Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Journal of Approximation Theory 125 (2003) 106–115

Journal of Approximation Theory

http://www.elsevier.com/locate/jat

Best approximation and interpolation of $(1+(ax)^2)^{-1}$ and its transforms

D.S. Lubinsky

The School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

Received 16 December 2002; accepted in revised form 15 September 2003

Communicated by Tamás Erdélyi

Abstract

We show that Lagrange interpolants at the Chebyshev zeros yield best relative polynomial approximations of $(1+(ax)^2)^{-1}$ on $[-1,1]$, and more generally of

$$\int_0^\infty \frac{d\mu(a)}{1+(ax)^2},$$

where μ is a suitably restricted measure. We use this to study relative approximation of $(1+x^2)^{-1}$ on an increasing sequence of intervals, and Lagrange interpolation of $|x|^\gamma$. Moreover, we show how it gives a simple proof of identities for some trigonometric sums.

Keywords: Interpolation; Best approximation

1. Results

While looking for relative approximations to $(1+x^2)^{-1}$ on a growing sequence of intervals, the author noticed the following simple (new?) result on explicit best relative approximation. Throughout this paper, $L_m[f]$ denotes the Lagrange interpolation polynomial to the function f at the zeros of T_m, the Chebyshev polynomial of degree m.

E-mail address: lubinsky@math.gatech.edu.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2003.09.002

Proposition 1. *Let m be an even positive integer, U be an even real polynomial of degree $\leqslant m$, and let $a>0$. Let*

$$f_a(x) = (1+(ax)^2)^{-1}, \quad x \in [-1,1].$$

Then $L_m[Uf_a]$ is a polynomial of degree $\leqslant m-2$ and

$$\begin{aligned} \|L_m[f_aU]/f_a - U\|_{L_\infty[-1,1]} &= \inf\{\|P/f_a - U\|_{L_\infty[-1,1]} : \deg(P) \leqslant m-1\} \\ &= |U(i/a)/T_m(i/a)|. \end{aligned} \tag{1}$$

Moreover, for all x,

$$L_m[f_aU](x)/f_a(x) - U(x) = (-1)^{1+m/2} T_m(x) U(i/a)/|T_m(i/a)|. \tag{2}$$

From this, with $U(x)=1$, one can readily derive a result on relative approximation of $(1+x^2)^{-1}$ on a growing sequence of intervals, with a lower bound on the circle centre 0, radius $\frac{1}{2}$. The author needed the latter in studying eigenvalues of Hankel matrices:

Corollary 2. *Let $(a_m)_{m=1}^\infty$ be an increasing sequence of positive numbers with limit ∞. There exist polynomials S_m of degree $\leqslant m-1$, $m \geqslant 1$, with*

$$\limsup_{m\to\infty} \|(1+x^2)S_m(x) - 1\|_{L_\infty[-a_m,a_m]} < 1 \tag{3}$$

iff

$$\liminf_{m\to\infty} m/a_m > 0. \tag{4}$$

Moreover, assuming this last condition, there exists $C>0$ and polynomials S_m of degree $\leqslant m$ satisfying (3) *and for $|z| = \frac{1}{2}$,*

$$|S_m(z)| \geqslant C. \tag{5}$$

We can also readily derive closed-form expressions for some trigonometric sums: the second one below sometimes appears in number theoretic contexts.

