

$(C, 1)$ Means of Orthonormal Expansions for Exponential Weights

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Communicated by Vilmos Totik

Received November 2, 1998; accepted in revised form September 16, 1999

Let $s_m[f]$ denote the m th partial sum of the orthonormal expansion of $f: \mathbb{R} \rightarrow \mathbb{R}$ with respect to the orthonormal polynomials for the weight $W^2(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. We show that for some C independent of f and n ,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W \phi_n^{-2/3} \right\|_{L_\infty(\mathbb{R})} \leq C \|fW\|_{L_\infty(\mathbb{R})}$$

where

$$\phi_n(x) := \left(\left| 1 - \frac{x}{a_n} \right| + n^{-2/3} \right)$$

and a_n denotes the n th Mhaskar–Rahmanov–Saff number for $Q(x) = \frac{1}{2}|x|^\alpha$. The novelty is the presence of the factor $\phi_n^{-2/3}$, which is large close to $\pm a_n$; that factor was absent in the classic results of G. Freud. Related results are proved for more general exponential weights on $(-1, 1)$ or \mathbb{R} . © 2000 Academic Press

1. INTRODUCTION AND RESULTS

Let I denote either $(-1, 1)$ or \mathbb{R} . Let $W: I \rightarrow (0, \infty)$ be such that all the power moments

$$\int_I x^n W^2(x) dx, \quad n \geq 0,$$

are finite. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \dots, \quad \gamma_n > 0, \quad n \geq 0,$$

satisfying

$$\int_I p_n p_m W^2 = \delta_{mn}.$$

For $f: I \rightarrow \mathbb{R}$ such that $f(x) x^j W^2(x) \in L_1(I)$, $j \geq 0$, we may form the formal orthonormal expansion

$$f \leftrightarrow \sum_{j=0}^{\infty} b_j p_j,$$

where

$$b_j := b_j(f) := \int_I f p_j W^2, \quad j \geq 0. \quad (1)$$

The m th partial sum of this expansion is denoted by

$$s_m[f] := \sum_{j=0}^{m-1} b_j(f) p_j, \quad m \geq 1. \quad (2)$$

A classic result of G. Freud, proved using the still more classic de la Vallée Poussin argument, asserts that for a class of weights including the exponential weights

$$(W(x) =) W_\alpha(x) := \exp(-\frac{1}{2}|x|^\alpha), \quad \alpha > 1, \quad (3)$$

there is strong $(C, 1)$ summability of the orthonormal expansions:

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n |s_m[f]| \right) W_\alpha \right\|_{L_\infty(\mathbb{R})} \leq C \|f W_\alpha\|_{L_\infty(\mathbb{R})} \quad (4)$$

with C independent of f and n . This inequality was the basis of Freud's methods for proving weighted Jackson theorems, see [6, 7, 23]. Strictly speaking Freud considered only $\alpha \geq 2$, but later work established that his proofs could be extended to all $\alpha > 1$. For $\alpha < 1$, the polynomials are not dense in a suitable weighted space, while the boundary case $\alpha = 1$ is not fully understood as regards Jackson theorems. See [19, 23] for further orientation.

In this paper, we show that it is possible to strengthen (4) in the sense that one can insert a factor that is large near the largest zero of p_n in the

left-hand side of (4). To further elucidate this, we require the notion of the Mhaskar–Rahmanov–Saff number. We shall assume throughout that our weight has the form

$$W = e^{-Q}, \quad (5)$$

where $Q: I \rightarrow \mathbb{R}$ is even and convex. The n th Mhaskar–Rahmanov–Saff number a_n is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}}, \quad n \geq 1. \quad (6)$$

One of its properties is that

$$\|PW\|_{L_\infty(I)} = \|PW\|_{L_\infty(-a_n, a_n)}, \quad P \in \mathcal{P}_n, \quad (7)$$

where \mathcal{P}_n denotes the polynomials of degree $\leq n$. For example, for $W = W_\alpha$, it is easily seen that

$$a_n = Cn^{1/\alpha}, \quad n \geq 1,$$

where C may be expressed in terms of gamma functions (see [19, 20, 26]).

We shall show that one may insert a factor $(|1 - (|x|/a_n)| + n^{-2/3})^{-1/3}$ in the left-hand side of (4), for a class of weights including W_α , $\alpha > 1$; moreover, when we drop the absolute value in (4), that is when we consider ordinary $(C, 1)$ summability, then we may replace $-1/3$ by $-2/3$. The most general class of Freud weights that we have in mind is given in:

DEFINITION 1. Freud Weights \mathcal{F} .

Let $W = e^{-Q}$, where $Q: \mathbb{R} \rightarrow \mathbb{R}$ is even, continuous and Q'' is continuous in $(0, \infty)$. Assume moreover, that $Q' > 0$ in $(0, \infty)$, and that for some $A, B > 1$,

$$A \leq 1 + \frac{xQ''(x)}{Q'(x)} \leq B, \quad x \in (0, \infty). \quad (8)$$

Then we write $W \in \mathcal{F}$.

Note that for $W = W_\alpha$, (8) holds with $A = B = \alpha$. In addition to Freud weights on the real line, we consider a class of Erdős weights, for which the exponent Q grows faster than any polynomial:

DEFINITION 2. Erdős Weights \mathcal{E} .

Let $W = e^{-Q}$, where $Q: \mathbb{R} \rightarrow \mathbb{R}$ is even, continuous and Q'' is continuous in $(0, \infty)$. Assume that $Q' > 0$, $Q'' \geq 0$ in $(0, \infty)$, and that the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \quad x \in (0, \infty) \quad (9)$$

is increasing in $(0, \infty)$ with

$$\lim_{x \rightarrow 0+} T(x) > 1; \quad \lim_{x \rightarrow \infty} T(x) = \infty. \quad (10)$$

Assume moreover that for some $C_j > 0$, $j = 1, 2, 3$,

$$C_1 \leq T(x) \frac{Q(x)}{xQ'(x)} \leq C_2, \quad x \geq C_3.$$

Then we write $W \in \mathcal{E}$.

The archetypal example of $W \in \mathcal{E}$ is

$$W(x) = W_{k,\alpha}(x) = \exp(-\exp_k(|x|^\alpha)) \quad (11)$$

where $\alpha > 1$ and $k \geq 1$ and

$$\exp_k := \underbrace{\exp(\exp(\dots \exp(\dots)))}_{k \text{ times}}$$

denotes the k th iterated exponential. We also set

$$\exp_0(x) := x.$$

See [12, 13] for further orientation on Erdős weights.

The third class of weights we consider is a class of exponential weights on $(-1, 1)$:

DEFINITION 3. Exponential Weights on $(-1, 1)$ $\mathcal{E}\mathcal{X}\mathcal{P}$.

Let $W = e^{-Q}$, where $Q: (-1, 1) \rightarrow \mathbb{R}$ is even and Q'' is continuous in $(-1, 1)$. Assume that $Q' > 0$, $Q'' \geq 0$ in $(0, 1)$, and that the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \quad x \in (0, 1) \quad (12)$$

is increasing in $(0, 1)$ with

$$\lim_{x \rightarrow 0+} T(x) > 1. \quad (13)$$

Assume moreover that for some $C_1 > 0$, $C_2 > 0$,

$$C_1 \leq T(x) \frac{Q(x)}{Q'(x)} \leq C_2, \quad x \text{ close enough to } 1 \quad (14)$$

and that for some $A > 2$ and x close enough to 1,

$$T(x) \geq \frac{A}{1-x^2}. \quad (15)$$

Then we write $W \in \mathcal{E}\mathcal{X}\mathcal{P}$.

The archetypal example of $W \in \mathcal{E}\mathcal{X}\mathcal{P}$ is

$$W(x) = W^{k, \alpha}(x) := \exp(-\exp_k((1-x^2)^{-\alpha})), \quad x \in (-1, 1) \quad (16)$$

where $k \geq 0$, $\alpha > 0$. For further orientation on $\mathcal{E}\mathcal{X}\mathcal{P}$, see [10].

It is possible to treat the classes \mathcal{F} , \mathcal{E} , $\mathcal{E}\mathcal{X}\mathcal{P}$ in a more general and unified framework [11], but we prefer here to quote already published results. In any event, it is possible to describe simultaneously several features of the (C, 1) means of the orthonormal expansions for all three classes of weights: this requires some additional notation. We set

$$\delta_n := (nT(a_n))^{-2/3}, \quad n \geq 1, \quad (17)$$

and define the functions

$$\phi_n(x) := \left| 1 - \frac{|x|}{a_n} \right| + \delta_n \quad (18)$$

and

$$\psi_n(x) := \frac{\phi_n(x) + T(a_n)^{-1}}{\sqrt{\phi_n(x)}} = \frac{\left| 1 - \frac{|x|}{a_n} \right| + \delta_n + T(a_n)^{-1}}{\sqrt{\left| 1 - \frac{|x|}{a_n} \right| + \delta_n}}. \quad (19)$$

The function ψ_n plays a role in describing the spacing between successive zeros of p_n , the growth of Christoffel functions, and related quantities, in much the same way as does the function $1 - x^2 + n^{-2}$ for Jacobi weights and their generalizations on $(-1, 1)$. Note that for Freud weights, T is bounded above and below by positive constants, so δ_n behaves like $n^{-2/3}$.

