Towards a Representation for the Bernstein Constant

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Abstract

Let $\alpha > 0$ be not an integer. S.N. Bernstein established the limit

$$\Lambda_\alpha := \lim_{n\to\infty} n^{2\alpha} E_n\left[x^\alpha; [0,1]\right],$$

where $E_n\left[x^\alpha; [0,1]\right]$ denotes the error in best approximation of x^α on $[0,1]$ by polynomials of degree $\leq n$. We outline the proof of a representation for Λ_α involving integrals of entire functions of exponential type.

1 Introduction

Let $\alpha > 0$ be not an even integer. In papers published in 1913 and 1938, S.N. Bernstein [1], [2] established the limit

$$\Lambda^*_\alpha = \lim_{n\to\infty} n^\alpha E_n\left[|x|^\alpha; [-1,1]\right],$$

where $E_n\left[|x|^\alpha; [-1,1]\right]$ denotes the error in best L_∞ approximation of $|x|^\alpha$ on $[-1,1]$ by polynomials of degree $\leq n$. The most studied case of this limit is $\alpha = 1$. Bernstein's first proof for this case was in the 1913 paper, and was long and difficult. Later he obtained a much simpler proof, for all α, involving dilations of the interval, making essential use of the homogeneity of $|x|^{\alpha}$, so that for $\lambda > 0$,

$$|\lambda x|^{\alpha} = \lambda^{\alpha} |x|^{\alpha} .$$

This enabled Bernstein to relate the error in approximation on $[-\lambda, \lambda]$ to that on $[-1, 1]$. It also yielded a formulation of the limit as the error in approximation on the whole real axis by entire functions of exponential type. Bernstein's dilation argument showed that

$$\Lambda_{\alpha}^{*} = \inf \left\{ \| \, |x|^{\alpha} - F \, \|_{L_\infty(\mathbb{R})} \colon F \text{ is an entire function of exponential type} \right\} .$$

However, Bernstein never resolved the value of Λ_{α}^{*}. Bernstein speculated that

$$\Lambda_{1}^{*} = \lim_{n \to \infty} n E_n \left[|x| \, ; [-1, 1] \right] = \frac{1}{2\sqrt{\pi}} = 0.28209\ 47917...,$$

Some 70 years later, this was disproved by Carpenter and Varga [13], using high precision scientific computation. They showed that

$$\Lambda_{1}^{*} = 0.28016\ 94990...$$

They also showed numerically that the normalized error $2nE_{2n} \left[|x| \, ; [-1, 1] \right]$ should admit an asymptotic expansion in negative powers of n.

Surprisingly, the much deeper analogous problem of rational approximation has already been solved, by H. Stahl in [9]. He proved, using sophisticated methods of potential theory and other complex analytic tools, that

$$\lim_{n \to \infty} e^{\pi \sqrt{n}} R_n \left[|x| \, ; [-1, 1] \right] = 8,$$

where $R_n \left[|x| \, ; [-1, 1] \right]$ denotes the error in best L_∞ approximation of $|x|$ on $[-1, 1]$ by rational functions with numerator and denominator degree $\leq n$. Later he extended this to $|x|^{\alpha}$ -see [8].

Although Λ_{α}^{*} is not known explicitly, the ideas of Bernstein have been refined, and greatly extended. M. Ganzburg has shown limit relations of this type for large classes of functions, in one and several variables, even when weighted norms are involved [3], [4]. He and others such as Raitsin [7] have considered not only uniform, but also L_p norms.

Vasiliev [11] extended Bernstein's results in another direction, replacing the interval $[-1, 1]$ by fairly general compact sets. Totik [10] has put

Vasiliev's results in final form, using sophisticated estimates for harmonic measures. For example, if K is a compact set containing 0 in its interior, then the Vasiliev-Totik result has the form

$$\lim_{n\to\infty} n^{\alpha} E_n \left[|x|^{\alpha}; K \right] = \left(\pi \omega_K (0) \right)^{-\alpha} \Lambda_{\alpha}^{*},$$

where ω_K denotes the equilibrium density of the set K (in the sense of classical potential theory).

In this paper, we announce a representation for Λ_{α}^{*}. While not fully explicit, since it involves a certain entire function which we are not yet certain is a classical special function, we believe it is a major step in determining explicit expressions for Λ_{α}^{*}. It is also the first representation of its type.

