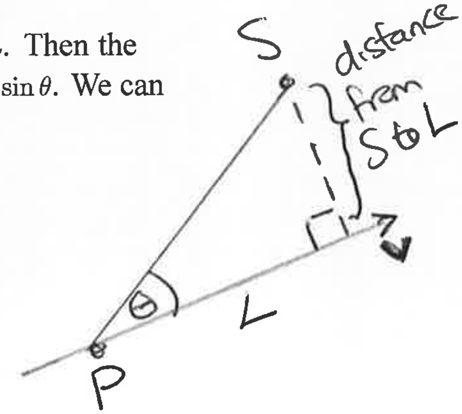
$$x = -3 + 4t;\, y = 2 - 3t;\, z = -3 + 7t, \quad -\infty < t < \infty.$$

If we only wanted to parametrize the line segment joining $P\,(-3, 2, -3)$ and $Q\,(1, -1, 4)$, instead we use

$$x = -3 + 4t;\, y = 2 - 3t;\, z = -3 + 7t, \; t \in [0, 1]\,.$$

The Distance from a Point to a Line in Space
Let L be a line passing through a point P and parallel to a vector $\mathbf{v}$. To find the distance from a point S to the line L, we drop a perpendicular/normal from S to L. The length of this normal is the distance we want. Now let's

find a formula for it. Suppose that $\overrightarrow{PS}$ makes an angle θ with L. Then the distance is the length of the perpendicular/normal, namely $\left|\overrightarrow{PS}\right| \sin\theta$. We can rewrite this as



$$
\begin{aligned}
& \left|\overrightarrow{PS}\right| (1) \sin\theta \\
= & \left|\overrightarrow{PS}\right| \left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| \sin\theta \\
= & \left|\overrightarrow{PS} \times \frac{\mathbf{v}}{|\mathbf{v}|}\right| \\
= & \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|}.
\end{aligned}
$$

Summary
The distance d is the magnitude of this vector, so

$$d = \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|}.$$

Example
Find the distance from the point $S(1,1,5)$ to the line L with parametric equations

$$x = 1 + t;\ y = 3 - t;\ z = 2t.$$

Solution
L passes through $P(1,3,0)$ and has direction vector $\mathbf{v} = (1,-1,2)$. The vector from P to S is

$$\overrightarrow{PS} = (1-1, 1-3, 5-0) = (0,-2,5).$$

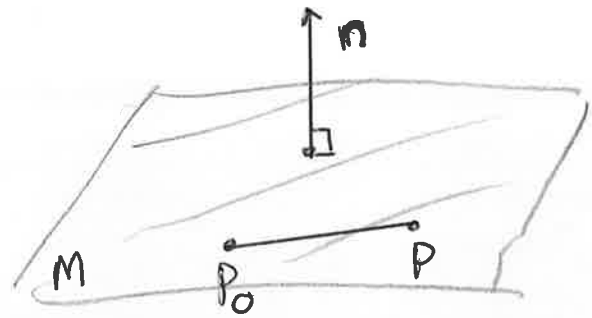
Then

$$
\begin{aligned}
\overrightarrow{PS} \times \mathbf{v} = \det & \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{bmatrix} \\
= & \ \mathbf{i} \det \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \\
= & \ \mathbf{i}(-4+5) - \mathbf{j}(0-5) + \mathbf{k}(0+2) = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k} = (1,5,2).
\end{aligned}
$$

The distance from S to L is

$$\frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation for a Plane in Space
A plane in space is determined by a point in the plane and a normal to the plane. Suppose that M is a plane that passes through $P(x_0, y_0, z_0)$ and is



normal to

$$\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} = (A, B, C).$$

(Note that $\mathbf{n}$ does not need to be a unit vector). Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$, that is

$$\mathbf{n}\cdot\overrightarrow{P_0P} = 0.$$

This is equivalent to

$$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

that is, M is the set of all points (x, y, z) with

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equivalently, if we set

$$D = Ax_0 + By_0 + Cz_0,$$

then

$$Ax + By + Cz = D.$$

Let us summarize all these:

Summary: Equations for a Plane
The plane M through $P(x_0, y_0, z_0)$ normal to $\mathbf{n} = (A, B, C)$ may be described by
Vector equation:

$$\mathbf{n}\cdot\overrightarrow{P_0P} = 0.$$

Component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Example
Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = (5, 2, -1) = 5\mathbf{i}+2\mathbf{j} - \mathbf{k}$.
Solution
The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0,$$

or equivalently,

$$5x + 15 + 2y - z + 7 = 0$$
$$\Leftrightarrow 5x + 2y - z = -22.$$

Example
Find an equation for the plane through $A(0, 0, 1)$, $B(2.0.0)$, $C(0, 3, 0)$.
Solution
Let us find two vectors in the plane, and hence a normal to the plane:

$$\overrightarrow{AB} = (2, 0, -1) \text{ and } \overrightarrow{AC} = (0, 3, -1).$$

A normal to the plane is

$$\begin{aligned}
\mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\
&= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{bmatrix} \\
&= \mathbf{i}\,(0+3) - \mathbf{j}\,(-2-0) + \mathbf{k}\,(6-0) \\
&= \mathbf{3i} + \mathbf{2j} + \mathbf{6k}.
\end{aligned}$$

Then the component equation is

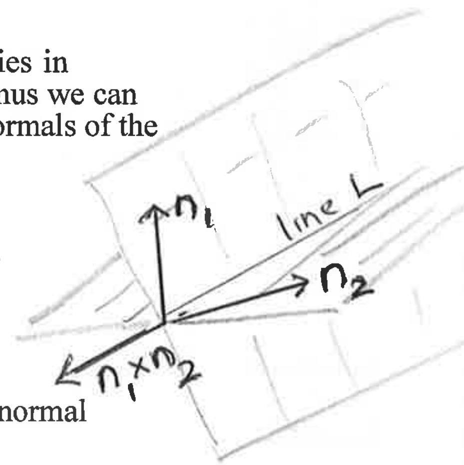
$$3\,(x-0) + 2(y-0) + 6(z-1) = 0,$$

or

$$3x + 2y + 6z = 6.$$

Lines of Intersection
We know that two lines are parallel iff they have the same direction. In the same way, two planes are parallel iff they have the same normal (or equivalently, the one normal is a constant times the other).

Two planes that are not parallel intersect in a line. That line lies in both planes, so is perpendicular to the normals of each plane. Thus we can find the direction of the line by taking the cross product of the normals of the two planes.



Example
Find a vector parallel to the line of intersection of the two planes

$$\begin{aligned}
3x - 6y - 2z &= 15 \\
2x + y - 2z &= 5.
\end{aligned}$$

Solution
The first plane has normal $\mathbf{n}_1 = (3, -6, -2)$ and the second has normal $\mathbf{n}_2 = (2, 1, -2)$. The line of intersection will be parallel to

$$\begin{aligned}
\mathbf{v} &= \mathbf{n}_1 \times \mathbf{n}_2 = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{bmatrix} \\
&= \mathbf{i}\,(12+2) - \mathbf{j}\,(-6+4) + \mathbf{k}\,(3+12) \\
&= 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k} = (14, 2, 15)\,.
\end{aligned}$$

Example
Use the above to find parametric equations for the line of intersection of the two planes
Solution
We already have the direction $\mathbf{v}$ of the line. We must find a point $P\,(x_0, y_0, z_0)$

that lies in both planes, at it will lie on the line. So we want

$$\begin{aligned} 3x_0 - 6y_0 - 2z_0 &= 15 \\ 2x_0 + y_0 - 2z_0 &= 5. \end{aligned}$$