By a minor modification of the classical de la Vallée Poussin argument for L_∞ and then via standard duality and interpolation techniques, we prove:

THEOREM 4. Let $W \in \mathcal{F}$, \mathcal{E} or $\mathcal{E}\mathcal{X}\mathcal{P}$. Let $1 \leq p < \infty$ and let

$$\Psi_n(x) := \max\{\psi_n^{1/2}(x), \psi_n^{2/3}(x)\}, \quad x \in I. \quad (20)$$

Then for some C independent of f and n ,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W \Psi_n^{1-1/p} \right\|_{L_p(I)} \leq C \|f W \Psi_n^{-1/p}\|_{L_p(I)}. \quad (21)$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n |s_m[f]| \right) W \Psi_n \right\|_{L_\infty(I)} \leq C \|f W\|_{L_\infty(I)}. \quad (22)$$

We note that if one uses the classical de la Vallée Poussin argument, one has to omit the $\psi_n^{2/3}$ in (20); our modification permits the inclusion of this factor.

In [16], strong $(C, 1)$ means of orthonormal expansions for Erdős weights were investigated; there for $p = \infty$, instead of Ψ_n in (22) there was a factor $T^{-1/4}$ in the left-hand side. Since one can show that

$$T^{-1/4} \leq C \psi_n^{1/2} \leq C \Psi_n$$

for the class \mathcal{E} , the above result constitutes an improvement of the result in [16].

To acquire some perspective on how Theorem 4 relates to Freud's (4), we specialize to Freud weights. Here $\Psi_n/\phi_n^{-1/3}$ is bounded above and below by positive constants and we obtain:

COROLLARY 5. Let $W \in \mathcal{F}$. Let $1 \leq p < \infty$. Then for some C independent of f and n ,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W \phi_n^{-(1-1/p)/3} \right\|_{L_p(I)} \leq C \|f W \phi_n^{1/(3p)}\|_{L_p(I)}. \quad (23)$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n |s_m[f]| \right) W \phi_n^{-1/3} \right\|_{L_\infty(I)} \leq C \|f W\|_{L_\infty(I)}. \quad (24)$$

Thus under the same hypotheses as Freud, one may insert the factor $\phi_n^{-1/3}$, which is large near $\pm a_n$. The obvious question is whether or not $1/3$ is sharp. If one assumes more about the orthonormal polynomials, it is not.

Recall that the orthonormal polynomials $\{p_n\}$ satisfy the three term recurrence relation

$$xp_{n-1}(x) = \alpha_n p_n(x) + \alpha_{n-1} p_{n-2}(x), \quad n \geq 1 \quad (25)$$

where we set $p_{-1} := 0$ and

$$\alpha_n := \gamma_{n-1}/\gamma_n, \quad n \geq 1. \quad (26)$$

It is known for large classes of Freud weights [14] that

$$\alpha_n = \frac{1}{2}a_n(1 + o(1)), \quad n \rightarrow \infty. \quad (27)$$

Assuming somewhat more allows us to improve on the $1/3$ in (23) and (24):

THEOREM 6. *Let $W \in \mathcal{F}$ and assume that for some $\beta > 0$,*

$$\alpha_n = \frac{1}{2}a_n(1 + O(n^{-\beta})) \quad n \rightarrow \infty. \quad (28)$$

Let

$$\kappa := \min \left\{ \frac{2}{3}, \frac{1}{3} + \frac{\beta}{2}, \frac{5}{12} + \frac{\beta}{4} \right\}. \quad (29)$$

Let $1 \leq p < \infty$. Then for some C independent of f and n ,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W \phi_n^{-(1-1/p)\kappa} \right\|_{L_p(I)} \leq C \|fW\phi_n^{\kappa/p}\|_{L_p(I)}. \quad (30)$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W \phi_n^{-\kappa} \right\|_{L_\infty(I)} \leq C \|fW\|_{L_\infty(I)}. \quad (31)$$

For W_α , $\alpha > 1$, (28) is known with $\beta = \min\{\alpha, 2\}$. This was recently proved by Kriecherbauer and McLaughlin [8], thereby improving results of Rakhmanov [25]. For α a positive even integer, more complete asymptotics are known, [3], [18]. Likewise when Q is a polynomial, more complete asymptotics are known [1, 3]. Thus we may deduce:

COROLLARY 7. *For $W = W_\alpha$, $\alpha > 1$, and $1 \leq p < \infty$,*

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W_\alpha \phi_n^{-(2/3)(1-1/p)} \right\|_{L_p(I)} \leq C \|fW_\alpha \phi_n^{2/3p}\|_{L_p(I)}. \quad (32)$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W_\alpha \phi_n^{-2/3} \right\|_{L_\infty(I)} \leq C \|f W_\alpha\|_{L_\infty(I)}. \quad (33)$$

It is an interesting problem to determine the sharp power of ϕ_n in (33).

This paper is organised as follows: in Section 2, we present the de la Vallee Poussin argument, and its minor modification, which leads to the proof of Theorem 4 and hence Corollary 5. In Section 3, we present an estimate on sums of squares of $p_{m+1} - p_{m-1}$, under the assumption (28). In Section 4, we prove Theorem 6 and deduce Corollary 7.

2. PROOF OF THEOREM 4

We begin by recalling the classic de la Vallee Poussin argument. (This has been clearly presented often [7], [23],... but we do need the details). Let $f: I \rightarrow \mathbb{R}$, $x \in I$ and $\rho_n > 0$. We let

$$f_n(t) := \begin{cases} f(t), & |t-x| \leq \rho_n \\ 0, & |t-x| > \rho_n \end{cases} \quad (34)$$

and

$$F_n(t) := \frac{f(t) - f_n(t)}{x-t} = \begin{cases} \frac{f(t)}{x-t}, & |t-x| > \rho_n \\ 0, & |t-x| \leq \rho_n. \end{cases} \quad (35)$$

Then we may split for $m \leq n$,

$$s_m[f](x) = s_m[f_n](x) + s_n[F_n(\cdot)(x - \cdot)](x). \quad (36)$$

Let

$$K_m(x, t) := \sum_{j=0}^{m-1} p_j(x) p_j(t) \quad (37)$$

so that

$$s_m[f_n](x) = \int_I K_m(x, t) f_n(t) W^2(t) dt. \quad (38)$$

The de la Vallee Poussin/Freud Estimate for $s_m[f_n](x)$.

$$|s_m[f_n](x)| \leq \|fW\|_{L_\infty(I)} \int_{I \cap [x-\rho_n, x+\rho_n]} |K_m(x, t)| W(t) dt \quad (39)$$

$$\begin{aligned} &\leq \|fW\|_{L_\infty(I)} \sqrt{2\rho_n} \sqrt{\int_I K_m^2(x, t) W^2(t) dt} \\ &= \|fW\|_{L_\infty(I)} \sqrt{2\rho_n} \sqrt{\sum_{j=0}^{m-1} p_j^2(x)}, \end{aligned} \quad (40)$$

by the Cauchy-Schwarz inequality and then orthogonality. Recall now the Christoffel function:

$$\lambda_m^{-1}(W^2, x) := \sum_{j=0}^{m-1} p_j^2(x). \quad (41)$$

Since λ_m^{-1} clearly increases with m , we deduce that (note that the λ_{n+1} simplifies later calculations)

$$\frac{1}{n} \sum_{m=1}^n |s_m[f_n](x)| W(x) \leq \|fW\|_{L_\infty(I)} \sqrt{2\rho_n} \sqrt{\lambda_{n+1}^{-1}(W^2, x) W^2(x)}. \quad (42)$$

The de la Vallee Poussin/Freud Estimate for $s_m[F_n(\cdot)(x - \cdot)](x)$. We need the Christoffel Darboux formula

$$K_m(x, t) = \alpha_m \frac{p_m(x) p_{m-1}(t) - p_{m-1}(x) p_m(t)}{x - t}. \quad (43)$$

We see then that

$$\begin{aligned} s_m[F_n(\cdot)(x - \cdot)](x) &= \int_I K_m(x, t) F_n(t)(x - t) W^2(t) dt \\ &= \alpha_m [p_m(x) b_{m-1}(F_n) - p_{m-1}(x) b_m(F_n)]. \end{aligned} \quad (44)$$

