A VARIATIONAL PRINCIPLE FOR CORRELATION FUNCTIONS FOR UNITARY ENSEMBLES, WITH APPLICATIONS

D. S. LUBINSKY

Abstract. In the theory of random matrices for unitary ensembles associated with Hermitian matrices, $m-$point correlation functions play an important role. We show that they possess a useful variational principle. Let μ be a measure with support in the real line, and K_n be the nth reproducing kernel for the associated orthonormal polynomials. We prove that for $m \geq 1$,

$$\det \left[K_n \left(\mu, x_i, x_j \right) \right]_{1 \leq 1, j \leq m} = m! \sup_P \frac{P^2 \left(\underline{x} \right)}{\int P^2 \left(\underline{t} \right) d\mu^{\times m} \left(\underline{t} \right)}$$

where the sup is taken over all alternating polynomials P of degree $\leq n-1$, in m variables $\underline{x}= \left(x_1, x_2, ..., x_m \right)$. Moreover, $\mu^{\times m}$ is the $m-$fold Cartesian product of μ. As a consequence, the suitably normalized $m-$ point correlation functions are *monotone decreasing in the underlying measure* μ. We deduce pointwise, one-sided, universality for arbitrary compactly supported measures, and other limits.

Orthogonal Polynomials, Random Matrices, Unitary Ensembles, Correlation Functions, Christoffel functions. 15B52, 60B20, 60F99, 42C05, 33C50

1. Introduction[1]

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu \left(x \right)$ finite for $j = 0, 1, 2, ...$. Then we may define orthonormal polynomials

$$p_n \left(x \right) = \gamma_n x^n + ..., \gamma_n > 0,$$

satisfying

$$\int p_n p_m d\mu = \delta_{mn}.$$

Date: February 19, 2012.

[1]Research supported by NSF grant DMS1001182 and US-Israel BSF grant 2008399

The nth *reproducing kernel* is

$$K_n(\mu, x, t) = \sum_{j=0}^{n-1} p_j(x) p_j(t)$$

and the nth *Christoffel function* is

$$\lambda_n(\mu, x) = 1/K_n(\mu, x, x) = 1/\sum_{j=0}^{n-1} p_j^2(x). \tag{1.1}$$

It admits an extremal property that is very useful in investigating asymptotics of orthogonal polynomials [24], [29]:

$$\lambda_n(\mu, x) = \inf_{\deg(P)<n} \frac{\int P(t)^2 d\mu(t)}{P^2(x)}.$$

Equivalently,

$$K_n(\mu, x, x) = \sup_{\deg(P)<n} \frac{P^2(x)}{\int P(t)^2 d\mu(t)}. \tag{1.2}$$

We shall prove a direct generalization for $\det [K_n(\mu, x_i, x_j)]_{1 \leq i,j \leq m}$, a determinant that plays a key role in analysis of random matrices.

Random Hermitian matrices rose to prominence, with the work of Eugene Wigner, who used their eigenvalues as a model for scattering theory of heavy nuclei. One places a probability distribution on the entries of an n by n Hermitian matrix. When expressed in "spectral form", that is, as a probability distribution on the (real) eigenvalues $x_1, x_2, ..., x_n$, it has the form

$$\mathcal{P}^{(n)}(x_1, x_2, ..., x_n) = \frac{\left(\prod_{1 \leq j < k \leq n} (x_k - x_j)^2\right) d\mu(x_1) d\mu(x_2) ... d\mu(x_n)}{\int ... \int \left(\prod_{1 \leq j < k \leq n} (t_k - t_j)^2\right) d\mu(t_1) ... d\mu(t_n)},$$

[5, p. 102]. Given $1 \leq m \leq n$, we define the $m-$point correlation function

$$\begin{aligned} & R_m^n(\mu; x_1, x_2, ..., x_m) \\ = & \frac{n!}{(n-m!)} \int ... \int \mathcal{P}^{(n)}(x_1, x_2, ..., x_n) d\mu(x_{m+1}) ... d\mu(x_n). \end{aligned} \tag{1.3}$$

Thus R_m^n is, up to normalization, a marginal distribution, where we integrate out $t_{m+1}, t_{m+2}, ..., t_n$. Note that we exclude from R_m^n, a factor of $\mu'(x_1)\mu'(x_2)...\mu'(x_m)$, which is used in [5]. It is a well established fact [5, p. 112] that

$$R_m^n(\mu; x_1, x_2, ..., x_m) = \det\left[K_n(\mu, x_i, x_j)\right]_{1\le i,j\le m}. \tag{1.4}$$

Again, we emphasize that in [5], as distinct from this paper, μ' is absorbed into K_n. Since much of the interest lies in asymptotics as $n\to\infty$, for fixed m, it is obviously easier to handle asymptotics of this fixed size determinant, than to deal with the $n-m$ fold integral in (1.3).