Corollary 3. *Let $n \geqslant 1$ and $a>0$. Then*

$$\sum_{j=1}^n \frac{(-1)^j \sin\left(j-\frac{1}{2}\right)\frac{\pi}{n}}{1+\left(a\cos\left(j-\frac{1}{2}\right)\frac{\pi}{2n}\right)^2} = \frac{2n(-1)^n}{a^2\left|T_{2n}\left(\frac{i}{a}\right)\right|}. \tag{6}$$

In particular,

$$\sum_{j=1}^n (-1)^j \tan\left(j-\frac{1}{2}\right)\frac{\pi}{2n} = (-1)^n n. \tag{7}$$

One of the features of Theorem 1 is that the alternation points are independent of a in f_a. Thus we may integrate with respect to a, the main idea of this paper:

Theorem 4. *Let μ be a non-negative Borel measure on $[0, \infty)$ satisfying*

$$0 < \int_0^\infty \frac{d\mu(a)}{1+a^2} < \infty. \tag{8}$$

Let U be an even real polynomial of degree $\leqslant m$, with no zeros on the imaginary axis, except possibly at 0. Let

$$F(x) := U(x) \int_0^\infty \frac{d\mu(a)}{1+(ax)^2}, \quad x \in [-1,1] \setminus \{0\} \tag{9}$$

and

$$w(x) := 1 \Big/ \int_0^\infty \left| \frac{U(i/a)}{T_m(i/a)} \right| \frac{d\mu(a)}{[1+(ax)^2]}, \quad x \in [-1,1] \setminus \{0\}. \tag{10}$$

Then $L_m[F]$ is a polynomial of degree $\leqslant m-2$ and

$$||(L_m[F] - F)w||_{L_\infty[-1,1]} = 1 = \inf\{||(P-F)w||_{L_\infty[-1,1]} : \deg(P) \leqslant m-1\}. \tag{11}$$

Here we interpret $w(0)$ as its limit 0 and $(Fw)(0)$ as its limit 1 if μ has infinite mass on $[0, \infty)$ and $U(0) \neq 0$. In all other cases, we interpret $F(0)$ and $w(0)$ as their limiting values at 0.

We note that one can relax the positivity of the measure μ and the restrictions on U. All one really needs is that μ and U are such that w is finite and non-zero, except possibly at 0. Perhaps initially, this theorem appears artificial—but the ideas of its proof can be used to easily study asymptotics of errors of Lagrange interpolation to the functions

$$g_a(x) := |x|^a. \tag{12}$$

Corollary 5. *Let $\gamma > 0$ and not be an even integer. Let 2ℓ be the largest even integer $\leqslant \gamma$ and*

$$\begin{aligned} A_\gamma &:= \int_0^\infty \frac{y^{\gamma-1}}{\cosh(y)} dy \Big/ \int_0^\infty \frac{y^{\gamma-2\ell-1}}{1+y^2} dy \\ &= \frac{2^{2-\gamma}}{\pi} \left| \sin \frac{\gamma\pi}{2} \right| \Gamma(\gamma) \sum_{j=0}^\infty \left(j + \frac{1}{2} \right)^{-\gamma} (-1)^j. \end{aligned} \tag{13}$$

(a) *Then*

$$\lim_{n\to\infty} (2n)^\gamma ||x^2 L_{2n}[g_{\gamma-2}](x) - |x|^\gamma||_{L_\infty[-1,1]} = A_\gamma. \tag{14}$$

Moreover, if (ξ_m) is any increasing sequence of positive numbers with limit ∞, we have uniformly for $|x| \in \left[\frac{\xi_{2n}}{2n}, 1\right]$, as $n \to \infty$,

$$(2n)^\gamma \{x^2 L_{2n}[g_{\gamma-2}](x) - |x|^\gamma\} = (-1)^{n+1+\ell} T_{2n}(x)(A_\gamma + o(1)). \tag{15}$$

(b)
$$\lim_{n\to\infty} (2n)^\gamma ||L_{2n}[g_\gamma](x) - |x|^\gamma||_{L_\infty[-1,1]} = \lim_{n\to\infty} (2n)^\gamma |L_{2n}[g_\gamma](0)| = A_\gamma. \tag{16}$$

For $\gamma = 1$, $A_\gamma = 1$. Thus, the polynomials $x^2 L_{2n}[g_{\gamma-2}](x)$ fare worse than the best polynomial approximations of degree n that give the Bernstein constant 0.28016….[5, p. 749ff; 6, p. 4]. This is not surprising, as the best polynomial approximations to $|x|$ have positive constant coefficients [2, p. 79, no. 27]. What is interesting, however, is that $x^2 L_{2n}[g_{\gamma-2}](x) - |x|^\gamma$ has $2n - O(1)$ points of "almost" alternation as $n \to \infty$. This suggests that A_γ might be the analogue of the Bernstein constant when we best approximate $|x|^\gamma$ by polynomials that vanish at 0.