One option is to just set $z_0 = 0$, and then solve the simultaneous equations

$$\begin{aligned} 3x_0 - 6y_0 &= 15 \\ 2x_0 + y_0 &= 5. \end{aligned}$$

You can check that this has the (unique) solution

$$(x_0, y_0) = (3, -1)$$

so

$$(x_0, y_0, z_0) = (3, -1, 0).$$

Thus the parametric form of the line of intersection, which we know passes through $(3, -1, 0)$ and has direction $v = (14, 2, 15)$ is

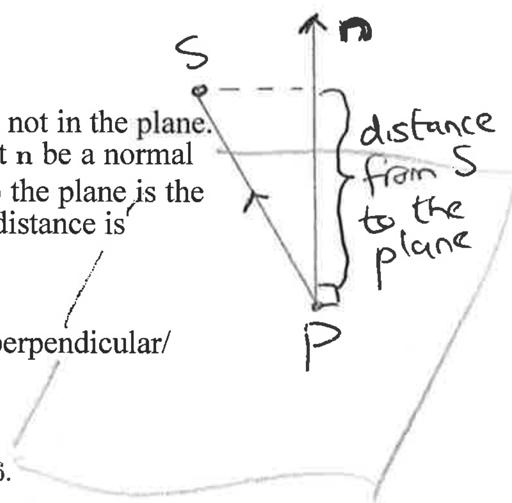
$$x = 3 + 14t;\ y = -1 + 2t;\ z = 0 + 15t.$$

The Distance from a Point to a Plane

Let P be a point on a plane with normal $\mathbf{n}$. Let S be a point not in the plane. We can find the distance from S to the plane as follows: let $\mathbf{n}$ be a normal to the plane. We project $\overrightarrow{PS}$ onto $\mathbf{n}$. The distance from S to the plane is the magnitude of the component of this projection: that is, the distance is

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|.$$

Note that this is the same magnitude as the length of the perpendicular/ normal that we can drop from S to the plane.



Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution

A normal to the plane is

$$\mathbf{n} = (3, 2, 6).$$

Next, we need a point in the plane. For any given two of (x, y, z), we can solve for the third. Suppose for example, we set $y = z = 0$, so that $3x + 2(0) + 6(0) = 6 \Rightarrow x = 2$. Then as our point P, we take $P(2, 0, 0)$, so that

$$\overrightarrow{PS} = (1 - 2, 1 - 0, 3 - 0) = (-1, 1, 3).$$

Also,

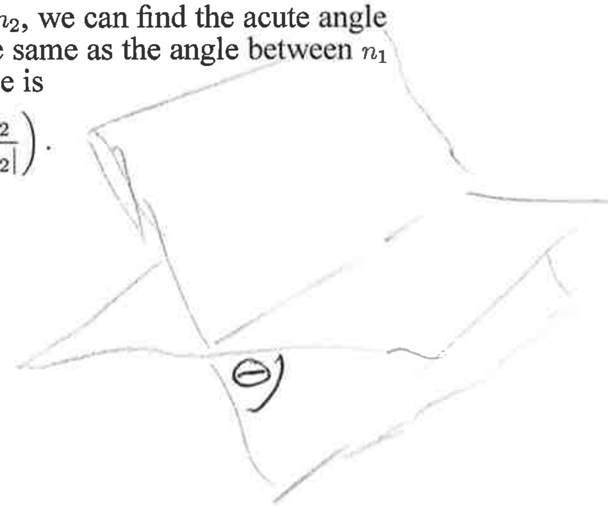
$$|\mathbf{n}| = |(3, 2, 6)| = \sqrt{9 + 4 + 36} = 7.$$

So the distance is

$$\begin{aligned} d &= \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| \\ &= \left|(-1,1,3) \cdot \frac{(3,2,6)}{7}\right| \\ &= \frac{1}{7}\left|(-1,1,3) \cdot (3,2,6)\right| \\ &= \frac{1}{7}(-3+2+18) = \frac{17}{7}. \end{aligned}$$

Angle Between Planes
If two planes that intersect have normals n_1, n_2, we can find the acute angle between the planes by observing that it is the same as the angle between n_1 and n_2 (using some geometry). Thus the angle is

$$\theta = \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right).$$



Section 12.6: Cylinders and Quadric Surfaces

We shall study several types of surfaces

Cylinders
We normally think of a cylinder as a "tube" whose cross section is a circular region. We allow a more general type of cylinder:

The formal definition is a bit confusing: a **cylinder** is a surface generated by moving a straight line along a given curve in a plane, while holding the line parallel to a fixed line. The curve is called a **generating curve**.