R_m^n can be used to describe the local spacing of m-tuples of eigenvalues. For example, if $m=2$, and $B\subset\mathbb{R}$ is measurable, then [5, p. 117]

$$\int_B\int_B R_2^n(\mu; t_1, t_2)\, d\mu(t_1)\ \ d\mu(t_2)$$

is the expected number of pairs (t_1, t_2) of eigenvalues, with both $t_1, t_2\in B$.

Of course there are other settings for random matrices that do not involve orthogonal polynomials. There one considers a class of matrices (such as normal matrices or symmetric matrices) where the elements of the matrix are independently distributed, or there are appropriate bounds on the dependence. The methods are quite different, but remarkably, similar limiting results arise [8], [9], [10], [12], [31].

The formulation of our main result involves $\mathcal{AL}_n^m$, the alternating polynomials of degree at most n in m variables. We say that $P\in\mathcal{AL}_n^m$ if

$$P(x_1, x_2, ..., x_m) = \sum_{0\le j_1, j_2, ..., j_m\le n} c_{j_1 j_2 ... j_m} x_1^{j_1} x_2^{j_2} ... x_m^{j_m}, \tag{1.5}$$

so that P is a polynomial of degree $\le n$ in each of its m variables, and in addition is *alternating*, so that for every pair (i,j) with $1\le i<j\le m$,

$$P(x_1, ..., x_i, ..., x_j, ..., x_m) = -P(x_1, ..., x_j, ..., x_i, ..., x_m). \tag{1.6}$$

Thus swapping variables changes the sign. Sometimes, these are called *skew-symmetric* polynomials.

Observe that if P_i is a univariate polynomial of degree $\le n$ for each $i=1,2,...,m$, then

$$P(t_1, t_2, ..., t_m) = \det\left[P_i(t_j)\right]_{1\le i,j\le m} \in \mathcal{AL}_n^m. \tag{1.7}$$

The set of such determinants of polynomials is a proper subset of $\mathcal{AL}_n^m$. It is well known, and easy to see, that every alternating polynomial is the product of a Vandermonde determinant and a symmetric polynomial. Thus $P \in \mathcal{AL}_n^m$ iff

$$P(t_1, t_2, ..., t_m) = \left(\prod_{1 \le i < j \le m} (t_j - t_i) \right) S(t_1, t_2, ..., t_m),$$

where S is symmetric, and of degree $\le n - m + 1$ in each variable.

Given a fixed m, we shall use the notation

$$\underline{x} = (x_1, x_2, ..., x_m), \ \underline{t} = (t_1, t_2, ..., t_m)$$

while $\mu^{\times m}$ denotes the $m-$fold Cartesian product of μ, so that

$$d\mu^{\times m}(\underline{t}) = d\mu(t_1) \, d\mu(t_2) \ldots d\mu(t_m). \tag{1.8}$$

We prove:

Theorem 1.1
Let $m \ge 1, n \ge m + 1$. Let $\underline{x} = (x_1, x_2, ..., x_m)$ be an m-tuple of real numbers. Then

$$\det [K_n(\mu, x_i, x_j)]_{1 \le i,j \le m} = m! \sup_{P \in \mathcal{AL}_{n-1}^m} \frac{(P(\underline{x}))^2}{\int (P(\underline{t}))^2 \, d\mu^{\times m}(\underline{t})}. \tag{1.9}$$

The sup is attained for

$$P(\underline{t}) = \det [K_n(\mu, x_i, t_j)]_{1 \le i,j \le m}. \tag{1.10}$$

We could also just take the supremum in (1.9) over the strictly smaller class of determinants of the form (1.7). An immediate, but important consequence is

Corollary 1.2
$R_m^n(\mu; x_1, x_2, ..., x_m)$ is a monotone decreasing function of μ, and a monotone increasing function of n.