After the results of this paper were obtained, the interesting paper of Ganzburg [3] appeared. There representations and asymptotics are obtained for errors in Lagrange interpolation that are similar to, but not the same, as some in this paper. In particular, limit (16) is given there, as well as a representation for the error in interpolation of $(1-x)^a$ that is similar in spirit to ours for $|x|^\gamma$. However, the main idea of this paper seems to be entirely new—namely, that integrating in Theorem 4 and Corollary 5 with respect to a positive measure $d\mu(a)$ allows easy analysis for a fair range of functions.

2. Proofs

We begin with

Proof of Proposition 1. Since f_a, U and T_m are even, so is the unique Lagrange interpolation polynomial $L_m[f_a U]$. The latter has degree $\leqslant m-1$, so has degree $\leqslant m-2$. But then

$$L_m[f_a U]/f_a - U$$

is a polynomial of degree $\leqslant m$, and has zeros at the zeros of T_m, so for some constant c,

$$L_m[f_a U]/f_a - U = cT_m.$$

To determine c, we evaluate this last identity at i/a:

$$-U(i/a) = cT_m(i/a) \Rightarrow c = -U(i/a)/T_m(i/a).$$

Next, recall that if $\phi(z) = z + \sqrt{z^2-1}, z \notin [-1,1]$, then

$$T_m(z) = \tfrac{1}{2}(\phi(z)^m + \phi(z)^{-m}),$$

so

$$\begin{aligned} T_m(i/a) &= \frac{(-1)^{m/2}}{2}\left(\left[\frac{1}{a} + \sqrt{1+\frac{1}{a^2}}\right]^m + \left[\frac{1}{a} + \sqrt{1+\frac{1}{a^2}}\right]^{-m}\right) \\ &= (-1)^{m/2}|T_m(i/a)|. \end{aligned} \tag{17}$$

Now (2) follows. The best approximation property (1) follows from (2), and the equioscillation theorem applied to weighted approximation [1, p. 52]. Indeed, $L_m[f_a U]/f_a - U = cT_m$ equioscillates $m+1$ times in $[-1,1]$. $\square$

Proof of Corollary 2. By the substitution

$$S_m(x) = P_m(a_m^{-1}x), \quad m \geqslant 1,$$

we see that the existence of polynomials satisfying (3) reduces to the existence of polynomials P_m of degree $\leqslant m$, with

$$\limsup_{m\to\infty} \|P_m(x)/f_{a_m}(x) - 1\|_{L_\infty[-1,1]} < 1.$$

Proposition 1 with $U \equiv 1$ shows that the error in relative approximation of f_{a_m} is the same for polynomials of degree $\leqslant m-1$ or $m-2$ if m is even. Thus, from (1), (P_m) exists iff

$$\liminf_{m\to\infty} \left| T_m\left(\frac{i}{a_m}\right)\right| > 1.$$

In turn since

$$\tfrac{1}{2}(s + s^{-1}) > 1 \quad \text{for } s \in (1, \infty),$$

(17) shows that this reduces to

$$\liminf_{m\to\infty} \left[\frac{1}{a_m} + \sqrt{1 + \frac{1}{a_m^2}}\right]^m > 1$$

and hence (4). If (4) is true, we can use $S_m(x) = L_m[f_{a_m}](x/a_m)$. From (2), we see that for $|z| = \dfrac{1}{2}$,

$$|(1+z^2)S_m(z) - 1| = \left|\frac{T_m(z/a_m)}{T_m(i/a_m)}\right| \leqslant \left|\frac{T_m(i/(2a_m))}{T_m(i/a_m)}\right|.$$