Let us abbreviate $b_m(F_n)$ as b_m . We deduce that

$$\begin{aligned} &\frac{1}{n} \sum_{m=1}^n |s_m[F_n(\cdot)(x - \cdot)](x)| \\ &\leq \frac{1}{n} \left(\max_{1 \leq m \leq n} \alpha_m \right) \sum_{m=1}^n (|p_m(x)| |b_{m-1}| + |p_{m-1}(x)| |b_m|) \\ &\leq \left(\max_{1 \leq m \leq n} \alpha_m \right) \frac{2}{n} \sqrt{\sum_{m=0}^n p_m^2(x)} \sqrt{\sum_{m=0}^n b_m^2} \end{aligned}$$

by the Cauchy–Schwarz inequality. Using Bessel’s inequality for orthonormal expansions, we continue this as

$$\begin{aligned}
 &\leq \left(\max_{1 \leq m \leq n} \alpha_m \right) \frac{2}{n} \sqrt{\lambda_{n+1}^{-1}(W^2, x)} \sqrt{\int_I F_n^2 W^2} \\
 &\leq \left(\max_{1 \leq m \leq n} \alpha_m \right) \frac{2}{n} \sqrt{\lambda_{n+1}^{-1}(W^2, x)} \|fW\|_{L_\infty(I)} \sqrt{\int_{|t-x| \geq \rho_n} \frac{dt}{(t-x)^2}} \\
 &= \left(\max_{1 \leq m \leq n} \alpha_m \right) \frac{2\sqrt{2}}{n} \sqrt{\lambda_{n+1}^{-1}(W^2, x) \rho_n^{-1}} \|fW\|_{L_\infty(I)}. \tag{45}
 \end{aligned}$$

The de la Vallee Poussin estimate for the strong $(C, 1)$ means of $s_m[f]$. Combining (36), (42) and (45) gives

$$\begin{aligned}
 \frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) &\leq \|fW\|_{L_\infty(I)} \sqrt{\lambda_{n+1}^{-1}(W^2, x) W^2(x)} \\
 &\quad \times \left(\sqrt{2\rho_n} + \left(\max_{1 \leq m \leq n} \alpha_m \right) 2 \sqrt{\frac{2}{n^2 \rho_n}} \right). \tag{46}
 \end{aligned}$$

Choosing

$$\rho_n := \frac{\max_{1 \leq m \leq n} \alpha_m}{n}$$

gives

$$\begin{aligned}
 \frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \\
 \leq 5 \|fW\|_{L_\infty(I)} \sqrt{\lambda_{n+1}^{-1}(W^2, x) W^2(x)} \sqrt{\frac{\max_{1 \leq m \leq n} \alpha_m}{n}}. \tag{47}
 \end{aligned}$$

We turn to a minor modification of the de la Vallee Poussin/Freud estimate before proving Theorem 4:

A simple alternative estimate for $s_m[f_n](x)$. Now for $|t-x| \leq \rho_n$, and $m \leq n$, the Cauchy–Schwarz inequality gives

$$\begin{aligned}
 |K_m(x, t)| &\leq \sqrt{K_m(x, x)} \sqrt{K_m(t, t)} \\
 &\leq \sqrt{K_{n+1}(x, x)} \sqrt{K_{n+1}(t, t)} \\
 &= \sqrt{\lambda_{n+1}^{-1}(W^2, x)} \sqrt{\lambda_{n+1}^{-1}(W^2, t)}.
 \end{aligned}$$

Then from (39),

$$|s_m[f_n](x)| W(x) \leq \|fW\|_{L_\infty(I)} 2\rho_n \sqrt{\lambda_{n+1}^{-1}(W^2, x) W^2(x)} \\ \times \sqrt{\max_{|t-x| \leq \rho_n} \lambda_{n+1}^{-1}(W^2, t) W^2(t)}. \quad (48)$$

Then instead of (46), we obtain

$$\frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \\ \leq \|fW\|_{L_\infty(I)} \sqrt{\lambda_{n+1}^{-1}(W^2, x) W^2(x)} \\ \times \left\{ 2\rho_n \sqrt{\max_{|t-x| \leq \rho_n} \lambda_{n+1}^{-1}(W^2, t) W^2(t)} + \left(\max_{1 \leq m \leq n} \alpha_m \right) 2 \sqrt{\frac{2}{n^2 \rho_n}} \right\}.$$

Choosing

$$\rho_n := \left(\frac{\max_{1 \leq m \leq n} \alpha_m}{n} \sqrt{\lambda_{n+1}(W^2, x)/W^2(x)} \right)^{2/3} \quad (49)$$

gives

$$\frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \\ \leq 3 \|fW\|_{L_\infty(I)} \left(\frac{\max_{1 \leq m \leq n} \alpha_m}{n} \lambda_{n+1}^{-1}(W^2, x) W^2(x) \right)^{2/3} \\ \times \left[\max_{|t-x| \leq \rho_n} \left(\frac{\lambda_{n+1}^{-1}(W^2, t) W^2(t)}{\lambda_{n+1}^{-1}(W^2, x) W^2(x)} \right)^{1/2} + 1 \right]. \quad (50)$$

Thus far, we have the estimates (47) and (50) for the strong (C, 1) means. Before we can choose which to apply, we need technical estimates for λ_{n+1}^{-1} , for α_m and so on. We use the standard notation \sim for sequences of real numbers: we write

$$c_n \sim d_n$$

if there exists positive constant C_1, C_2 independent of n such that for the relevant range of n ,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions. Moreover, in the sequel, C, C_1, C_2, \dots denote positive constants independent of n, x, f . The same symbol does not necessarily denote the same constant in different occurrences.

LEMMA 8. *Let $W \in \mathcal{F}, \mathcal{E}$ or $\mathcal{E}\mathcal{X}\mathcal{P}$. Then*

(a)

$$\max_{1 \leq m \leq n} \alpha_m \sim \alpha_n \sim a_n. \quad (51)$$

(b) *Let $\eta, L > 0$. There exists n_0 such that uniformly for $n \geq n_0$ and for $|x| \leq a_n(1 + L\delta_n)$,*

$$\lambda_n(W^2, x) \sim \frac{a_n}{n} W^2(x) \psi_n(x). \quad (52)$$

(c) *Let $\eta > 0$. There exists n_0 such that uniformly for $n \geq n_0$ and for $|x| \leq a_n(1 + L\delta_n)$,*

$$|t - x| \leq \eta \frac{a_n}{n} \psi_n(x) \Rightarrow \psi_n(t) \sim \psi_n(x) \quad \text{and} \quad \phi_n(t) \sim \phi_n(x). \quad (53)$$

The constants in \sim are independent of n, x, t .

(d) *There exists n_0 such that for $n \geq n_0$*

$$\frac{1}{2} \leq \frac{m}{n} \leq 2 \Rightarrow \left| 1 - \frac{a_m}{a_n} \right| \sim \frac{1}{T(a_n)} \left| 1 - \frac{m}{n} \right|. \quad (54)$$

Moreover,

$$T(a_n) \sim T(a_{2n}); \quad \delta_n \sim \delta_{2n}; \quad \delta_n^{-1/2} = o(n). \quad (55)$$

(e) *Let $L > 0$. There exists n_0 such that uniformly for $n \geq n_0$ and for $|x| \leq a_n(1 + L\delta_n)$,*

$$\psi_n(x) \sim \psi_{n+1}(x); \quad \phi_n(x) \sim \phi_{n+1}(x). \quad (56)$$

(f) *Let $L > 0, 0 < p \leq \infty$. There exist C and n_0 such that for $n \geq n_0$ and for $P \in \mathcal{P}_n$,*

$$\|PW\|_{L_p(I)} \leq C \|PW\|_{L_p(-a_n(1-L\delta_n), a_n(1-L\delta_n))}. \quad (57)$$

Moreover, if $r > 1$, there exist $C_1, C_2 > 0$ such that for $n \geq 1$ and for $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(I \setminus [a_{-m}, a_m])} \leq C_1 \exp(-C_2 n T(a_n)^{-1/2}) \|PW\|_{L_p(I)}. \quad (58)$$

Proof. (a) We note that since a_m increases with m , it suffices to show that

$$\alpha_m \sim a_m, \quad m \geq 1.$$

For $W \in \mathcal{F}$, this is Theorem 12.3(b) in [9, p. 529]; for $W \in \mathcal{E}$, this is (10.33) in [12, p. 285]; for $W \in \mathcal{E}\mathcal{X}\mathcal{P}$, this follows from a far more general result of Rakhmanov [24] that for $W > 0$ a.e. in $[-1, 1]$, $\alpha_m \rightarrow \frac{1}{2}$, $m \rightarrow \infty$.