Despite an extensive literature search, I have not found Theorem 1.1 or Corollary 1.2 in the (extensive!) literature for random matrices. At the very least, they must be new to those interested in universality limits, because of the applications they have there. We shall present some in Section 2.

The proof of Theorem 1.1 is based on multivariate orthogonal polynomials built from μ. Given $m \ge 1$, and non-negative integers $j_1, j_2, ..., j_m$, we define

$$
\begin{aligned}
& T_{j_1,j_2,...,j_m}(x_1,x_2,...,x_m) \\
= \;& \det\left(p_{j_i}(x_k)\right)_{1\le i,k\le m} \\
= \;& \det\begin{bmatrix} p_{j_1}(x_1) & p_{j_1}(x_2) & \dots & p_{j_1}(x_m) \\ p_{j_2}(x_1) & p_{j_2}(x_2) & \dots & p_{j_2}(x_m) \\ \vdots & \vdots & \ddots & \vdots \\ p_{j_m}(x_1) & p_{j_m}(x_2) & \dots & p_{j_m}(x_m) \end{bmatrix}.
\end{aligned}
\tag{1.11}
$$

We show that the $\{T_{j_1,j_2,...,j_m}\}_{j_1<j_2<...<j_m}$ form an orthogonal family with respect to $\mu^{\times m}$, and moreover, the $m-$point correlation function admits an expansion as a sum of squares of $\{T_{j_1,j_2,...,j_m}\}$, just as does K_n in terms of squares of the orthonormal polynomials. We shall need an associated reproducing kernel,

$$
K_n^m(\mu,\underline{x},\underline{t}) = \frac{1}{m!}\sum_{1\le j_1<j_2<...<j_m\le n} T_{j_1,j_2,...,j_m}(\underline{x})\,T_{j_1,j_2,...,j_m}(\underline{t}). \tag{1.12}
$$

Theorem 1.3

(a) Let $0\le j_1<j_2<...<j_m$ *and* $0\le k_1<k_2<...<k_m$. *Then*

$$
\begin{aligned}
& \int T_{j_1,j_2,...,j_m}(\underline{t})\,T_{k_1,k_2,...,k_m}(\underline{t})\,d\mu^{\times m}(\underline{t}) \\
= \;& m!\delta_{j_1k_1}\delta_{j_2k_2}...\delta_{j_mk_m}.
\end{aligned}
$$

(1.13)

(b) For $P\in\mathcal{AL}_{n-1}^m$, *and* $\underline{x}\in\mathbb{R}^n$,

$$
P(\underline{x}) = \int P(\underline{t})\,K_n^m(\mu,\underline{x},\underline{t})\,d\mu^{\times m}(\underline{t}). \tag{1.14}
$$

(c) For $\underline{x},\underline{t}\in\mathbb{R}^n$,

$$
\begin{aligned}
& \det\left[K_n(\mu,x_i,t_j)\right]_{1\le i,j\le m} \\
= \;& m!K_n^m(\mu,\underline{x},\underline{t}).
\end{aligned}
\tag{1.15}
$$

In particular,

$$
\begin{aligned}
& \det\left[K_n(\mu,x_i,x_j)\right]_{1\le i,j\le m} \\
= \;& \sum_{1\le j_1<j_2<...<j_m\le n}\left(T_{j_1,j_2,...,j_m}(\underline{x})\right)^2.
\end{aligned}
\tag{1.16}
$$

Remarks

(a) Note that in the case $m=1$, (1.16) reduces to (1.1) for $K_n(\mu,x,x)$.

After an extensive literature search, we found that (1.16) already appears for general m in [8, Section 1.5.3]. We may also express it as

$$\begin{aligned} &\det\left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m} \\ &= \frac{1}{m!}\sum_{1\leq j_1,j_2,\ldots,j_m\leq n}\left(T_{j_1,j_2,\ldots,j_m}\left(\underline{x}\right)\right)^2, \end{aligned} \tag{1.17}$$

as $T_{j_1,j_2,\ldots,j_m}$ vanishes if any two indices j_i are equal.
(b) The expression (1.15) may also be thought of as a Christoffel-Darboux formula, for it expresses the sum (1.12) in a compact form involving an $m\times m$ determinant.

