AVERAGES OF RATIOS OF CHRISTOFFEL FUNCTIONS FOR COMPACTLY SUPPORTED MEASURES

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ABSTRACT. Let μ be a compactly supported positive measure on the real line, with associated Christoffel functions $\lambda_n(d\mu, \cdot)$. Let g be a measurable function that is bounded above and below by positive constants on $\text{supp}[\mu]$. We show that if g is continuous on some compact set J, then for a.e. $x \in J$, we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \left| \frac{\lambda_n(g\ d\mu, x)}{\lambda_n(d\mu, x)} - g(x) \right| = 0.$$

This is proved using an analogous limit for means of Nevai operators. The main idea is a new averaged maximal function estimate for the "tail" in Nevai's operators. Similar results are established without continuity of g, but with a local assumption on μ. An essential feature of all results is that there are no global restrictions, such as regularity of μ.

Orthogonal Polynomials on the real line, Christoffel functions, ratio asymptotics, Nevai's operators. 42C05

1. INTRODUCTION[1]

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, ...$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

In analysis and applications of orthogonal polynomials, the reproducing kernel

$$K_n(d\mu, x, y) = \sum_{j=0}^{n-1} p_j(x) p_j(y)$$

plays a key role. For $y = x$, K_n becomes the reciprocal of the Christoffel function

$$\lambda_n(d\mu, x) = \frac{1}{K_n(d\mu, x, x)}.$$

There is the classic extremum property

$$\lambda_n(d\mu, x) = \inf_{\deg(P)\le n-1} \frac{\int P^2 d\mu}{P^2(x)}.$$

We shall omit μ, and just write $K_n(x, y)$ and $\lambda_n(x)$, when no confusion can arise.

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Ratio asymptotics for orthogonal polynomials associated with two different measures are a major topic in orthogonal polynomials. They were studied extensively by Maté, Nevai, and Totik [7], [8] as part of a program to extend Szegő's theory. Many others have taken up this topic - for example, Lopez [4] and Simon [12]. One of the essential limits within this topic is

$$\lim_{n\to\infty}\frac{\lambda_n\left(g\ d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}=g\left(x\right), \tag{1.1}$$

for appropriate functions g, and in an appropriate sense. For example, if μ is supported on $[-1,1]$, and $\mu'>0$ a.e. on $[-1,1]$, while $g^{\pm1}$ is bounded on $\mathrm{supp}[\mu]$, and g is continuous at x, then (1.1) holds at x. This follows from results of Nevai [6], [9].

Paul Nevai [9] introduced the operators

$$G_n\left[f\right]\left(x\right)=\int\frac{K_n^2\left(d\mu,x,t\right)}{K_n\left(d\mu,x,x\right)}f\left(t\right)d\mu\left(t\right),$$

as a means to establish (1.1). They are now called the *Nevai operators*, and have been studied for their own intrinsic interest, and have been widely generalized [1], [2], [3], [10]. They have turned out to be useful for orthogonal polynomials on the unit circle, and for questions in approximation theory. In most results to date, restrictions have had to be placed on the measure.

In a recent paper [5], we proved that $\lambda_n\left(g\ d\mu,x\right)/\lambda_n\left(d\mu,x\right)$ converges in measure to g in $\{\mu'>0\}=\{x:\mu'(x)>0\}$. More precisely, if meas denotes linear Lebesgue measure, we proved:

Theorem A
Let μ be a compactly supported measure on the real line with infinitely many points in its support. Let $g:\mathbb{R}\to(0,\infty)$ be a $d\mu$ measurable function such that $g^{\pm1}$ are bounded on $\mathrm{supp}[\mu]$. *Then*

$$\frac{\lambda_n\left(g\ d\mu,\cdot\right)}{\lambda_n\left(d\mu,\cdot\right)}\to g \text{ in measure in } \{\mu'>0\}. \tag{1.2}$$

Moreover, for every $p>0$,

$$\lim_{n\to\infty}\int_{\{\mu'>0\}}\left|\frac{\lambda_n\left(g\ d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}-g\left(x\right)\right|^p dx=0. \tag{1.3}$$