(b) For $W \in \mathcal{F}$, Theorem 1.1 in [9, p. 465] states that

$$\lambda_n(W^2, x)/W^2(x) \sim \frac{a_n}{n} \phi_n^{-1/2}(x) \sim \frac{a_n}{n} \left(\left| 1 - \frac{|x|}{a_n} \right| + n^{-2/3} \right)^{-1/2}$$

for the relevant range of n and x . Note that for $W \in \mathcal{F}$, $A \leq T \leq B$, where A, B are as in (8), so

$$\psi_n(x) = \frac{\phi_n(x) + T(a_n)^{-1}}{\sqrt{\phi_n(x)}} \sim \frac{1}{\sqrt{\phi_n(x)}}.$$

Thus we have (52) in this case. Next, if $W \in \mathcal{E}$, Theorem 1.2 in [12, p. 204] implies that

$$\lambda_n(W^2, x)/W^2(x) \sim \frac{a_n}{n} \max\{\sqrt{\phi_n(x)}, [T(a_n) \sqrt{\phi_n(x)}]^{-1}\} \quad (59)$$

for the relevant range of n and x . This is easily recast in the form (52). Finally, if $W \in \mathcal{E}\mathcal{X}\mathcal{P}$, Theorem 1.2 in [10, p. 7] again implies (59) and hence (52).

(c) In view of the form of ψ_n , it clearly suffices to show that $\phi_n(t) \sim \phi_n(x)$ for the relevant range n, t, x . Let us denote the zeros of $p_n(x) = p_n(W^2, x)$ by

$$x_{nm} < x_{n-1, n} < \cdots < x_{2n} < x_{1n}.$$

It is known for all three classes of weights that uniformly in n and j ,

$$\phi_n(x_{jn}) \sim \phi_n(x_{j-1, n}) \text{ and hence } \psi_n(x_{jn}) \sim \psi_n(x_{j-1, n}). \quad (60)$$

For $W \in \mathcal{F}$, this is (11.10) in [9, p. 521]; for $W \in \mathcal{E}$, this is (9.9) in [12, p. 265]; and for $W \in \mathcal{E}\mathcal{X}\mathcal{P}$, this is (10.12), in [10, p. 111]. Next, for all three classes of weights it is known that uniformly in n and j ;

$$\begin{aligned} x_{j-1, n} - x_{j+1, n} &\sim \lambda_{jn} W^{-2}(x_{jn}) \\ &\sim \frac{a_n}{n} \psi_n(x_{jn}); \quad \left| 1 - \frac{x_{1n}}{a_n} \right| \leq C\delta_n. \end{aligned} \quad (61)$$

For $W \in \mathcal{F}$, this follows from (b) above and Corollary 1.2 in [9, pp. 466–467]; for $W \in \mathcal{E}$, this follows from Corollary 1.3 in [12, p. 205]; and for $W \in \mathcal{E}\mathcal{X}\mathcal{P}$, this follows from Corollary 1.4 in [10, p. 9]. The monotonicity of ϕ_n in $[0, a_n]$ or $[-a_n, 0]$ and (60) and (61) then give the result.

(d) For $W \in \mathcal{F}$, these follow from Lemma 5.2(c) in [9, p. 478] (recall that $T \sim 1$ and $\delta_n \sim n^{-2/3}$ for this case); for $W \in \mathcal{E}$, these follow from Lemma 2.2 in [12, pp. 208–209]; and for $W \in \mathcal{E}\mathcal{X}\mathcal{P}$, these follow from Lemma 3.2 in [10, p. 24–25].

(e) This follows easily from (d), which shows that

$$\left| 1 - \frac{a_n}{a_{n+1}} \right| \sim \frac{1}{nT(a_n)} = o(\delta_n), \quad n \rightarrow \infty.$$

(f) For $W \in \mathcal{F}$, (57) is Theorem 1.8 in [9, p. 469] while (58) follows easily from (7.14) in [9, p. 486] and (10.2) in [9, p. 512]; for $W \in \mathcal{E}$, (57) is Theorem 1.5 in [12, p. 206] while (58) follows from (4.18) in [12, p. 228] and (5.2) in [12, p. 231]; and for $W \in \mathcal{E}\mathcal{X}\mathcal{P}$, (57) is Theorem 1.7 in [10, p. 12] while (58) follows from (5.18) in [10, p. 53] and (6.2) in [10, p. 57]. ■

We proceed to:

Proof of Theorem 4 for $p = \infty$. Let us substitute the estimates of the last lemma in (47): we obtain for $|x| \leq a_n$,

$$\frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \leq C \|fW\|_{L_\infty(I)} \psi_n^{-1/2}(x). \quad (62)$$

Next, provided we choose ρ_n by (49), so that by Lemma 8(a), (b), (e),

$$\rho_n \sim \frac{a_n}{n} \psi_n(x)^{1/3} \quad (63)$$

we have also from (50) and Lemma 8(a), (b), (e),

$$\begin{aligned} & \frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \\ & \leq C \|fW\|_{L_\infty(I)} \psi_n^{-2/3}(x) \left[\max_{|t-x| \leq \rho_n} \left(\frac{\psi_n(t)}{\psi_n(x)} \right)^{-1/2} + 1 \right]. \end{aligned}$$

Now if

$$\psi_n(x) \geq \frac{1}{2},$$

then (63) shows that

$$\rho_n \leq C \frac{a_n}{n} \psi_n(x)$$

and then from Lemma 8(c),

$$\max_{|t-x| \leq \rho_n} \left(\frac{\psi_n(t)}{\psi_n(x)} \right)^{-1/2} \leq C.$$

Thus

$$\psi_n(x) \geq \frac{1}{2} \Rightarrow \frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \leq C \|fW\|_{L_\infty(I)} \psi_n^{-2/3}(x).$$

This and (62) show that

$$\frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \max\{\psi_n^{1/2}, \psi_n^{2/3}\}(x) \leq C \|fW\|_{L_\infty(I)}.$$

When $\psi_n(x) < \frac{1}{2}$, (62) shows that this inequality persists as then $\psi_n^{2/3}(x) < \psi_n^{1/2}(x)$. Thus

$$\max_{|x| \leq a_n} \frac{1}{n} \sum_{m=1}^n |s_m[f](x)| W(x) \max\{\psi_n^{1/2}, \psi_n^{2/3}\}(x) \leq C \|fW\|_{L_\infty(I)}. \quad (64)$$

To extend this to the rest of I , we use infinite-finite range inequalities in the following way: let us suppose that there are polynomials R_n with the following properties:

- (i) R_n has degree $O(\delta_n^{-1/2})$;
- (ii) $R_n \sim \Psi_n = \max\{\psi_n^{1/2}, \psi_n^{2/3}\}$ in $[-a_n, a_n]$;
- (iii) $R_n \geq C\Psi_n$ in $I \setminus [-a_n, a_n]$.

We now use a device of J. Szabados [27] to apply the infinite-finite range inequalities: for any $\varepsilon_m = \pm 1$, (64) gives

$$\max_{|x| \leq a_n} \left| \frac{1}{n} \sum_{m=1}^n \varepsilon_m s_m[f](x) R_n(x) \right| W(x) \leq C \|fW\|_{L_\infty(I)}.$$

The expression is the $||$ is a polynomial of degree at most $[n + C\delta_n^{-1/2}]$ for some C (here $[x]$ denotes the integer part of x). But by (54) and then the third relation in (55),

$$\left| 1 - \frac{a_n}{a_{[n + C\delta_n^{-1/2}]}} \right| \sim \frac{\delta_n^{-1/2}}{nT(a_n)} = \delta_n \sim \delta_{[n + C\delta_n^{-1/2}]}$$

so for some $L > 0$, if n is large enough,

$$\begin{aligned} & \max_{|x| \leq a_{[n+C\delta_n^{-1/2}]}(1-L\delta_{[n+C\delta_n^{-1/2}]})} \left| \frac{1}{n} \sum_{m=1}^n \varepsilon_m s_m[f](x) R_n(x) \right| W(x) \\ & \leq C \|fW\|_{L_\infty(I)}. \end{aligned}$$

The infinite-finite range inequality (57) shows that as the choice $\varepsilon_m = \pm 1$ is arbitrary,

$$\max_{x \in I} \frac{1}{n} \sum_{m=1}^n |s_m[f](x)| R_n(x) W(x) \leq C \|fW\|_{L_\infty(I)}.$$

Finally, since $\Psi_n = O(R_n)$ in I , we obtain (22). It remains to give

The Construction of the $\{R_n\}$ satisfying (i), (ii), (iii) above. Now $\Psi_n \sim \psi_n^{2/3} + \psi_n^{1/2}$, and $\psi_n = \sqrt{\phi_n} + 1/(T(a_n) \sqrt{\phi_n})$ so it suffices to show the following: given $b \in \mathbb{R}$, there exist polynomials R_n^* such that

- (i*) R_n^* has degree $O(\delta_n^{-1/2})$;
- (ii*) $R_n^* \sim \phi_n^b$ in $[-a_n, a_n]$;
- (iii*) $R_n^* \geq C\phi_n^b$ in $I \setminus [-a_n, a_n]$.