One consequence of the variational principle is a lower bound for ratios of correlation functions:

Theorem 1.4
Let $m\geq 2, n\geq m+1$, and $x_1, x_2, \ldots, x_m$ be distinct real numbers. Define a measure ν by

$$d\nu\left(t\right) = d\mu\left(t\right)\prod_{j=2}^{m}\left(t-x_j\right)^2.$$

Then

$$\begin{aligned} K_n\left(\mu, x_1, x_1\right) &\geq \frac{\det\left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m}}{\det\left[K_n\left(\mu, x_i, x_j\right)\right]_{2\leq i,j\leq m}} \\ &\geq \frac{1}{m}K_{n-m+1}\left(\nu, x_1, x_1\right)\prod_{j=2}^{m}\left(x_1-x_j\right)^2. \end{aligned} \tag{1.18}$$

The upper bound is a well known consequence of inequalities for positive definite matrices. It is the lower bound that is new.

This paper is organised as follows: in Section 2, we state some applications of Theorem 1.1 to asymptotics and universality limits. In Section 3, we first prove Theorem 1.3, and then deduce Theorem 1.1 and Corollary 1.2, followed by Theorem 1.4. Theorems 2.1, 2.2, and 2.3 are proved in Section 4. Theorem 2.4 is proved in Section 5, and Theorem 2.5 and Corollary 2.6 in Section 6.

2. Applications to Asymptotics and Universality Limits

The extremal property (1.2) is essential in proving the following: if μ is any measure with support in $[-1,1]$, then at every Lebesgue point

x of μ in $(-1,1)$,

$$\liminf_{n\to\infty} \frac{1}{n} K_n(\mu,x,x)\,\mu'(x) \geq \frac{1}{\pi\sqrt{1-x^2}}. \tag{2.1}$$

Here μ' is understood as the Radon-Nikodym derivative of the absolutely continuous part of μ. This is more commonly formulated for Christoffel functions as

$$\limsup_{n\to\infty} n\lambda_n(\mu,x) \leq \mu'(x)\,\pi\sqrt{1-x^2}.$$

Barry Simon calls this the *Maté-Nevai-Totik upper bound*. See, for example [22], [29, Thm. 5.11.1, p. 334], [32].

Under additional conditions, including regularity of μ, there is equality in (2.1), with a full limit. We say that μ is *regular in the sense of Stahl, Totik, and Ullman*, or just *regular,* if the leading coefficients $\{\gamma_n\}$ of its orthonormal polynomials satisfy

$$\lim_{n\to\infty} {\gamma_n}^{1/n} = \frac{1}{\operatorname{cap}(\operatorname{supp}[\mu])}. \tag{2.2}$$

Here $\operatorname{cap}(\operatorname{supp}[\mu])$ is the logarithmic capacity of the support of μ. We shall need only a very simple criterion for regularity, namely a version of the Erdős-Turán criterion: if the support of μ consists of finitely many intervals, and $\mu' > 0$ a.e. with respect to Lebesgue measure in that support, then μ is regular [30, p. 102].

In 1991, Maté, Nevai and Totik [22] showed that if μ is a regular measure with support $[-1,1]$, and in some subinterval I of $(-1,1)$, we have

$$\int_I \log\mu' > -\infty, \tag{2.3}$$

then for a.e. $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu,x,x)\,\mu'(x) = \frac{1}{\pi\sqrt{1-x^2}}. \tag{2.4}$$

Totik gave a far reaching extension of this to measures with compact support J [32], [33]. Here one needs the equilibrium measure ν_J for the compact set J, as well as its Radon-Nikodym derivative, which we denote by ω_J. Thus ν_J is the unique probability measure that minimizes the energy integral

$$\int\int \log\frac{1}{|s-t|}\,d\nu(s)\,d\nu(t)$$

amongst all probability measures ν with support in J [25], [26]. If I is some subinterval of J, then ν_J is absolutely continuous in I, and moreover, $\omega_J > 0$ in the interior I^o of I. In the special case $J = [-1,1]$,

$d\nu_J(x) = \omega_J(x)\,dx = \frac{dx}{\pi\sqrt{1-x^2}}$. Totik showed that if μ is regular, and in some subinterval I of J, we have (2.3), then for a.e. $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, x)\,\mu'(x) = \omega_J(x)\,. \tag{2.5}$$

Further developments are explored in Simon's monograph [29].