The novelty was the lack of restrictions on μ, especially the lack of a global condition. While convergence in measure implies that subsequences converge a.e., it does not imply anything about a.e. convergence of the full sequence. In this paper, we prove that when g is continuous in a compact set J, then averages converge a.e., again without any local or global assumptions on the measure:

Theorem 1.1
Let μ be a compactly supported measure on the real line with infinitely many points in its support. Let $g:\mathbb{R}\to(0,\infty)$ be a $d\mu$ measurable function such that $g^{\pm1}$ are bounded on $\mathrm{supp}[\mu]$. *Let J be a compact subset of* $\mathrm{supp}[\mu]$ *such that g is continuous at each point of J. Then for a.e. $x\in J$, we have for all $p>0$,*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\left|\frac{\lambda_n\left(g\ d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}-g\left(x\right)\right|^p=0. \tag{1.4}$$

We shall also prove a result assuming less on g but locally more on μ. Recall that a Lebesgue point of a function g is a point x such that

$$\lim_{h\to 0+}\frac{1}{2h}\int_{x-h}^{x+h}|g(t)-g(x)|\,dt=0.$$

Theorem 1.2
Let μ be a compactly supported measure on the real line with infinitely many points in its support. Let $g:\mathbb{R}\to(0,\infty)$ be a $d\mu$ measurable function such that $g^{\pm 1}$ are bounded on $\text{supp}[\mu]$. *Let I be an open interval in which μ is absolutely continuous, and for some $C>1$,*

$$C^{-1}\le \mu'\le C \text{ a.e. in } I. \tag{1.5}$$

Then at each Lebesgue point $x\in I$ of g, and in particular for a.e. $x\in I$, we have (1.4) for all $p>0$.

Theorem 1.1 and 1.2 follow from convergence results for averages of Nevai operators. As mentioned above, their convergence has been studied by many authors. One of the very first results, due to Nevai [9, Thm. 2, p. 74], remains the most relevant. We quote a special case, Nevai considered the more general Nevai-Blumenthal class of measures:

Theorem B
Assume that μ is a measure supported on $[-1,1]$ with $\mu'>0$ a.e. there. Let $f:[-1,1]\to\mathbb{R}$ be $d\mu$ measurable and bounded on $[-1,1]$. Let f be continuous at a given $x\in(-1,1)$. Then

$$\lim_{n\to\infty}G_n[f](x)=f(x). \tag{1.6}$$

In [5], we proved that $G_n[f]$ converges to f in measure in $\{\mu'>0\}$ under conditions similar to those in Theorem 1.1. In this paper, we prove

Theorem 1.3
Let μ be a compactly supported measure on the real line with infinitely many points in its support. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded $d\mu$ measurable function. Let J be a compact subset of $\text{supp}[\mu]$ *such that f is continuous at each point of J. Then for a.e. $x\in J$, we have for all $p>0$,*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}|G_n[f](x)-f(x)|^p=0. \tag{1.7}$$

We can assume less on f but more on μ, as in Theorem 1.2:

Theorem 1.4
Let μ be a compactly supported measure on the real line with infinitely many points in its support. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded $d\mu$ measurable function. Let I be an interval in which μ is absolutely continuous, and for some $C>1$, (1.5) holds. Then at each Lebesgue point $x\in I$ of f, and in particular for a.e. $x\in I$, we have (1.7) for all $p>0$.