We need only do this for $|b| < \frac{1}{2}$ (raising to suitable powers gives the general case). We use the Christoffel functions for the ultraspherical weight

$$u(x) := (1-x^2)^{-b-(1/2)}, \quad x \in (-1, 1).$$

Let us set

$$\begin{aligned} m & := m(n) := [\delta_n^{-1/2}]; \\ R_n^\#(x) & := m^{-1} \lambda_m^{-1}(u, x). \end{aligned}$$

It is well known that uniformly in m , x [21, p. 120]

$$R_n^\#(x) \sim (|1-|x|| + m^{-2})^b \quad \text{in } [-1, 1]. \quad (65)$$

Then it is easily seen that

$$R_n^*(x) := R_n^\# \left(\frac{x}{a_n} \right)$$

satisfies (i*), (ii*). To verify (iii*), it suffices to show that

$$R_n^\#(x) \geq C(x-1 + m^{-2})^b, \quad x \in (1, \infty).$$

(Recall that $R_n^\#$ is even). Let ℓ denote the least integer $\geq b/2$. Let $p_j^\#$ denote the j th orthonormal polynomial for u , so that its zeros lie in $(-1, 1)$, and for some integer j_0 and $C_1 > 0$, $p_j^\#$ has at least ℓ zeros in $[1 - (C_1 j)^{-2}, 1]$ for $j \geq j_0$. See, for example, [21, Thm. 22, p. 167]. Then for $j \geq j_0$ and $x > 1$,

$$\frac{p_j^\#(x)}{p_j^\#(1)} = \prod_{y: p_j^\#(y)=0} \left(1 + \frac{x-1}{1-y}\right) \geq (1 + (C_1 j)^2 (x-1))^\ell.$$

Let $\eta \in (0, 1)$. Then for $x > 1$, $m \geq j_0/\eta$,

$$\begin{aligned} \lambda_m^{-1}(u, x) &> \sum_{j=[\eta m]+1}^{m-1} (p_j^\#(x))^2 \\ &\geq (1 + (C_1 \eta m)^2 (x-1))^b \sum_{j=[\eta m]+1}^{m-1} (p_j^\#(1))^2. \end{aligned}$$

It follows easily from the fact that $|b| < \frac{1}{2}$ and from the estimate

$$k^{-1} \lambda_k^{-1}(u, 1) \sim k^{-2b}, \quad k \geq 1,$$

that if η is small enough,

$$\sum_{j=[\eta m]+1}^{m-1} (p_j^\#(1))^2 \sim \sum_{j=0}^{m-1} (p_j^\#(1))^2 = \lambda_m^{-1}(u, 1) \sim m^{1-2b}$$

and hence that for $x > 1$,

$$\begin{aligned} R_n^\#(x) &= m^{-1} \lambda_m^{-1}(u, x) \\ &\geq (1 + (c_1 \eta m)^2 (x-1))^b m^{-2b} \geq C(x-1 + m^{-2})^b, \end{aligned}$$

as desired. \blacksquare

The extension from $p = \infty$ to $1 \leq p < \infty$ is entirely standard [6], but we provide the details:

The proof of Theorem 4 for $p = 1$. Now

$$\begin{aligned} \left\| \frac{1}{n} \sum_{m=1}^n s_m[f] W \right\|_{L_1(I)} &= \sup_{\|gW\|_{L_\infty(I)} \leq 1} \int_I \left(\frac{1}{n} \sum_{m=1}^n s_m[f] W \right) gW \\ &= \sup_{\|gW\|_{L_\infty(I)} \leq 1} \frac{1}{n} \sum_{m=1}^n \int_I s_m[f] gW^2 \\ &= \sup_{\|gW\|_{L_\infty(I)} \leq 1} \frac{1}{n} \sum_{m=1}^n \int_I f s_m[g] W^2 \end{aligned}$$

by self-adjointness of s_m (this follows easily from orthogonality). We continue this as

$$\begin{aligned} &\leq \sup_{\|gW\|_{L_\infty(I)} \leq 1} \frac{1}{n} \sum_{m=1}^n \int_I |fW\Psi_n^{-1}| |s_m[g] W\Psi_n| \\ &\leq \sup_{\|gW\|_{L_\infty(I)} \leq 1} \int_I |fW\Psi_n^{-1}| \left\| \frac{1}{n} \sum_{m=1}^n |s_m[g] W\Psi_n| \right\|_{L_\infty(I)} \\ &\leq C \int_I |fW\Psi_n^{-1}| \end{aligned}$$

by our result for $p = \infty$. ■

Finally, we use weighted interpolation to treat the case $1 < p < \infty$:

Proof of Theorem 4 for $1 < p < \infty$. One applies a theorem of E. M. Stein [2, p. 213] on interpolation in weighted spaces. More specifically, if

$$\tau[f] := \frac{1}{n} \sum_{m=1}^n s_m[f]$$

and

$$\begin{array}{llll} q_0 := 1; & p_0 := 1; & v_0 := W; & u_0 := W\Psi_n^{-1}; \\ q_1 := \infty; & p_1 := \infty; & v_1 := W\Psi_n; & u_1 := W, \end{array}$$

we have shown that for $i = 0, 1$ and some C independent of f, n, i

$$\|\tau[f] v_i\|_{L_{q_i}(I)} \leq C \|fu_i\|_{L_{p_i}(I)}$$

and hence if $0 < \theta < 1$ and

$$\begin{aligned} \frac{1}{p} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = 1 - \theta; & \frac{1}{q} &= \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = 1 - \theta; \\ u &:= u_0^{1-\theta} u_1^\theta; & v &:= v_0^{1-\theta} v_1^\theta \end{aligned}$$

then

$$\|\tau[f] v\|_{L_q(I)} \leq C \|fu\|_{L_p(I)}.$$

This is easily reformulated as (21). ■

Deduction of Corollary 5. Suppose first that $1 \leq p < \infty$. Recall that for Freud weights $T \sim 1$ so in $[-a_n, a_n]$,

$$\begin{aligned}\psi_n &= \frac{\phi_n + T(a_n)^{-1}}{\sqrt{\phi_n}} \sim \frac{1}{\sqrt{\phi_n}} \geq C \\ \Rightarrow \Psi_n &= \max\{\psi_n^{1/2}, \psi_n^{2/3}\} \sim \psi_n^{2/3} \sim \phi_n^{-1/3}.\end{aligned}$$

Moreover, $\Psi_n \geq C\phi_n^{-1/3}$ in $I \setminus [-a_n, a_n]$. Then

$$\begin{aligned}\left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W\phi_n^{-(1-1/p)/3} \right\|_{L_p(I)} \\ \leq C \left\| \left(\frac{1}{n} \sum_{m=1}^n s_m[f] \right) W\Psi_n^{1-1/p} \right\|_{L_p(I)} \\ \leq C \|fW\Psi_n^{-1/p}\|_{L_p(I)} \leq C \|fW\phi_n^{1/(3p)}\|_{L_p(I)}.\end{aligned}$$

Here we have used (21). The case $p = \infty$ is easier. \blacksquare

3. ESTIMATE OF AN ORTHONORMAL POLYNOMIAL SUM

In this section, we prove:

THEOREM 9. *Let $W \in \mathcal{F}$ and assume that for some $\beta > 0$,*

$$\frac{\alpha_n}{a_n} = \frac{1}{2} + O(n^{-\beta}).$$

Then for $n \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned}\sum_{m=1}^n (p_{m+1} - p_{m-1})^2(x) W^2(x) \\ \leq C \frac{n}{a_n} \phi_n(x)^{\min\{3/2, 1/2 + 3\beta/2, 3/4(1+\beta)\}}.\end{aligned} \tag{66}$$

We begin with a simple consequence of the recurrence relation:

LEMMA 10.

$$\begin{aligned}
 & \sum_{m=1}^n \alpha_m (p_m - p_{m-1})^2(x) \\
 &= \sum_{m=1}^{n-1} p_m^2(x) (\alpha_m + \alpha_{m+1} - x) + p_0^2(x) (\alpha_1 - x) \\
 & \quad + \alpha_n p_n(x) (p_n - p_{n-1})(x).
 \end{aligned} \tag{67}$$

Proof. Recall the recurrence relation

$$x p_{m-1}(x) = \alpha_m p_m(x) + \alpha_{m-1} p_{m-2}(x).$$

Multiplying this by $p_{m-1}(x)$ and adding for $m = 1, 2, \dots, n$ gives

$$x \sum_{m=1}^n p_{m-1}^2(x) = \sum_{m=1}^n \alpha_m p_m(x) p_{m-1}(x) + \sum_{m=1}^n \alpha_{m-1} p_{m-2}(x) p_{m-1}(x).$$