Now let us denote the zeros of T_m by $x_{jm}, 1 \leqslant j \leqslant m$, and recall that $\sin\frac{\pi}{2m}$ is the positive zero closest to 0. Since the zeros of T_m are symmetric about 0, we see that

$$\left|\frac{T_m(i/(2a_m))}{T_m(i/a_m)}\right| = \prod_{x_{jm}>0} \left(\frac{\frac{1}{4} + (a_m x_{jm})^2}{1 + (a_m x_{jm})^2}\right) \leqslant \frac{\frac{1}{4} + \left(a_m \sin\frac{\pi}{2m}\right)^2}{1 + \left(a_m \sin\frac{\pi}{2m}\right)^2} \leqslant C_1 < 1,$$

in view of the fact that a_m/m is bounded above. So for $|z| = \frac{1}{2}$,

$$|(1+z^2)S_m(z)| \geqslant 1 - C_1 > 0,$$

and (5) follows. $\square$

Proof of Corollary 3. Let m be even and as above,

$$x_{jm} = \cos\left(\left(j - \frac{1}{2}\right)\frac{\pi}{m}\right), \quad 1 \leqslant j \leqslant m,$$

denote the zeros of T_m. The standard formulas for Lagrange interpolation applied to f of Theorem 1 and to the constant 1 give

$$L_m[f_a](x) = \sum_{j=1}^{m} \frac{f_a(x_{jm})T_m(x)}{T_m'(x_{jm})(x - x_{jm})},$$

$$1 = \sum_{j=1}^{m} \frac{T_m(x)}{T_m'(x_{jm})(x - x_{jm})}.$$

Then (2) with $U \equiv 1$ gives

$$\frac{(-1)^{1+m/2}}{|T_m(i/a)|} = \frac{L_m[f_a](x)/f_a(x) - 1}{T_m(x)} = \sum_{j=1}^{m} \frac{1}{T_m'(x_{jm})(x - x_{jm})}\left[\frac{f_a(x_{jm})}{f_a(x)} - 1\right]$$
$$= a^2 \sum_{j=1}^{m} \frac{(x + x_{jm})}{T_m'(x_{jm})(1 + (ax_{jm})^2)}.$$

Since the left-hand side is constant, the term involving x on the right-hand side vanishes. Moreover, T_m' is odd as T_m is even, so we see that

$$(-1)^{1+m/2}/|T_m(i/a)| = 2a^2 \sum_{x_{jm}>0} \frac{x_{jm}}{T_m'(x_{jm})(1 + (ax_{jm})^2)}.$$

Since

$$T_m'(x_{jm}) = \frac{(-1)^{j-1} m}{\sqrt{1 - x_{jm}^2}},$$

we obtain on writing $m = 2n$,

$$(-1)^{1+n}/|T_{2n}(i/a)| = \frac{a^2}{n} \sum_{j=1}^{n} \frac{(-1)^{j-1} x_{jm} \sqrt{1 - x_{jm}^2}}{1 + (ax_{jm})^2}.$$

Then (6) follows using a little manipulation. If we let $a \to \infty$ in the above identity, we obtain (7). $\square$

Proof of Theorem 4. If we multiply (2) by f_a and then integrate with respect to $d\mu(a)$, we obtain

$$L_m[F](x) - F(x) = (-1)^{1+m/2} T_m(x) \int_0^\infty \frac{U(i/a)}{|T_m(i/a)|} \frac{d\mu(a)}{[1 + (ax)^2]}. \tag{18}$$