Example
Start with the parabola $y = x^2$ (in the plane $z = 0$). This generates the cylinder

$$(x, y, z) = (x, x^2, z)$$

where now x and z take arbitrary values.

Quadric Surfaces
A **quadric surface** is the graph in space of a second degree equation in (x, y, z). We study quadric surfaces defined by the equation

$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

where A, B, C, D, E are constants.

Example: The Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

cuts the coordinate axes at $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0, \pm c)$. Note that in the plane $x = 0$, we obtain

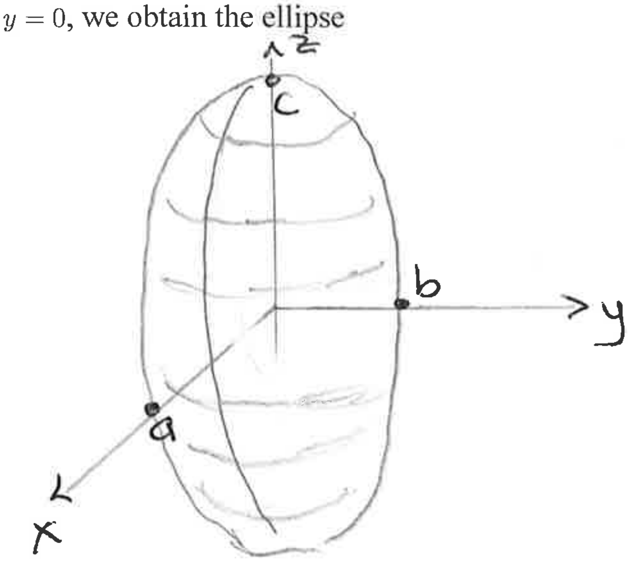
$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which is indeed an ellipse. Similarly in the plane $y = 0$, we obtain the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

and in the plane $z = 0$, we obtain the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Example The Elliptical Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}.$$

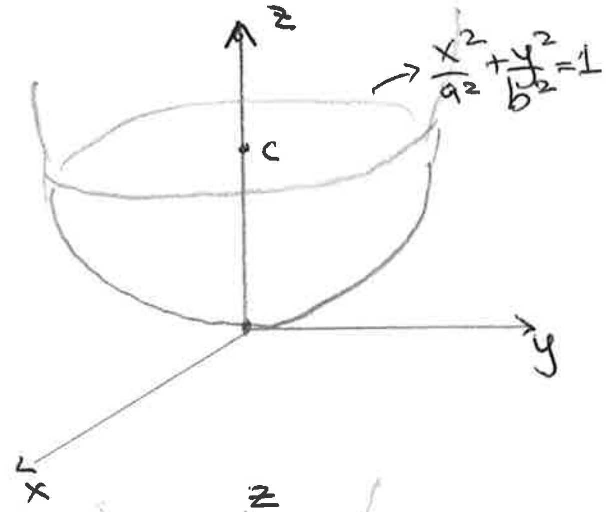
This gives an ellipse in the plane $z = c$, namely,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

but gives a parabola

$$z = \frac{c}{a^2}x^2$$

in the plane $y = 0$.