Changing the index of summation from m to $m-1$ in the sum on the left and the second sum on the right gives

$$x \sum_{m=0}^{n-1} p_m^2(x) = 2 \sum_{m=1}^n \alpha_m p_m(x) p_{m-1}(x) - \alpha_n p_n(x) p_{n-1}(x),$$

recall $p_{-1} = 0$. Then

$$\begin{aligned}
 & \sum_{m=1}^n \alpha_m (p_m - p_{m-1})^2(x) \\
 &= \sum_{m=1}^n \alpha_m p_m^2(x) + \sum_{m=1}^n \alpha_m p_{m-1}^2(x) - 2 \sum_{m=1}^n \alpha_m p_m(x) p_{m-1}(x) \\
 &= \sum_{m=1}^{n-1} (\alpha_m + \alpha_{m+1}) p_m^2(x) + \alpha_n p_n^2(x) \\
 & \quad + \alpha_1 p_0^2(x) - x \sum_{m=0}^{n-1} p_m^2(x) - \alpha_n p_n(x) p_{n-1}(x).
 \end{aligned}$$

Then (67) follows. ■

Surprisingly the most troublesome term on the right-hand side of (67) is the third term. This is handled in the following lemma: there and in the sequel, we assume that $W \in \mathcal{F}$, that (28) holds, and we shall use the estimates [9, Cor. 1.4, p. 467]

$$|p_n W|(x) \leq C a_n^{-1/2} \phi_n^{-1/4}(x), \quad x \in \mathbb{R}, \quad n \geq 1 \tag{68}$$

and [9, Lemma 5.2(a), p. 478]

$$\frac{a_{2n}}{a_n} \geq C > 1, \quad n \geq 1. \quad (69)$$

LEMMA 11. For $x \in [0, a_n]$,

$$|p_n - p_{n-1}|(x) W(x) \leq C n^{\max\{0, (1-\beta)/2\}} a_n^{-1/2} \phi_n^{1/4}(x). \quad (70)$$

Proof. We consider two ranges of x :

(i) $x \in [0, \frac{1}{2}a_n]$

Here $\phi_n(x) \sim 1$ and the desired estimate follows from (68).

(ii) $x \in (\frac{1}{2}a_n, a_n]$

We use the Dombrowski–Fricke identity [4, 5, 22] in the form

$$\begin{aligned} \Gamma_n(x) &:= \frac{1}{\alpha_n^2} \sum_{k=0}^{n-1} (\alpha_{k+1}^2 - \alpha_k^2) p_k^2(x) \\ &= (p_n - p_{n-1})^2(x) + 2p_{n-1}(x) p_n(x) \left(1 - \frac{x}{2\alpha_n}\right). \end{aligned}$$

This gives

$$\begin{aligned} &\Gamma_n(x) W^2(x) \\ &= ((p_n - p_{n-1}) W)^2(x) + 2(p_{n-1} p_n W^2)(x) \left(\left[1 - \frac{x}{a_n}\right] + O(n^{-\beta}) \right) \\ &= ((p_n - p_{n-1}) W)^2(x) + O(a_n^{-1} \phi_n(x)^{1/2}) \\ &\quad + O(a_n^{-1} \phi_n(x)^{-1/2} n^{-\beta}). \end{aligned} \quad (71)$$

Here we have used (68), (56) and our hypothesis (28). Next, that hypothesis gives for $0 \leq k \leq n-1$,

$$\begin{aligned} \alpha_{k+1}^2 - \alpha_k^2 &= \alpha_{k+1}^2 \left(1 - \left[\frac{a_k}{a_{k+1}}\right]^2 \left[\frac{\alpha_k/a_k}{\alpha_{k+1}/a_{k+1}}\right]^2\right) \\ &= \alpha_{k+1}^2 \left(1 - \left[1 + O\left(\frac{1}{k+1}\right)\right]^2 [1 + O((k+1)^{-\beta})]^2\right) \\ &\leq C a_n^2 (k+1)^{-\min\{1, \beta\}}. \end{aligned}$$

Here we have used not only (28) but also (54) (recall $T \sim 1$ for $W \in \mathcal{F}$). Then from (51), we obtain

$$\Gamma_n(x) \leq C \sum_{k=0}^{n-1} (k+1)^{-\min\{1, \beta\}} p_k^2(x).$$

Recall that $x > \frac{1}{2}a_n$. Now from (69), there exists ε_1 independent of n , such that for large enough n ,

$$\frac{1}{2}a_n \geq a_{2\varepsilon_1 n}.$$

We then use (58) of Lemma 8 applied to W^2 rather than W to deduce that

$$\begin{aligned} W^2(x) & \sum_{k=0}^{[\varepsilon_1 n]-1} (k+1)^{-\min\{1, \beta\}} p_k^2(x) \\ & \leq W^2(x) \sum_{k=0}^{[\varepsilon_1 n]-1} p_k^2(x) \\ & = W^2(x) \lambda_{[\varepsilon_1 n]}^{-1}(W^2, x) \\ & \leq C_1 \exp(-C_2 n) \sup_{t \in \mathbb{R}} W^2(t) \lambda_{[\varepsilon_1 n]}^{-1}(W^2, t) \\ & \leq C_3 \exp(-C_4 n). \end{aligned}$$

Next,

$$\begin{aligned} W^2(x) \sum_{k=[\varepsilon_1 n]}^{n-1} (k+1)^{-\min\{1, \beta\}} p_k^2(x) & \leq Cn^{-\min\{1, \beta\}} W^2(x) \lambda_n^{-1}(W^2, x) \\ & \leq Cn^{-\min\{1, \beta\}} \frac{n}{a_n} \phi_n(x)^{1/2}, \end{aligned}$$

recall that $\psi_n \sim \phi_n^{-1/2}$ for Freud weights. Thus, the last two estimates yield

$$\Gamma_n(x) W^2(x) \leq Cn^{-\min\{1, \beta\}} \frac{n}{a_n} \phi_n(x)^{1/2} \quad (72)$$

and hence from (71),

$$\begin{aligned} & ((p_n - p_{n-1}) W)^2(x) \\ & \leq C \left[n^{-\min\{1, \beta\}} \frac{n}{a_n} \phi_n(x)^{1/2} + a_n^{-1} \phi_n(x)^{-1/2} n^{-\beta} \right]. \end{aligned} \quad (73)$$

Now by definition of ϕ_n ,

$$n^{-1} \leq C\phi_n^{3/2} \quad (74)$$

and it then follows that the first term in the right-hand side of (73) is the larger one (apart from a constant), so we obtain (70). \blacksquare

For future use, we record the estimate effectively proved in the above lemma: for $\Delta > 0$,

$$W^2(x) \sum_{k=0}^{n-1} (k+1)^{-\Delta} p_k^2(x) \leq Cn^{1-\Delta} a_n^{-1} \phi_n(x)^{1/2} \quad x \in [\frac{1}{2}a_n, a_n]. \quad (75)$$

The next step in the proof of Theorem 9 is:

LEMMA 12. For $x \in [0, a_n]$,

$$\begin{aligned} & \sum_{m=1}^{n-1} \alpha_m (p_m - p_{m-1})^2(x) W^2(x) \\ & \leq n \phi_n(x)^{\min\{3/2, 1/2 + (3/2)\beta, 3/4(\beta+1)\}}. \end{aligned} \quad (76)$$

Proof. For $x \in [0, \frac{1}{2}a_n]$, the estimate follows easily from (51), (52) since $\phi_n(x) \sim 1$. We now assume that $x \in [\frac{1}{2}a_n, a_n]$. We use (28). Now for $m \leq n-1$,

$$\alpha_m + \alpha_{m+1} - x \leq \frac{a_m}{2} + \frac{a_{m+1}}{2} - x + O\left(\frac{a_{m+1}}{m^\beta}\right) \leq a_n - x + O\left(\frac{a_n}{m^\beta}\right),$$

so

$$\begin{aligned} & W^2(x) \sum_{m=1}^{n-1} p_m^2(x) (\alpha_m + \alpha_{m+1} - x) \\ & \leq (a_n - x) W^2(x) \lambda_n^{-1}(W^2, x) + Ca_n W^2(x) \sum_{m=1}^{n-1} \frac{p_m^2(x)}{m^\beta} \\ & \leq Cn \phi_n(x)^{3/2} + Cn^{1-\beta} \phi_n(x)^{1/2} \end{aligned}$$

by (52), (75). Using (74), we continue this as

$$\begin{aligned} & W^2(x) \sum_{m=1}^{n-1} p_m^2(x) (\alpha_m + \alpha_{m+1} - x) \\ & \leq Cn \phi_n(x)^{\min\{3/2, 1/2 + (3/2)\beta\}}. \end{aligned} \quad (77)$$

Next, combining (68) and (70) gives

$$\begin{aligned} & W^2(x) \alpha_n |p_n(x)(p_n - p_{n-1}(x))| \\ & \leq Cn^{\max\{0, (1-\beta)/2\}} \leq Cn \cdot n^{-\min\{1, (\beta+1)/2\}} \\ & \leq Cn \phi_n(x)^{(3/2) \min\{1, (\beta+1)/2\}}. \end{aligned}$$