Since $L_m[f_a]$ has degree $\leqslant m - 2$, so does $L_m[F]$. Next, as U is even and has real coefficients, $U(i/a)$ is real valued for $a > 0$. Moreover, as U has no zeros on the imaginary axis, except possibly at 0, $U(i/a)$ is of one sign for $a > 0$, that is, has the same sign as $U(i)$. Hence,

$$(L_m[F] - F)w = (-1)^{1+m/2} \operatorname{sign}(U(i)) T_m,$$

where w is given by (10). Now we consider three cases:

(I) μ *has finite total mass*: Then we see from (9) and (10) that $F(0)$ may be defined by (9) and that F is continuous in $[-1, 1]$, while w is positive and continuous in $[-1, 1]$. Hence we may apply the standard alternation theorem for weighted approximation [1, p. 52] to obtain (11).

(II) *μ has infinite total mass but* $U(0)=0$: Let $B>0$. We write

$$F(x)=\frac{U(x)}{x^2}\left(x^2\int_0^B\frac{d\mu(a)}{1+(ax)^2}+\int_B^\infty\frac{(ax)^2}{1+(ax)^2}\frac{d\mu(a)}{a^2}\right).$$

Here

$$x^2\int_0^B\frac{d\mu(a)}{1+(ax)^2}\leqslant x^2\int_0^B d\mu(a)\to 0,\quad x\to 0.$$

Moreover,

$$\int_B^\infty\frac{(ax)^2}{1+(ax)^2}\frac{d\mu(a)}{a^2}\leqslant\int_B^\infty\frac{d\mu(a)}{a^2}.$$

From (8), we see that for large B this last right-hand side is small. Since $U(x)/x^2\to U''(0)/2$ as $x\to 0$, we obtain

$$\lim_{x\to 0}F(x)=0.$$

Since $U(i/a)=O(a^{-2})$ and $|T_m(i/a)|\to 1$ as $a\to\infty$, we also see that

$$w(x)\to 1\Big/\int_0^\infty\frac{|U(i/a)|}{|T_m(i/a)|}d\mu(a)=w(0),$$

a finite positive value. Again we can apply the usual alternation theorem [1, p. 52].

(III) *μ has infinite total mass and* $U(0)\neq 0$: In this case,

$$\lim_{x\to 0}w(x)=0;\quad \lim_{x\to 0}F(x)=\infty.$$

The fact that w vanishes at 0 prevents us from applying the usual alternation theorem. However, for $x\neq 0$,

$$\begin{aligned}\frac{U(x)}{|U(0)|}\frac{1}{F(x)w(x)}&=\int_0^\infty\frac{|U(i/a)|}{|U(0)|\,|T_m(i/a)|}\frac{d\mu(a)}{[1+(ax)^2]}\Big/\int_0^\infty\frac{d\mu(a)}{[1+(ax)^2]}\\&=\int_0^\infty(1+\varepsilon(a))\frac{d\mu(a)}{[1+(ax)^2]}\Big/\int_0^\infty\frac{d\mu(a)}{[1+(ax)^2]}\\&=1+\int_0^\infty\varepsilon(a)\frac{d\mu(a)}{[1+(ax)^2]}\Big/\int_0^\infty\frac{d\mu(a)}{[1+(ax)^2]},\end{aligned}$$

where $\varepsilon(a)\to 0$ as $a\to\infty$. Fix $B>0$. We see that

$$\begin{aligned}&\int_0^B|\varepsilon(a)|\frac{d\mu(a)}{[1+(ax)^2]}\Big/\int_0^\infty\frac{d\mu(a)}{[1+(ax)^2]}\\&\quad\leqslant\int_0^B|\varepsilon(a)|d\mu(a)\Big/\int_0^\infty\frac{d\mu(a)}{[1+(ax)^2]}\to 0,\quad x\to 0.\end{aligned}$$

Moreover,

$$\int_B^\infty |\varepsilon(a)| \frac{d\mu(a)}{[1+(ax)^2]} \Big/ \int_0^\infty \frac{d\mu(a)}{[1+(ax)^2]} \leqslant \sup\{|\varepsilon(a)|:\ a \geqslant B\}.$$