It is a fairly straightforward consequence of this last relation, and the Christoffel-Darboux formula, that for $m \geq 2$ and a.e. $(x_1, x_2, ..., x_m) \in I^m$,

$$\lim_{n\to\infty} \frac{1}{n^m} \det\left[K_n(\mu, x_i, x_j)\right]_{1\leq i,j\leq m} = \prod_{j=1}^{m} \frac{\omega_J(x_j)}{\mu'(x_j)}. \tag{2.6}$$

The right-hand side is interpreted as ∞ if any $\mu'(x_j) = 0$. Thus, the matrix $\left[K_n(\mu, x_i, x_j)\right]_{1\leq i,j\leq m}$ behaves essentially like its diagonal. We shall prove this in Section 4. Without having to assume regularity, or (2.3), we can use Theorem 1.1 to prove one-sided versions of (2.6).

For measures μ with compact support J, and $x \in J$, we let

$$\omega_\mu(x) = \inf\left\{\omega_L(x) : L \subset J \text{ is compact, } \mu_{|L} \text{ is regular, } x \in L.\right\} \tag{2.7}$$

Since ν_L decreases as L increases, one can roughly think of ω_μ as the density of the equilibrium measure of the largest set to whose restriction μ is regular. In the sequel, J^o denotes the interior of J.

Theorem 2.1

Let μ have compact support J, of positive Lebesgue measure, and let ω_J denote the equilibrium density of J. Let $m \geq 1$.
(a) For Lebesgue a.e. $(x_1, x_2, ..., x_m) \in (J^o)^m$,

$$\liminf_{n\to\infty} \frac{1}{n^m} \det\left[K_n(\mu, x_i, x_j)\right]_{1\leq i,j\leq m} \geq \prod_{j=1}^{m} \frac{\omega_J(x_j)}{\mu'(x_j)}. \tag{2.8}$$

The right-hand side is interpreted as ∞ if any $\mu'(x_j) = 0$.
(b) Suppose that I is a compact subset of J consisting of finitely many intervals, for which (2.3) holds. Then for Lebesgue a.e. $(x_1, x_2, ..., x_m) \in I^m$,

$$\limsup_{m\to\infty} \frac{1}{n^m} \det\left[K_n(\mu, x_i, x_j)\right]_{1\leq i,j\leq m} \leq \prod_{j=1}^{m} \frac{\omega_\mu(x_j)}{\mu'(x_j)}. \tag{2.9}$$

A perhaps more impressive application of Theorem 1.1 is to universality limits in the bulk, which describe local spacing of eigenvalues of random Hermitian matrices [5], [6], [12], [23]. One of the more standard formulations, for a measure μ supported on $[-1,1]$, is

$$\begin{aligned} & \lim_{n\to\infty}\left(\frac{\mu'(x)\,\pi\sqrt{1-x^2}}{n}\right)^m R_m^n\left(\mu; x+a_1\frac{\pi\sqrt{1-x^2}}{n},...,x+a_m\frac{\pi\sqrt{1-x^2}}{n}\right) \\ = & \lim_{n\to\infty}\left(\frac{\mu'(x)\,\pi\sqrt{1-x^2}}{n}\right)^m \det\left[K_n\left(\mu; x+a_i\frac{\pi\sqrt{1-x^2}}{n}, x+a_j\frac{\pi\sqrt{1-x^2}}{n}\right)\right]_{1\le i,j\le m} \\ = & \det\left(S\left(a_i-a_j\right)\right)_{1\le i,j\le m}, \end{aligned}$$

where

$$S(t)=\frac{\sin \pi t}{\pi t} \tag{2.10}$$

is the sine (or sinc) kernel. There is a vast literature for universality limits, especially in the case where μ is replaced by varying weights. A great many methods have been applied, including classical asymptotics for orthonormal polynomials, Riemann Hilbert techniques, and theory of entire functions of exponential type [1], [2], [5], [6], [7], [11], [12] [16], [19], [28], [29], [33].