Finally,

$$W^2(x) p_0^2(x) |\alpha_1 - x| \leq C \leq Cn\phi_n(x)^{3/2}.$$

Combining the last three estimates and (67) gives the result. \blacksquare

We turn to

The Proof of Theorem 9. Firstly for $x \in [0, \frac{1}{4}a_n]$,

$$\sum_{m=1}^n (p_{m+1} - p_{m-1})^2(x) W^2(x) \leq 4\lambda_{n+2}^{-1}(W^2, x) W^2(x) \leq C \frac{n}{a_n}$$

and then (66) follows as $\phi_n(x) \sim 1$. We now assume that $x \in [\frac{1}{4}a_n, a_n]$. Let $\varepsilon_0 \in (0, \frac{1}{2})$. Since

$$(p_{m+1} - p_{m-1})^2 \leq 2(p_{m+1} - p_m)^2 + 2(p_m - p_{m-1})^2,$$

we obtain from (51) and then Lemma 12 and (56), that

$$\begin{aligned} & \sum_{m=\lceil \varepsilon_0 n \rceil}^n (p_{m+1} - p_{m-1})^2(x) W^2(x) \\ & \leq \frac{C}{a_n} \left[\sum_{m=\lceil \varepsilon_0 n \rceil}^n \alpha_{m+1} (p_{m+1} - p_m)^2(x) W^2(x) \right. \\ & \quad \left. + \sum_{m=\lceil \varepsilon_0 n \rceil}^n \alpha_m (p_m - p_{m-1})^2(x) W^2(x) \right] \\ & \leq C \frac{n}{a_n} \phi_n(x)^{\min\{3/2, 1/2 + (3/2)\beta, 3/4(\beta+1)\}}. \end{aligned}$$

If we choose ε_0 small enough, then it follows as in the proof of (72) of Lemma 11 that the contribution of the terms with $m < \lceil \varepsilon_0 n \rceil$ is negligible. Thus we have the desired estimate (66) for $x \in [0, a_n]$ and hence for all $x \in [-a_n, a_n]$, recall that $(p_{m+1} - p_{m-1})^2$ is even. To extend the estimate to the whole real line, one uses the same trick as in the proof of Theorem 4 for $p = \infty$: one approximates powers of ϕ_n by polynomials R_n of degree $O(\delta_n^{-1/2}) = O(n^{1/3})$, and then uses infinite-finite range inequalities. \blacksquare

We shall actually apply not Theorem 9, but a simple consequence thereof:

COROLLARY 13. Let $W \in \mathcal{F}$ and assume that for some $\beta > 0$,

$$\frac{\alpha_n}{a_n} = \frac{1}{2} + O(n^{-\beta}).$$

Then for $n \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned} & \sum_{m=1}^n \left(p_{m+1} - \frac{\alpha_m}{\alpha_{m+1}} p_{m-1} \right)^2(x) W^2(x) \\ & \leq C \frac{n}{a_n} \phi_n(x)^{\min\{3/2, 1/2 + 3\beta/2, 3/4(1+\beta)\}}. \end{aligned} \quad (78)$$

Proof. We have

$$\begin{aligned} \left(p_{m+1} - \frac{\alpha_m}{\alpha_{m+1}} p_{m-1} \right)^2 & \leq 2(p_{m+1} - p_{m-1})^2 + 2 \left(1 - \frac{\alpha_m}{\alpha_{m+1}} \right)^2 p_{m-1}^2 \\ & \leq 2(p_{m+1} - p_{m-1})^2 + C m^{-2 \min\{1, \beta\}} p_{m-1}^2. \end{aligned}$$

Here for $x \in [\frac{1}{2}a_n, a_n]$, (75) gives

$$\begin{aligned} W^2(x) \sum_{m=1}^n m^{-2 \min\{1, \beta\}} p_{m-1}^2(x) & \leq C n^{1-2 \min\{1, \beta\}} a_n^{-1} \phi_n(x)^{1/2} \\ & \leq C \frac{n}{a_n} \phi_n(x)^{3 \min\{1, \beta\} + 1/2} \end{aligned}$$

and the rest of the details follow as before. ■

4. PROOF OF THEOREM 6

Throughout, we assume the hypotheses of Theorem 6. We shall also assume that the sequence ρ_n decays to 0 no faster than some negative power of n . The proof is based on:

An alternative estimate for $s_m[F_n(\cdot)(x - \cdot)](x)$. The alternative estimate involves a simple change of indices in summation, that has been employed several times before (for example in [17]); we do not know who

first used it. Recall the notation (1), (44) and the abbreviation $b_m = b_m(F_n)$. Then

$$\begin{aligned}
 & \frac{1}{n} \sum_{m=1}^n s_m [F_n(\cdot)(x - \cdot)](x) \\
 &= \frac{1}{n} \left[\sum_{m=1}^n \alpha_m p_m b_{m-1} - \sum_{m=1}^n \alpha_m p_{m-1} b_m \right] \\
 &= \frac{1}{n} \left[\sum_{m=0}^{n-1} \alpha_{m+1} p_{m+1} b_m - \sum_{m=1}^n \alpha_m p_{m-1} b_m \right] \\
 &= \frac{1}{n} \left[\sum_{m=1}^{n-1} b_m (\alpha_{m+1} p_{m+1} - \alpha_m p_{m-1}) + \alpha_1 p_1 b_0 - \alpha_n p_{n-1} b_n \right] \\
 &:= T^{(1)} + T^{(2)} + T^{(3)}. \tag{79}
 \end{aligned}$$

Estimation of $T^{(1)}$. Here

$$\begin{aligned}
 |T^{(1)}| &= \frac{1}{n} \left| \sum_{m=1}^{n-1} b_m (\alpha_{m+1} p_{m+1} - \alpha_m p_{m-1}) \right| \\
 &\leq \frac{1}{n} \sqrt{\sum_{m=1}^{n-1} b_m^2} \sqrt{\sum_{m=1}^{n-1} (\alpha_{m+1} p_{m+1} - \alpha_m p_{m-1})^2} \\
 &\leq C \|fW\|_{L_\infty(\mathbb{R})} \sqrt{\frac{a_n^2}{n^2 \rho_n}} \sqrt{\sum_{m=1}^n \left(p_{m+1} - \frac{\alpha_m}{\alpha_{m+1}} p_{m-1} \right)^2},
 \end{aligned}$$

exactly as in the de la Vallée Poussin estimate for $s_m[F_n(\cdot)(x - \cdot)](x)$ (see (44–45)). Using Corollary 13, we continue this as

$$\begin{aligned}
 |T^{(1)}W|(x) &\leq C \|fW\|_{L_\infty(\mathbb{R})} \sqrt{\frac{a_n}{n\rho_n}} \sqrt{\phi_n(x)^{\min\{3/2, 1/2 + 3\beta/2, 3/4(1 + \beta)\}}}, \\
 &x \in \mathbb{R}. \tag{80}
 \end{aligned}$$

Estimation of $T^{(2)}$. Next,

$$|b_0| = \left| \int_I F_n p_0 W^2 \right| \leq \|fW\|_{L_\infty(\mathbb{R})} \gamma_0 \int_{|t-x| \geq \rho_n} \frac{W(t)}{|t-x|} dt.$$

We consider separately two ranges of x :

(I) x such that $a_n \phi_n(x) \geq 1$.

Then we estimate

$$\begin{aligned} |b_0| &\leq \|fW\|_{L_\infty(\mathbb{R})} \gamma_0 \left(W(0) \int_{\rho_n \leq |t-x| < a_n \phi_n(x)} \frac{dt}{|t-x|} \right. \\ &\quad \left. + \frac{1}{a_n \phi_n(x)} \int_{|t-x| \geq a_n \phi_n(x)} W(t) dt \right) \\ &\leq C \|fW\|_{L_\infty(\mathbb{R})} \left(\log^+ \left(\frac{a_n \phi_n(x)}{\rho_n} \right) + 1 \right). \end{aligned}$$

Here we set

$$\log^+ t := \max\{0, \log t\}.$$

(II) x such that $a_n \phi_n(x) < 1$

Then

$$\left| 1 - \frac{x}{a_n} \right| < \frac{1}{a_n} < \frac{1}{2}$$

for large enough n , so that $|t-x| \leq 1 \Rightarrow W(t)$ is geometrically small:

$$|t-x| \leq 1 \Rightarrow W(t) \leq \exp(-C_1 n).$$

(See [9, Lemma 5.1(c), p. 477]). Then we estimate

$$\begin{aligned} |b_0| &\leq \|fW\|_{L_\infty(\mathbb{R})} \gamma_0 \left(\exp(-C_1 n) \log^+ \left(\frac{1}{\rho_n} \right) + \int_{|t-x| \geq 1} W(t) dt \right) \\ &\leq C \|fW\|_{L_\infty(\mathbb{R})} \left(\log^+ \left(\frac{a_n \phi_n(x)}{\rho_n} \right) + 1 \right). \end{aligned}$$