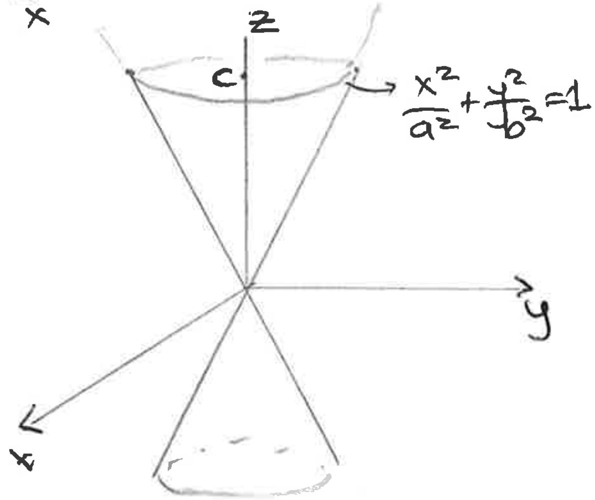
Example The Elliptical Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

This gives an ellipse in the plane $z = c$ and lines

$$z = \pm\frac{c}{a}x$$

in the plane $y = 0$.



Example The Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

This gives an ellipse in the plane $z = c$,

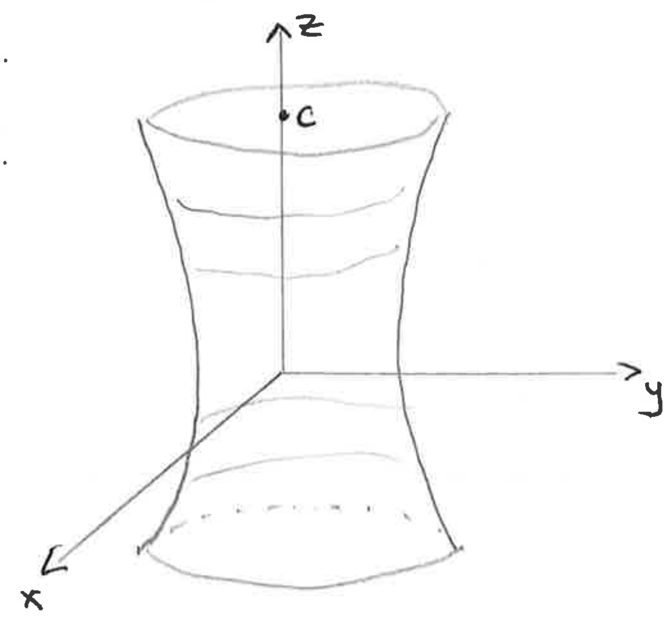
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

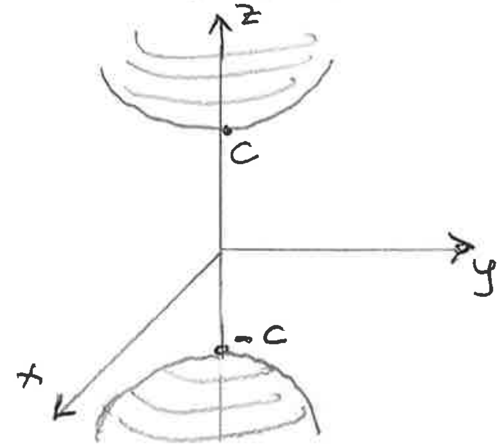
It clearly gives an ellipse for any fixed value of z. For example, in the plane $z = 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In the plane $x = 0$, this gives the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$





Example The Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Note that we need $\frac{z^2}{c^2} - 1 \geq 0$, so there is no part of the surface in the range $z \in (-c, c)$ and consequently it splits into two pieces. If $z = c\sqrt{2}$ or $z = -c\sqrt{2}$, we obtain the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In the $x = 0$ plane, this gives the hyperbola

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1.$$

Example The hyperbolic paraboloid

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0.$$

has symmetry with respect to the plane $x = 0$ and $y = 0$. In the plane $x = 0$, we obtain the parabola

$$z = \frac{c}{b^2} y^2$$

and in the plane $y = 0$,

$$z = -\frac{c}{a^2} x^2.$$