It follows that

$$\lim_{x\to 0}(Fw)(x) = \mathrm{sign}(U(0)) =: (Fw)(0).$$

Then for any polynomial P, we see that Pw vanishes at 0, so

$$\|(P-F)w\|_{L_\infty[-1,1]} \geqslant |Fw|(0) = 1$$

and (11) persists. $\square$

In the proof of Corollary 5, we need

Lemma. *Let* $r>0$ *and*

$$\rho_m(r) = \int_0^\infty \frac{a^{-r-1}}{|T_m(i/a)|}\,da.$$

Then as $m\to\infty$,

$$\rho_m(r) = m^{-r}\int_0^\infty \frac{y^{r-1}}{\cosh(y)}\,dy(1+o(1)). \tag{19}$$

Proof. We split

$$\rho_m(r) = \left(\int_0^{\frac{3}{2}} + \int_{\frac{3}{2}}^{m^{3/4}} + \int_{m^{3/4}}^\infty\right)\frac{a^{-r-1}}{|T_m(i/a)|}\,da =: I_1 + I_2 + I_3.$$

To handle I_1, we use the lower bound

$$|T_m(i/a)| \geqslant 2^{m-1}a^{-m}$$

which follows readily from (17). So, for large enough m,

$$I_1 \leqslant 2^{1-m}\int_0^{\frac{3}{2}} a^{m-r-1}\,da = O\left(\left(\frac{3}{4}\right)^m\right).$$

Next, in I_3, we use the asymptotic

$$|T_m(i/a)| = \cosh\left(\frac{m}{a} + O\left(\frac{m}{a^2}\right)\right),$$

which holds uniformly for $m\geqslant 1$ and $a\in[1,\infty)$, and follows easily from (17). Thus

$$I_3 = \int_{m^{3/4}}^\infty \frac{a^{-r-1}}{\cosh\left(\frac{m}{a}+O\left(\frac{m}{a^2}\right)\right)}\,da$$

$$= m^{-r} \int_0^{m^{1/4}} \frac{y^{r-1}}{\cosh\left(y + O\left(\frac{y^2}{m}\right)\right)} dy$$
$$= m^{-r} \int_0^{\infty} \frac{y^{r-1}}{\cosh(y)} dy(1 + o(1)),$$

by the substitution $y = m/a$. Finally,

$$I_2 \leqslant \frac{1}{|T_m(i/m^{3/4})|} \int_{\frac{3}{2}}^{m^{3/4}} a^{-r-1} da \leqslant \frac{1}{\cosh(m^{1/4} + O(m^{-1/2}))} \int_{\frac{3}{2}}^{\infty} a^{-r-1} da.$$

The estimates above give (19). $\square$

Proof of Corollary 5. (a) Let 2ℓ be the least even integer $\leqslant \gamma$ and $\Delta = \gamma - 2\ell \in (0, 2)$. We use $U(x) = x^{2\ell}$ and

$$d\mu(a) = a^{1-\Delta} da/C_0, \quad a \in (0, \infty),$$

where

$$C_0 := \int_0^{\infty} \frac{y^{\Delta-1}}{1+y^2} dy = \int_0^{\infty} \frac{x^{1-\Delta}}{1+x^2} dx.$$

Since $\Delta \in (0, 2)$, (8) is valid. Then F given by (9) has

$$F(x) = x^{2\ell} \int_0^{\infty} \frac{a^{1-\Delta}}{1+(ax)^2} da/C_0$$
$$= |x|^{2\ell-1+\Delta-1} \int_0^{\infty} \frac{y^{1-\Delta}}{1+y^2} dy/C_0 = |x|^{\gamma-2} = g_{\gamma-2}(x)$$

so (18) implies for positive even m,

$$L_m[g_{\gamma-2}](x) - |x|^{\gamma-2} = (-1)^{1+m/2+\ell} T_m(x) \int_0^{\infty} \frac{1}{1+(ax)^2} \frac{a^{1-2\ell-\Delta}}{|T_m(i/a)|} da/C_0$$
$$= (-1)^{1+m/2+\ell} T_m(x) \int_0^{\infty} \frac{1}{1+(ax)^2} \frac{a^{1-\gamma}}{|T_m(i/a)|} da/C_0. \quad (20)$$