(Recall our hypothesis $\rho_n \geq n^{-C}$). Thus we have this estimate in all cases and hence

$$\begin{aligned} |T^{(2)}W|(x) &= \frac{1}{n} |\alpha_1 p_1(x) b_0| W(x) \\ &\leq \frac{C}{n} \|fW\|_{L_\infty(\mathbb{R})} \left(\log^+ \left(\frac{a_n \phi_n(x)}{\rho_n} \right) + 1 \right). \end{aligned} \quad (81)$$

Estimation of $T^{(3)}$. It is more difficult to estimate b_n :

$$\begin{aligned}
 |b_n| &= \left| \int_I F_n p_n W^2 \right| \\
 &\leq C \|fW\|_{L_\infty(\mathbb{R})} \left(a_n^{-1/2} \int_{\rho_n \leq |t-x| \leq (1/4)a_n} \frac{\phi_n(t)^{-1/4}}{|t-x|} dt \right. \\
 &\quad \left. + a_n^{-1} \int_{|t-x| > (1/4)a_n} |p_n W|(t) dt \right) \\
 &\leq C \|fW\|_{L_\infty(\mathbb{R})} \left(a_n^{-1/2} \int_{\rho_n \leq |t-x| \leq (1/4)a_n} \frac{\phi_n(t)^{-1/4}}{|t-x|} dt + a_n^{-1/2} \right). \quad (82)
 \end{aligned}$$

Here we have used an estimate for the L_1 norm of $p_n W$ from [15, Thm. 1, p.44]. In subsequent estimation, we consider $x \geq 0$, and consider two subcases:

(I) $x \in [0, 1/4 a_n]$

Here $|t-x| \leq \frac{1}{4} a_n \Rightarrow |t| \leq \frac{1}{2} a_n$, so that $\phi_n(t) \sim 1$ and we obtain

$$|b_n| < C \|fW\|_{L_\infty(\mathbb{R})} a_n^{-1/2} \left[\log^+ \left(\frac{a_n}{4\rho_n} \right) + 1 \right]. \quad (83)$$

(II) $x \in [\frac{1}{4} a_n, a_n(1 - n^{-2/3})]$

Here $|t-x| \leq \frac{1}{4} a_n \Rightarrow t \geq 0$ and $\phi_n(t) \sim 1 - (t/a_n)$ so that

$$\begin{aligned}
 &\int_{\rho_n \leq |t-x| \leq (1/4)a_n} \frac{\phi_n(t)^{-1/4}}{|t-x|} dt \\
 &\sim \int_{\rho_n \leq |t-x| < (1/4)a_n} \frac{\left| 1 - \frac{t}{a_n} \right|^{-1/4}}{a_n \left| \left(1 - \frac{t}{a_n} \right) - \left(1 - \frac{x}{a_n} \right) \right|} dt \\
 &= \left(1 - \frac{x}{a_n} \right)^{-1/4} \int_{(\rho_n/a_n(1 - (x/a_n))) \leq |s-1| \leq (1/4)(1 - (x/a_n))} \frac{|s|^{-1/4}}{|1-s|} ds \\
 &\leq C \left(1 - \frac{x}{a_n} \right)^{-1/4} \left[\log^+ \frac{a_n \left(1 - \frac{x}{\rho_n} \right)}{\rho_n} + 1 \right],
 \end{aligned}$$

by first the substitution $1 - (t/a_n) = s(1 - (x/a_n))$ and then some straightforward estimation. Together with our estimates (82), (83), this shows that for all $x \in [0, a_n(1 - n^{-2/3})]$,

$$|b_n| \leq C \|fW\|_{L_\infty(\mathbb{R})} a_n^{-1/2} \phi_n(x)^{-1/4} \left[\log + \frac{a_n \phi_n(x)}{\rho_n} + 1 \right].$$

Then for $|x| \leq a_n(1 - n^{-2/3})$,

$$\begin{aligned} |T^{(3)}W|(x) &= \frac{1}{n} |\alpha_n p_{n-1}(x) W(x) b_n| \\ &\leq \frac{C}{n} \|fW\|_{L_\infty(\mathbb{R})} \phi_n(x)^{-1/2} \left[\log + \frac{a_n \phi_n(x)}{\rho_n} + 1 \right]. \end{aligned}$$

We obtain from (79)–(81) and this last estimate that for $|x| \leq a_n(1 - n^{-2/3})$,

$$\begin{aligned} &\left| \frac{1}{n} \sum_{m=1}^n s_m [F_n(\cdot)(x - \cdot)](x) \right| W(x) \\ &\leq C \|fW\|_{L_\infty(\mathbb{R})} \left\{ \sqrt{\frac{a_n}{n\rho_n}} \phi_n(x)^{\min\{3/4, 1/4 + 3\beta/4, 3/8(1 + \beta)\}} \right. \\ &\quad \left. + n^{-1} \phi_n^{-1/2}(x) \left[\log + \frac{a_n \phi_n(x)}{\rho_n} + 1 \right] \right\}. \end{aligned} \quad (84)$$

We turn to

The Proof of Theorem 6. Combining (36), (48), (52), (84) gives for $|x| \leq a_n(1 - n^{-2/3})$,

$$\begin{aligned} \Gamma &:= \frac{1}{n} \left| \sum_{m=1}^n s_m [f](x) \right| W(x) \\ &\leq C \|fW\|_{L_\infty(\mathbb{R})} \left\{ \begin{aligned} &\sqrt{\frac{a_n}{n\rho_n}} \phi_n(x)^{\min\{3/4, 1/4 + 3\beta/4, 3/8(1 + \beta)\}} + n^{-1} \phi_n^{-1/2}(x) \\ &\quad \times \log + \left[\frac{a_n \phi_n(x)}{\rho_n} + 1 \right] \\ &\quad + \frac{n\rho_n}{a_n} \phi_n^{1/4}(x) \max_{|t-x| \leq \rho_n} \phi_n^{1/4}(t) \end{aligned} \right\}. \end{aligned} \quad (85)$$

(Recall that for Freud weights, $\psi_n \sim \phi_n^{-1/2}$). Now fix x such that $|x| \leq a_n(1 - n^{-2/3})$, fix $\Delta \in [0, \frac{1}{2}]$ and set

$$\rho_n := \frac{a_n}{n} \phi_n^\Delta(x) \leq C \frac{a_n}{n} \phi_n^{-1/2}(x) \leq C \frac{a_n}{n} \psi_n(x). \quad (86)$$

Then (53) shows that for $|t - x| \leq \rho_n$, $\phi_n(t) \sim \phi_n(x)$. So (85) becomes

$$\Gamma \leq C \|fW\|_{L_\infty(\mathbb{R})} \times \left\{ \begin{aligned} &\phi_n(x)^{-\Delta/2 + \min\{3/4, 1/4 + 3\beta/4, 3/8(1 + \beta)\}} \\ &+ n^{-1} \phi_n^{-1/2}(x) [\log^+ [n\phi_n^{1-\Delta}(x)] + 1] + \phi_n^{\Delta+1/2}(x) \end{aligned} \right\}. \quad (87)$$

The ratio of the second and third terms in the last right-hand side is

$$\begin{aligned} &n^{-1} \phi_n^{-1-\Delta}(x) [\log^+ [n\phi_n^{1-\Delta}(x)] + 1] \\ &\leq C \frac{\log n}{n} \phi_n^{-1-\Delta}(x) \leq C \frac{\log n}{n} (n^{-2/3})^{-1-\Delta} = o(1) \end{aligned}$$

as $\Delta < \frac{1}{2}$. It follows that the second term in the right-hand side of (87) is bounded by a constant times the third. Finally, we deduce for $|x| \leq a_n(1 - n^{-2/3})$,

$$\Gamma \leq C \|fW\|_{L_\infty(\mathbb{R})} \phi_n(x)^{\min\{-\Delta/2 + \min\{3/4, 1/4 + 3\beta/4, 3/8(1 + \beta)\}, \Delta + 1/2\}}$$

Choosing

$$\Delta := \frac{2}{3} \min \left\{ \frac{3}{4}, \frac{1}{4} + \frac{3\beta}{4}, \frac{3}{8}(1 + \beta) \right\} - \frac{1}{3} \in \left[0, \frac{1}{6} \right]$$

gives for $|x| \leq a_n(1 - n^{-2/3})$,

$$\begin{aligned} &\frac{1}{n} \left| \sum_{m=1}^n s_m[f](x) \right| \\ &\leq C \|fW\|_{L_\infty(\mathbb{R})} \phi_n(x)^{2/3 \min\{3/4, 1/4 + 3\beta/4, 3/8(1 + \beta)\} + 1/6}. \end{aligned}$$

We extend this estimate to the whole real line exactly as in the proof of Theorem 4. Then we obtain (31) for $p = \infty$. The extension to $p \in [1, \infty)$ follows as in the proof of Theorem 4 for that range of p . ■

Proof of Corollary 7. As we have noted, Kriecherbauer and McLaughlin proved that (28) holds for $W = W_\alpha$ with $\beta = \min\{\alpha, 2\}$. Then $\kappa = 2/3$ in (29) and the result follows. ■

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