For fixed measures μ with compact support J, the most general pointwise result is due to Totik [33]. It asserts that if μ is regular, while (2.3) holds in some interval I in the support, then for a.e. $x\in I$, and all real $a_1,a_2,...,a_m$, there are limits for the scaled reproducing kernels that immediately yield

$$\begin{aligned} & \lim_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R_m^n\left(\mu; x+\frac{a_1}{n\omega_J(x)},...,x+\frac{a_m}{n\omega_J(x)}\right) \\ = & \det\left(S\left(a_i-a_j\right)\right)_{1\le i,j\le m}. \end{aligned}$$

Simon had a similar result, proved using Jost functions [27], [28]. Totik uses the comparison method of the author [19], together with "polynomial pullbacks". Without any local or global restrictions on μ, the author showed [21] that universality holds in measure in $\{\mu'>0\}=\{x:\mu'(x)>0\}$.

We prove pointwise, almost everywhere, one-sided universality, without any local or global restrictions on μ:

Theorem 2.2
Let μ have compact support J, and let ω_J denote the equilibrium density of J. Let $m\ge 1$.

(a) For a.e. $x \in J^o \cap \{\mu' > 0\}$*, and for all real* $a_1, a_2, ..., a_m$*,*

$$\begin{aligned} &\liminf_{n\to\infty} \left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R_m^n\left(\mu; x + \frac{a_1}{n\omega_J(x)}, ..., x + \frac{a_m}{n\omega_J(x)}\right) \\ \geq\ &\det\left(S\left(a_i - a_j\right)\right)_{1\leq i,j\leq m}. \end{aligned}$$

(2.11)

(b) Suppose that I *is a compact subset of* J *consisting of finitely many intervals, for which (2.3) holds. Then for a.e.* $x \in I$*, and for all real* $a_1, a_2, ..., a_m$*,*

$$\begin{aligned} &\limsup_{n\to\infty} \left(\frac{\mu'(x)}{n\omega_\mu(x)}\right)^m R_m^n\left(\mu; x + \frac{a_1}{n\omega_\mu(x)}, ..., x + \frac{a_m}{n\omega_\mu(x)}\right) \\ \leq\ &\det\left(S\left(a_i - a_j\right)\right)_{1\leq i,j\leq m}. \end{aligned}$$

(2.12)

Pointwise universality at a given point x seems to usually require at least something like μ' being continuous at x, or x being a Lebesgue point of μ. Indeed, when μ' has a jump discontinuity, the universality limit is different from the sine kernel [13], and involves de Branges spaces [20]. In our next result, we show that one can still bound the behavior of the correlation function above and below near such a given x. It is noteworthy, though, that pure singularly continuous measures can exhibit sine kernel behavior [4].

Theorem 2.3
Let μ *have compact support* J*, be regular, and let* ω_J *denote the equilibrium density of* J*. Assume that the singular part* μ_s *of* μ *satisfies at a given* x *in the interior of* J*,*

$$\lim_{h\to 0+} \mu_s\left[x - h, x + h\right]/h = 0. \tag{2.13}$$

Assume moreover, that the derivative μ' *of the absolutely continuous part of* μ *satisfies*

$$0 < C_1 = \liminf_{t\to x} \mu'(t) \leq \limsup_{t\to x} \mu'(t) = C_2 < \infty. \tag{2.14}$$

Then for all real $a_1, a_2, ..., a_m$,

$$\begin{aligned} & C_2^{-m} \det \left(S\left(a_i - a_j\right)\right)_{1 \leq i,j \leq m} \\ \leq \; & \liminf_{n \to \infty} \left(\frac{1}{n\omega_J(x)}\right)^m R_m^n \left(\mu; x + \frac{a_1}{n\omega_J(x)}, ..., x + \frac{a_m}{n\omega_J(x)}\right) \\ \leq \; & \limsup_{n \to \infty} \left(\frac{1}{n\omega_J(x)}\right)^m R_m^n \left(\mu; x + \frac{a_1}{n\omega_J(x)}, ..., x + \frac{a_m}{n\omega_J(x)}\right) \\ \leq \; & C_1^{-m} \det \left(S\left(a_i - a_j\right)\right)_{1 \leq i,j \leq m} . \end{aligned}$$

(2.15)