Our main idea is an estimate for averages of "tail" functions. Let

$$\Phi_n(x,r) = \frac{\int_{|t-x|\geq r/n} K_n(x,t)^2\, d\mu(t)}{K_n(x,x)}. \tag{1.8}$$

Also, let

$$A_n(x) = p_{n-1}^2(x) + p_n^2(x) \tag{1.9}$$

and define the maximal function

$$\mathcal{M}[d\nu](x) = \sup_{h>0} \frac{1}{2h}\int_{x-h}^{x+h} d\nu \tag{1.10}$$

for positive measures ν on the real line. Our new estimate is that for a.e. $x \in \{\mu' > 0\}$,

$$\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \leq \frac{8\Gamma}{\sqrt{r}}\left(\frac{m\lambda_m(x)}{\mu'(x)}\right)^{1/2} \mathcal{M}[K_{2m}d\mu](x), \tag{1.11}$$

where

$$\Gamma = \sup_n \frac{\gamma_{n-1}}{\gamma_n}. \tag{1.12}$$

Also, K_{2m} is evaluated "along the diagonal", that is, we use $K_{2m}(t,t)$ inside the maximal function. The same convention is used in the sequel.

In [5], we proved a similar estimate for the single term $\Phi_n(x,r)$, but there the right-hand side involved $\mathcal{M}[A_n d\mu](x)$. Even when μ is locally nice, it is impossible to estimate $\mathcal{M}[A_n d\mu](x)$ pointwise, but $\mathcal{M}[K_{2m}d\mu](x)$ can be estimated pointwise.

In the sequel, $C, C_1, C_2, ...,$ denote positive constants independent of $n, x, t,$ and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. I^0 denotes the interior of an interval I. We prove the theorems in Sections 2 and 3.

2. Proof of Theorems 1.1 and 1.3

Recall our notation (1.8) to (1.12).

Theorem 2.1
Let μ be a measure on the real line with infinitely many points in its support. Let $r > 0$ and $m \geq 1$. Then

$$\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \leq \frac{8\Gamma}{r^{1/2}}\left(\frac{K_{2m}(x,x)}{K_m(x,x)}\right)^{1/2} \left(m\mathcal{M}[K_{2m}d\mu](x)\right)^{1/2}. \tag{2.1}$$

Moreover, at each Lebesgue point x of μ' with $\mu'(x) > 0$, and in particular for a.e. $x \in \{\mu' > 0\}$,

$$\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \leq \frac{8\Gamma}{r^{1/2}}\left(\frac{m\lambda_m(x)}{\mu'(x)}\right)^{1/2} \mathcal{M}[K_{2m}d\mu](x). \tag{2.2}$$

Proof

Observe that

$$\begin{aligned} |K_n(x,t)| &= \frac{\gamma_{n-1}}{\gamma_n}\left|\frac{p_n(x)\,p_{n-1}(t)-p_{n-1}(x)\,p_n(t)}{x-t}\right| \\ &\le \frac{\gamma_{n-1}}{\gamma_n}\frac{A_n(x)^{1/2}A_n^{1/2}(t)}{|x-t|}, \end{aligned}$$

by Cauchy-Schwarz. Then for $m \le n \le 2m-1$,

$$\begin{aligned} \Phi_n(x,r) &\le \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \int_{|t-x|\ge \frac{r}{n}} \frac{A_n(t)}{(t-x)^2} d\mu(t) \\ &\le \Gamma^2 \frac{A_n(x)}{K_m(x,x)} \int_{|t-x|\ge \frac{r}{2m}} \frac{A_n(t)}{(t-x)^2} d\mu(t). \end{aligned}$$

Using Cauchy-Schwarz, we obtain

$$\begin{aligned} &\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \\ \le\ & \frac{\Gamma}{K_m(x,x)^{1/2}} \sum_{n=m}^{2m-1} A_n(x)^{1/2}\left(\int_{|t-x|\ge \frac{r}{2m}} \frac{A_n(t)}{(t-x)^2} d\mu(t)\right)^{1/2} \\ \le\ & \frac{\Gamma}{K_m(x,x)^{1/2}} \left(\sum_{n=m}^{2m-1} A_n(x)\right)^{1/2} \left(\sum_{n=m}^{2m-1}\int_{|t-x|\ge \frac{r}{2m}} \frac{A_n(t)}{(t-x)^2} d\mu(t)\right)^{1/2} \\ \le\ & \frac{2\Gamma}{K_m(x,x)^{1/2}} \left(K_{2m}(x,x)\right)^{1/2} \left(\int_{|t-x|\ge \frac{r}{2m}} \frac{K_{2m}(t,t)}{(t-x)^2} d\mu(t)\right)^{1/2}. \qquad (2.3) \end{aligned}$$