So

$$x^2 L_m[g_{\gamma-2}](x) - |x|^{\gamma} = (-1)^{1+m/2+\ell} T_m(x) W_m(x),$$

where

$$W_m(x) = \int_0^{\infty} \frac{(ax)^2}{1+(ax)^2} \frac{a^{-1-\gamma}}{|T_m(i/a)|} da/C_0. \quad (21)$$

We see that

$$W_m(x) \leqslant \int_0^{\infty} \frac{a^{-1-\gamma}}{|T_m(i/a)|} da/C_0 = \rho_m(\gamma)/C_0$$

and hence, applying the lemma,

$$m^{\gamma}\|x^2 L_m[g_{\gamma-2}](x)-|x|^{\gamma}\|_{L_\infty[-1,1]}\leqslant m^{\gamma}\rho_m(\gamma)/C_0$$
$$\leqslant(1+o(1))\int_0^\infty \frac{y^{\gamma-1}}{\cosh(y)}dy/C_0=(1+o(1))A_\gamma,$$

with the notation (13). In the other direction, since

$$\frac{(ax)^2}{1+(ax)^2}=1-\frac{1}{1+(ax)^2}\geqslant 1-\frac{1}{(ax)^2},$$

we also obtain

$$W_m(x)\geqslant\rho_m(\gamma)/C_0-\rho_m(\gamma+2)/(C_0x^2)=m^{-\gamma}(A_\gamma+o(1)), \tag{22}$$

by the lemma, uniformly for $|x|\in\left[\frac{\xi_m}{m},1\right]$, if only (ξ_m) is a sequence increasing to ∞. Hence uniformly for such x,

$$m^{\gamma}\{x^2L_m[g_{\gamma-2}](x)-|x|^{\gamma}\}=(-1)^{1+m/2+\ell}T_m(x)m^{\gamma}\rho_m(\gamma)/C_0(1+o(1))$$
$$=(-1)^{1+m/2+\ell}T_m(x)A_\gamma(1+o(1)).$$

The second form of A_γ in (13) follows from [4, (3.241.2), p. 292] and [4, (3.523.3), p. 348].

(b) Replacing γ by $\gamma+2$ in (20), we see that for positive even m,

$$L_m[g_\gamma](x)-|x|^{\gamma}=(-1)^{1+m/2+\ell+1}T_m(x)V_m(x),$$

where now

$$V_m(x)=\int_0^\infty\frac{1}{1+(ax)^2}\frac{a^{-1-\gamma}}{|T_m(i/a)|}da/C_0$$
$$\leqslant\int_0^\infty\frac{a^{-1-\gamma}}{|T_m(i/a)|}da/C_0=\rho_m(\gamma)/C_0,$$

with equality iff $x=0$. Applying the lemma gives the result. □

References

[1] N.I. Achieser, Theory of Approximation, Dover, New York, 1992.

[2] E.W. Cheney, Introduction to Approximation Theory, Chelsea, New York, 1982.

[3] M. Ganzburg, The Bernstein constant and polynomial interpolation at the Chebyshev nodes, J. Approx. Theory 119 (2002) 193–213.

[4] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, San Diego, 1980.

[5] G.V. Milovanovic, D.S. Mitrinovic, Th.M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, World Scientific, Singapore, 1994.

[6] R. Varga, Scientific Computation on Mathematical Problems and Conjectures, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Vermont, 1990.