In many of the investigations of best approximation of $|x|^{\alpha}$, it turns out to be simpler to transform the problem to $[0,1]$, and we shall do likewise. We fix $\alpha > 0$, not a positive integer, and let

$$E_n \left[x^{\alpha}; [0,1] \right] := \inf \left\{ \| \, x^{\alpha} - P(x) \, \|_{L_{\infty}[0,1]} : \deg(P) \leq n \right\} \tag{1}$$

denote the error in best uniform approximation of x^{α} on $[0,1]$ by polynomials of degree $\leq n$. Because of symmetry considerations,

$$E_n \left[x^{\alpha}; [0,1] \right] = E_{2n} \left[|x|^{2\alpha}; [-1,1] \right],$$

so

$$\Lambda_{\alpha} := \lim_{n\to\infty} n^{2\alpha} E_n \left[x^{\alpha}; [0,1] \right] = 2^{-2\alpha} \Lambda_{2\alpha}^{*}. \tag{2}$$

Let $B_n(x)$ denote the best polynomial approximation of degree $\leq n$ to x^{α} in the uniform on $[0,1]$, so that B_n is a polynomial of degree $\leq n$, and

$$\| \, x^{\alpha} - B_n(x) \, \|_{L_{\infty}[0,1]} = \inf \left\{ \| \, x^{\alpha} - P(x) \, \|_{L_{\infty}[0,1]} : \deg(P) \leq n \right\}.$$

Let

$$0 < x_{1n} < x_{2n} < \ldots < x_{n+1,n} < 1$$

denote the zeros of the remainder or error function

$$R_n(x) = x^{\alpha} - B_n(x). \tag{3}$$

Because $\{1, x, x^2, \ldots, x^n, x^{\alpha}\}$ is a Chebyshev system, there are exactly $n+1$ such zeros, all simple, interlacing the $n+2$ alternation points. Moreover, let

$$X_n(z) = \prod_{j=1}^{n+1} (x - x_{jn}) \tag{4}$$

and

$$D_0 := \int_0^\infty \frac{s^{-\{\alpha\}}}{1+s} ds, \tag{5}$$

where $\{\alpha\}$ is the fractional part of α. Our result is:

Theorem 1
Let $\alpha > 0$, and not be an integer.
(I) Fix $j \geq 1$. There exists

$$x_j = \lim_{n\to\infty} n^2 x_{jn} > 0. \tag{6}$$

Moreover, $x_1 > 0$ and for $j \geq 2$,

$$x_j \in \left[\left(\left(j - \frac{3}{2} \right) \frac{\pi}{2} \right)^2, \left(\left(j - \frac{1}{2} \right) \frac{\pi}{2} \right)^2 \right]. \tag{7}$$

(II) Let

$$F(z) = \prod_{j=1}^{\infty} (1 - z/x_j). \tag{8}$$

Then F is an entire function such that $F(z^2)$ is of exponential type 2, and

$$\Lambda_\alpha = \lim_{n\to\infty} n^{2\alpha} E_n [x^\alpha; [0,1]] = \int_0^\infty \frac{t^{\alpha-1}}{F(-t)} dt / D_0. \tag{9}$$

(III) Uniformly for z in compact subsets of $\mathbb{C}\backslash(-\infty, 0]$,

$$\lim_{n\to\infty} n^{2\alpha} R_n \left(z/n^2 \right) = (-1)^{\alpha - \{\alpha\}+1} F(z) \int_0^\infty \frac{t^\alpha}{t+z} \frac{dt}{F(-t)} / D_0 \tag{10}$$

and uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} X_n \left(z/n^2 \right) / X_n(0) = F(z). \tag{11}$$

Recall here that $F(z^2)$ is of exponential type 2 means that for each $\varepsilon > 0$, we have

$$\sup_{|z|=r} \left| F\left(z^2\right) \right| \leq \exp\left((2+\varepsilon) r\right),$$

at least for large enough r, and 2 cannot be replaced by a smaller number.

In spirit, the asymptotics above are similar to asymptotics for orthogonal polynomials near the endpoints of the interval of orthogonality. For example, in describing asymptotic behaviour near 0 of orthogonal polynomials with respect to the Legendre weight 1 on $[0, 1]$, we would use the same scaling $z \to z/n^2$. Since Bessel functions play such a large role in describing this type of asymptotic for orthogonal polynomials, one might hope that Bessel (or cylinder) functions can be used to describe F. In any event, once F is known, we will have a fully explicit expression for the Bernstein constant.

We shall outline the proof of the result in Section 2. The full proof will be given elsewhere.

2 Ideas Behind the Proof

The basic idea that we use is to interpolate, for fixed a, the function

$$h_a(x) = \frac{1}{1+ax}, \tag{12}$$

and exploit the fact that $1/h_a$ is a linear polynomial. Then we multiply by $a^{-\{\alpha\}}$ and integrate with respect to a, using the elementary identity

$$\int_0^\infty h_a(x)\, a^{-\{\alpha\}} da = \int_0^\infty \frac{a^{-\{\alpha\}}}{1+ax} da = D_0 x^{\{\alpha\}-1}, \tag{13}$$

where D_0 is as in (5). This idea was already used in [5], and works for any set of interpolation points. Throughout this section, we write

$$p = \alpha - \{\alpha\} + 1. \tag{14}$$

Thus p is the least integer larger than α. Our representation for the remainder is:

Theorem 2
Let $n \geq p-1$. Then

$$R_n(x) = x^\alpha - B_n(x) = \frac{(-1)^p}{D_0} X_n(x) \int_0^\infty \frac{s^\alpha}{X_n(-s)} \frac{ds}{s+x}. \tag{15}$$

Proof

Fix $a \geq 0$ and let $U(x) = x^p$. Let $L_n[Uh_a]$ denote the Lagrange interpolation polynomial of degree $\leq n$ to Uh_a at the $n+1$ zeros of X_n. Then

$$L_n[Uh_a]/h_a - U$$

is a polynomial of degree $\leq n+1$, that has zeros at the $n+1$ zeros of X_n, so there exists a constant c such that

$$L_n[Uh_a]/h_a - U = cX_n.$$

If we evaluate both sides at $x = -1/a$ and use that $1/h_a$ vanishes at this point, we obtain

$$-U(-1/a) = cX_n(-1/a),$$

and hence

$$L_n[Uh_a]/h_a - U = \frac{(-1)^{p+1} a^{-p}}{X_n(-1/a)} X_n.$$

So for all real x,

$$L_n[Uh_a](x) - \frac{x^p}{1+ax} = X_n(x) \frac{(-1)^{p+1}}{X_n(-1/a)} \frac{a^{-p}}{1+ax}.$$

Now multiply by $a^{-\{\alpha\}}$, and integrate over $a \in [0, \infty)$, and use (13) to obtain

$$D_0 L_n\left[Ux^{\{\alpha\}-1}\right](x) - D_0 x^{p+\{\alpha\}-1} = X_n(x)(-1)^{p+1} \int_0^\infty \frac{a^{-p-\{\alpha\}}}{1+ax} \frac{da}{X_n(-1/a)}.$$

Our choice (14) of p and the fact that $U(x) = x^p$ give

$$\begin{aligned} L_n[x^\alpha](x) - x^\alpha &= \frac{(-1)^{p+1}}{D_0} X_n(x) \int_0^\infty \frac{a^{-\alpha-1}}{1+ax} \frac{da}{X_n(-1/a)} \\ &= \frac{(-1)^{p+1}}{D_0} X_n(x) \int_0^\infty \frac{s^\alpha}{x+s} \frac{ds}{X_n(-s)}, \end{aligned}$$

by the substitution $s = 1/a$. Finally, $B_n(x)$ is a polynomial of degree $\leq n$ that interpolates to x^α at the $n+1$ zeros of X_n, so by uniqueness of Lagrange interpolation,

$$B_n(x) = L_n[x^\alpha](x).$$

Multiplying through by -1 gives the result. ■

Armed with this representation, we can sketch the

Proof of Theorem 1

We can use comparison theorems for Chebyshev polynomials in different Markov systems [6], to show that the zeros of B_n interlace with those of $T_{n+1}(2x-1)$ and hence that for $j \geq 2$,

$$x_{jn} \in \left[\left(\sin \left(j - \frac{3}{2} \right) \frac{\pi}{2(n+1)} \right)^2, \left(\sin \left(j - \frac{1}{2} \right) \frac{\pi}{2(n+1)} \right)^2 \right].$$

Thus for each fixed $j \geq 2$, $n^2 x_{jn}$ is bounded above and below by positive constants independent of n.

Next, let $\mathcal{S}$ be an infinite subsequence of positive integers. We can use a diagonal choice argument (as is done in the proof of the Arzela-Ascoli Theorem) to choose a further subsequence $\mathcal{T}$ such that for each $j \geq 1$, there exists

$$x_j = \lim_{n \to \infty, n \in \mathcal{T}} n^2 x_{jn}.$$

Then x_j satisfies (7). Also

$$\begin{aligned} \lim_{n \to \infty, n \in \mathcal{T}} X_n\left(z/n^2\right)/X_n(0) &= \lim_{n \to \infty, n \in \mathcal{T}} \prod_{j=1}^{n+1} \left(1 - \frac{z}{n^2 x_{jn}}\right) \\ &= \prod_{j=1}^{\infty} \left(1 - \frac{z}{x_j}\right) = F(z), \end{aligned}$$

uniformly in compact subsets of $\mathbb{C}$. Using this limit in our representation (15) for the remainder gives the relation (10), but only as $n \to \infty$ through the subsequence $\mathcal{T}$. The difficult part is to show uniqueness of the limit F, which forces the limit to be independent of $\mathcal{S}$. To do this we show uniqueness of the limit form of the remainder. This is achieved using uniqueness results for entire functions of exponential type. Finally, we obtain (9) from (10) by setting $z = 0$ and using the fact that R_n attains its maximum modulus at 0, or equivalently that 0 is an alternation point. $\blacksquare$

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