Here, using the definition of the maximal function, we see that

$$\int_{|t-x|\le 2^{j+1}\frac{r}{m}} K_{2m}(t,t)\,d\mu(t) \le 2^{j+2}\frac{r}{m}\mathcal{M}\left[K_{2m}d\mu\right](x),$$

so

$$\begin{aligned} &\int_{|t-x|\ge \frac{r}{2m}} \frac{K_{2m}(t,t)}{(t-x)^2} d\mu(t) \\ \le\ & \sum_{j=-1}^{\infty} \int_{2^j\frac{r}{m}\le |t-x|\le 2^{j+1}\frac{r}{m}} \frac{K_{2m}(t,t)}{(2^j r/m)^2} d\mu(t) \\ \le\ & \frac{m}{r}\sum_{j=-1}^{\infty} \frac{4}{2^j}\mathcal{M}\left[K_{2m}d\mu\right](x) \\ =\ & \frac{16m}{r}\mathcal{M}\left[K_{2m}d\mu\right](x). \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \\ \le\ & \frac{8\Gamma}{r^{1/2}}\left(\frac{K_{2m}(x,x)}{K_m(x,x)}\right)^{1/2} \left(m\mathcal{M}\left[K_{2m}d\mu\right](x)\right)^{1/2}. \end{aligned}$$

Thus we have (2.1). Now every Lebesgue point x of μ' is also a Lebesgue point of $K_{2m}\mu'$, so at every such x,

$$\begin{aligned}\mathcal{M}\left[K_{2m}d\mu\right](x) &\geq \lim_{h\to 0+}\frac{1}{2h}\int_{x-h}^{x+h}K_{2m}(t,t)\,\mu'(t)\,dt \\ &= K_{2m}(x,x)\,\mu'(x).\end{aligned}$$

Thus, also at such x,

$$\begin{aligned}&\sum_{n=m}^{2m-1}\Phi_n(x,r)^{1/2} \\ \leq\ & \frac{8\Gamma}{r^{1/2}}\left(\frac{m}{K_m(x,x)\,\mu'(x)}\right)^{1/2}\mathcal{M}\left[K_{2m}d\mu\right](x) \\ =\ & \frac{8\Gamma}{r^{1/2}}\left(\frac{m\lambda_m(x)}{\mu'(x)}\right)^{1/2}\mathcal{M}\left[K_{2m}d\mu\right](x).\end{aligned}$$

■

In the sequel, given x, and $\delta > 0$, we use the local modulus of continuity,

$$\omega_x(f;\delta) = \sup\left\{|f(t) - f(x)| : |t-x| \leq \delta\right\}.$$

We also let

$$\|f\|_\infty = \sup\left\{|f(x)| : x \in \mathbb{R}\right\}.$$

Proof of Theorem 1.3
Let $m \geq 1$. Now if $m \leq n \leq 2m-1$,

$$\begin{aligned}&|G_n[f] - f|(x) \\ \leq\ & \left(\int_{|t-x|\leq\frac{r}{n}} + \int_{|t-x|>\frac{r}{n}}\right)|f(t) - f(x)|\,\frac{K_n^2(x,t)}{K_n(x,x)}d\mu(t) \\ \leq\ & \omega_x\left(f;\frac{r}{m}\right) + 2\|f\|_\infty\,\Phi_n(x,r).\end{aligned}$$