At the boundary of the support of the measure, (referred to as the edge of the spectrum in random matrix theory), the universality limit takes a different form [12], [15]. For fixed measures that behave like Jacobi weights near the endpoints, they involve the Bessel kernel of order $\alpha > -1$,

$$\mathbb{J}_\alpha(u, v) = \frac{J_\alpha(\sqrt{u})\sqrt{v}J'_\alpha(\sqrt{v}) - J_\alpha(\sqrt{v})\sqrt{u}J'_\alpha(\sqrt{u})}{2(u - v)}.$$

Here J_α is the usual Bessel function of the first kind and order α. Using a comparison method, the author proved [17] that if μ is a regular measure on $[-1, 1]$, and μ is absolutely continuous in some left neighborhood $(1-\eta, 1]$ of 1, and there $\mu'(t) = h(t)(1-t)^\alpha$, where $h(1) > 0$ and h is continuous at 1, then

$$\lim_{n \to \infty} \frac{1}{2n^2} \tilde{K}_n \left(\mu, 1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a, b), \tag{2.16}$$

uniformly for a, b in compact subsets of $(0, \infty)$. Here, and in the sequel,

$$\tilde{K}_n(\mu, x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(\mu, x, y).$$

When $\alpha \geq 0$, we may allow also $a, b = 0$. This has the immediate consequence that for $m \geq 2$, and $a_1, a_2, ..., a_m > 0$,

$$\begin{aligned} & \lim_{n \to \infty} \left(\frac{1}{2n^2}\right)^m R_m^n \left(\mu; 1 - \frac{a_1}{2n^2}, ..., 1 - \frac{a_m}{2n^2}\right) \left(\prod_{j=1}^m \mu'\left(1 - \frac{a_j}{2n^2}\right)\right) \\ = \; & \det \left(\mathbb{J}_\alpha(a_i, a_j)\right)_{1 \leq i,j \leq m} . \end{aligned}$$

(2.17)

Under weak conditions at the edge, we can prove one-sided universality:

Theorem 2.4

Let μ *have support contained in* $[-1, 1]$ *and let* 1 *be the right endpoint* *of that support. Assume that μ is absolutely continuous near 1, and for some $\alpha > -1$,*

(2.18)
$$0 < C_1 = \liminf_{t \to 1-} \mu'(t)(1-t)^{-\alpha} \leq \limsup_{t \to 1-} \mu'(t)(1-t)^{-\alpha} = C_2 < \infty.$$

Then for $a_1, a_2, ..., a_m > 0$,

$$\liminf_{n\to\infty} \left(\frac{1}{2n^2}\right)^m R_m^n\left(\mu; 1 - \frac{a_1}{2n^2}, ..., 1 - \frac{a_m}{2n^2}\right) \prod_{j=1}^m \mu'\left(1 - \frac{a_j}{2n^2}\right)$$
$$\geq \left(\frac{C_1}{C_2}\right)^m \det\left(\mathbb{J}_\alpha(a_i, a_j)\right)_{1\leq i,j\leq m}.$$

(2.19)

If $\alpha \geq 0$, we may also allow $a_1, a_2, ..., a_m \geq 0$.

We note that if in addition, μ has support $[-1, 1]$ and is regular, then we may replace the lim inf by lim sup, the asymptotic lower bound by an upper bound, provided we replace $(C_1/C_2)^m$ by $(C_2/C_1)^m$.

Our final result has a comparison or "localization" flavor, generalizing similar results for Christoffel functions. Recall that a set $J \subset \mathbb{R}$ is said to be regular for the Dirichlet problem [25], [30], if for every function f continuous on J, there exists a function harmonic in $\bar{\mathbb{C}}\backslash J$, continuous on $\mathbb{C}$, whose restriction to J is f. Of course, this is confusing when juxtaposed with the notion of a regular measure!