Then using the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$, we obtain

$$\begin{aligned}&\frac{1}{m}\sum_{n=m}^{2m-1}|G_n[f] - f|(x)^{1/2} \\ \leq\ & \omega_x\left(f;\frac{r}{m}\right)^{1/2} + \left(2\|f\|_\infty\right)^{1/2}\frac{1}{m}\sum_{n=m}^{2m-1}\Phi_n(x,r)^{1/2} \\ \leq\ & \omega_x\left(f;\frac{r}{m}\right)^{1/2} + \frac{8\sqrt{2}}{\sqrt{r}}\Gamma\,\|f\|_\infty^{1/2}\left(\frac{m\lambda_m(x)}{\mu'(x)}\right)^{1/2}\frac{1}{m}\mathcal{M}\left[K_{2m}d\mu\right](x).\end{aligned} \tag{2.4}$$

Choosing $r = \frac{m}{\log m}$, we obtain at every point x of continuity of f, that is also a Lebesgue point of μ',

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{n=m}^{2m-1} |G_n[f](x) - f(x)|^{1/2}$$

$$\text{(2.5)} \quad \leq \quad 8\sqrt{2}\Gamma \|f\|_\infty^{1/2} \limsup_{m\to\infty} \left(\frac{\log m}{m}\right)^{1/2} \left(\frac{m\lambda_m(x)}{\mu'(x)}\right)^{1/2} \frac{1}{m}\mathcal{M}[K_{2m}d\mu](x).$$

By the classical weak (1,1) inequality for maximal functions [11, p.137], for $\lambda > 0$,

$$\begin{aligned} & meas\left\{x \in \mathbb{R} : \frac{1}{m}\mathcal{M}[K_{2m}d\mu](x) \geq \lambda\right\} \\ \leq \quad & \frac{3}{\lambda}\int \frac{1}{m}K_{2m}d\mu = \frac{6}{\lambda}. \end{aligned}$$

Moreover, for a.e. $x \in \{\mu' > 0\}$, we have

$$\limsup_{m\to\infty} \frac{m\lambda_m(x)}{\mu'(x)}^{1/2} \leq \frac{1}{\omega(x)} < \infty,$$

where ω is the density of the equilibrium measure for an interval containing the support of μ, in the sense of potential theory [14], see also [13, p. 309, Theorem 1.7]. Now choose $m = 2^k$ and let

$$\mathcal{E}_k = \left\{x \in \{\mu' > 0\} : \frac{1}{2^k}\mathcal{M}[K_{2^{k+1}}d\mu](x) > k^2\right\},$$

so that

$$meas(\mathcal{E}_k) \leq 6k^{-2}.$$

Let

$$\mathcal{E} = \limsup_{k\to\infty} \mathcal{E}_k = \bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} \mathcal{E}_k,$$

so that $meas(\mathcal{E}) = 0$. Then for a.e. $x \in J \cap \{\mu' > 0\} \backslash \mathcal{E}$, we have

$$\begin{aligned} & \limsup_{k\to\infty} \frac{1}{2^k} \sum_{n=2^k}^{2^{k+1}-1} |G_n[f](x) - f(x)|^{1/2} \\ \leq \quad & 8\sqrt{2}\Gamma \|f\|_\infty^{1/2} \limsup_{k\to\infty} \left(\frac{\log 2^k}{2^k}\right)^{1/2} \left(\frac{2^k\lambda_{2^k}(x)}{\mu'(x)}\right)^{1/2} k^2 = 0. \end{aligned}$$

Finally, if $2^\ell \leq m < 2^{\ell+1}$, we see that

$$\begin{aligned} & \frac{1}{m}\sum_{n=1}^{m} |G_n[f](x) - f(x)|^{1/2} \\ \leq \quad & \frac{1}{2^\ell}\sum_{k=0}^{\ell}\sum_{n=2^k}^{2^{k+1}} |G_n[f](x) - f(x)|^{1/2} \\ = \quad & \frac{1}{2^\ell}\sum_{k=0}^{\ell} o\left(2^k\right) = o(1), \end{aligned}$$

with a slight abuse of notation. This gives the result for $p = \frac{1}{2}$. As $|G_n[f] - f| \leq 2\|f\|_\infty$, (1.7) follows immediately for any $p > \frac{1}{2}$. For $p < \frac{1}{2}$, one may use Hölder's inequality with appropriate parameters. ■

Remark
If f satisfies a Lipschitz condition of order $\beta > 0$, we can say more. It would follow easily from (2.4), with the choice $r = m^{\frac{\beta}{\beta+1}}$ that for a.e. $x \in J$, as $m \to \infty$,

$$\frac{1}{m}\sum_{n=m}^{2m-1} \left|G_n[f] - f\right|(x)^{1/2} \leq Cm^{-\frac{\beta}{2(\beta+1)}}.$$

Proof of Theorem 1.1
We use a special case of an inequality of Nevai [9, p. 76, Theorem 3]: by the extremal property of Christoffel functions,

$$\begin{aligned}\lambda_n(g\ d\mu, x) &\leq \int \left(\frac{K_n(x,t)}{K_n(x,x)}\right)^2 g(t)\,d\mu(t) \\ &= \lambda_n(d\mu, x)\,G_n[g](x).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\lambda_n(g\ d\mu, x)}{\lambda_n(d\mu, x)} - g(x) \\ \leq\ & G_n[g](x) - g(x) \\ \leq\ & \left|G_n[g](x) - g(x)\right|. \qquad (2.6)\end{aligned}$$

Conversely, let G_n^* denote Nevai's operator for the measure $g\ d\mu$. The above inequality applied to G_n^* and the function g^{-1} gives

$$\begin{aligned}&\frac{\lambda_n\left(g^{-1}\ g\ d\mu, x\right)}{\lambda_n(g\ d\mu, x)} - g^{-1}(x) \\ \leq\ & \left|G_n^*\left[g^{-1}\right](x) - g^{-1}(x)\right|.\end{aligned}$$

Multiplying by $\frac{\lambda_n(g\ d\mu,x)}{\lambda_n(d\mu,x)} g(x)$ gives

$$\begin{aligned}&g(x) - \frac{\lambda_n(g\ d\mu, x)}{\lambda_n(d\mu, x)} \\ \leq\ & \frac{\lambda_n(g\ d\mu, x)}{\lambda_n(d\mu, x)} g(x) \left|G_n^*\left[g^{-1}\right](x) - g^{-1}(x)\right| \\ \leq\ & \|g\|_\infty^2 \left|G_n^*\left[g^{-1}\right](x) - g^{-1}(x)\right|.\end{aligned}$$

Combined with (2.6), this gives

$$\begin{aligned}&\left|\frac{\lambda_n(g\ d\mu, x)}{\lambda_n(d\mu, x)} - g(x)\right| \\ \leq\ & \left|G_n[g](x) - g(x)\right| + \|g\|_\infty^2 \left|G_n^*\left[g^{-1}\right](x) - g^{-1}(x)\right|. \qquad (2.7)\end{aligned}$$

Averaging this over $n = 1, 2, ..., m$, and using Theorem 1.3 on the measures μ and $g\ d\mu$, gives the result. ■

3. Proof of Theorems 1.2 and 1.4

Lemma 3.1
Assume the hypotheses of Theorem 1.2. Let L be a closed subinterval of the interior of I.
(a) There exists $C_1 > 1$ such that for $n \geq 1$ and $x \in L$,

$$C_1^{-1} \leq K_n(x,x)/n \leq C_1. \tag{3.1}$$

(b) There exists $C_2 > 0$ such that for $n \geq 1$ and $x \in L$,

$$\mathcal{M}[K_n d\mu](x) \leq C_2 n. \tag{3.2}$$

Proof
(a) This follows directly from the assumption (1.5) and monotonicity of Christoffel functions in the underlying measure. See, for example, [9, p. 116, Theorem 20].
(b) Choose a closed interval J such that $L^0 \subset J$ and $J \subset I^0$. By (a), we have

$$C_3^{-1} \leq K_n(x,x)/n \leq C_3 \text{ in } J.$$

Let η denote the distance from L to $\mathbb{R}\backslash J$. For $0 < h < \eta$, and $x \in L$, this last inequality shows that

$$\frac{1}{2h}\int_{x-h}^{x+h} K_n(t,t)\,d\mu(t) \leq C_3 n.$$

For $h \geq \eta$, we have

$$\frac{1}{2h}\int_{x-h}^{x+h} K_n(t,t)\,d\mu(t) \leq \frac{1}{2\eta}\int K_n(t,t)\,d\mu(t) = \frac{n}{2\eta}.$$

Thus for $n \geq 1$ and $x \in L$,

$$\mathcal{M}[K_n d\mu](x) \leq \max\left\{C_3, \frac{1}{2\eta}\right\} n.$$

■

Proof of Theorem 1.4
Fix $r > 0$. Fix closed intervals L, J such that $L \subset J^0$ and $J \subset I^0$. For $x \in L$, and $t \in J$, Cauchy-Schwarz and Lemma 3.1 give

$$\frac{K_n^2(x,t)}{K_n(x,x)} \leq K_n(t,t) \leq C_3 n.$$

Choose n_0 such that

$$x + \frac{r}{n} \in J \text{ for } n \geq n_0 \text{ and } x \in L.$$

Then

$$\begin{aligned}
&|G_n[f](x) - f(x)| \\
\leq\ & \int_{x-\frac{r}{n}}^{x+\frac{r}{n}} |f(t) - f(x)| \frac{K_n^2(x,t)}{K_n(x,x)} \mu'(t)\,dt \\
& + 2\|f\|_\infty \int_{|t-x| \geq \frac{r}{n}} \frac{K_n^2(x,t)}{K_n(x,x)} d\mu(t) \\
\leq\ & C_4 n \int_{x-\frac{r}{n}}^{x+\frac{r}{n}} |f(t) - f(x)|\,dt + 2\|f\|_\infty \Phi_n(x,r).
\end{aligned}$$

Here C_4 is independent of n, x, r. Adding for $n = m, m+1, ..., 2m-1$, and using the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$, Theorem 2.1 and Lemma 3.1(a), (b) give

$$\frac{1}{m} \sum_{n=m}^{2m-1} \left| G_n [f](x) - f(x) \right|^{1/2} \leq \left(2C_4 m \int_{x-\frac{r}{m}}^{x+\frac{r}{m}} |f(t) - f(x)| \, dt \right)^{1/2} +$$

$$+ \left(2 \|f\|_\infty \right)^{1/2} \frac{8\Gamma}{m r^{1/2}} \left(\frac{K_{2m}(x,x)}{K_m(x,x)} \right)^{1/2} \left(m \mathcal{M} [K_{2m} d\mu](x) \right)^{1/2}$$

$$\leq \left(2C_4 m \int_{x-\frac{r}{m}}^{x+\frac{r}{m}} |f(t) - f(x)| \, dt \right)^{1/2} + \frac{C_5}{r^{1/2}} \|f\|_\infty^{1/2},$$

for $m \geq 1$ and $x \in L$. Here C_4 and C_5 are independent of x, r, m. If x is a Lebesgue point of f, we have

$$\lim_{h \to 0+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| \, dt = 0.$$

So at such points,

$$\limsup_{m \to \infty} \frac{1}{m} \sum_{n=m}^{2m-1} \left| G_n [f](x) - f(x) \right|^{1/2}$$
$$\leq \frac{C_5}{r^{1/2}} \|f\|_\infty^{1/2}.$$

As $r > 0$ is arbitrary, and C_5 is independent of r, we obtain

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=m}^{2m-1} \left| G_n [f](x) - f(x) \right|^{1/2} = 0.$$

We may now complete the proof exactly as in the proof of Theorem 1.3: first replace the average over $[m, 2m-1]$ by the average over $[1, m]$, and then replace the exponent $\frac{1}{2}$ by any $p > 0$. ■

Proof of Theorem 1.2

We can use the inequality (2.7) and Theorem 1.4 in the obvious way. Note that if x is a Lebesgue point of g, then it is also a Lebesgue point of g^{-1}, because $g^{\pm 1}$ are bounded in supp$[\mu]$. ■

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