Theorem 2.5
Let μ, ν have compact support J and both be regular. Assume that J is regular with respect to the Dirichlet problem. Let $\xi \in J$ and $\mu'(\xi), \nu'(\xi)$ be finite and positive, with

$$\lim_{dist(I,\xi)\to 0} \frac{\mu(I)}{\nu(I)} = \frac{\mu'(\xi)}{\nu'(\xi)}, \tag{2.20}$$

where the limit is taken over intervals I of length $|I|$, and $dist(I, \xi) = \sup\{|x - \xi| : x \in I\}$. Let $m \geq 1$. Assume that for $n \geq 1$,

$$\underline{y}_n = (y_{1n}, y_{2n}, ..., y_{mn})$$

is a vector of real numbers satisfying

$$\lim_{n\to\infty} \left(\max_{1\leq j\leq m} |y_{mj} - \xi|\right) = 0, \tag{2.21}$$

and

$$\lim_{\varepsilon\to 0+}\left(\limsup_{n\to\infty}\left|\frac{K^m_{[n(1\pm\varepsilon)]}\left(\nu,\underline{y}_n,\underline{y}_n\right)}{K^m_n\left(\nu,\underline{y}_n,\underline{y}_n\right)}-1\right|\right)=0. \tag{2.22}$$

Then

$$\lim_{n\to\infty}\frac{K^m_n\left(\mu,\underline{y}_n,\underline{y}_n\right)}{K^m_n\left(\nu,\underline{y}_n,\underline{y}_n\right)}=\left(\frac{\nu'(\xi)}{\mu'(\xi)}\right)^m. \tag{2.23}$$

Of course in (2.22), $[n(1\pm\varepsilon)]$ denotes the integer part of $n(1\pm\varepsilon)$. As an immediate consequence, we obtain:

Corollary 2.6
Let μ,ν have compact support J and be regular. Assume that J is regular with respect to the Dirichlet problem. Let $x\in J$ and $\mu'(x),\nu'(x)$ be finite and positive, with (2.20) holding at $\xi=x$. Assume that for given $m\geq 2$ and all real $a_1,a_2,...a_m$,

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{\nu'(x)}{n\omega_J(x)}\right)^m R^n_m\left(\nu;x+\frac{a_1}{n\omega_J(x)},...,x+\frac{a_m}{n\omega_J(x)}\right)\\ =\;&\det\left(S\left(a_i-a_j\right)\right)_{1\leq i,j\leq m}.\end{aligned}$$

(2.24)

Then for all real $a_1,a_2,...a_m$,

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R^n_m\left(\mu;x+\frac{a_1}{n\omega_J(x)},...,x+\frac{a_m}{n\omega_J(x)}\right)\\ =\;&\det\left(S\left(a_i-a_j\right)\right)_{1\leq i,j\leq m}.\end{aligned}$$

(2.25)

3. Proof of Theorems 1.1, 1.3, 1.4 and Corollary 1.2

We begin with

Proof of Theorem 1.3 (a)
We use σ and η to denote permutations of $(1,2,...,m)$ with respective

signs ε_σ and ε_η. We see that

$$\begin{aligned} I &= \int ... \int T_{j_1,j_2,...,j_m}(t_1,t_2,...,t_m)\, T_{k_1,k_2,...,k_m}(t_1,t_2,...,t_m)\, d\mu(t_1)...d\mu(t_m) \\ &= \sum_{\sigma,\eta} \varepsilon_\sigma \varepsilon_\eta \int ... \int \left(\prod_{i=1}^{m} p_{j_{\sigma(i)}}(t_i)\right)\left(\prod_{i=1}^{m} p_{k_{\eta(i)}}(t_i)\right) d\mu(t_1)...d\mu(t_m) \\ &= \sum_{\sigma,\eta} \varepsilon_\sigma \varepsilon_\eta \prod_{i=1}^{m} \delta_{j_{\sigma(i)} k_{\eta(i)}} \\ &= \sum_{\sigma,\eta} \varepsilon_\sigma \varepsilon_\eta \prod_{\ell=1}^{m} \delta_{j_\ell k_{\eta(\sigma^{-1}(\ell))}}, \end{aligned}$$

(3.1)

where σ^{-1} is the inverse of the permutation σ. For a term in this last sum to be non-zero, we need

$$j_\ell = k_{\eta(\sigma^{-1}(\ell))} \text{ for all } 1 \leq \ell \leq m. \tag{3.2}$$

Since $j_1 < j_2 < ... < j_m$ and $k_1 < k_2 < ... < k_m$, we see that this will fail unless

$$\eta\left(\sigma^{-1}(\ell)\right) = \ell \text{ for all } 1 \leq \ell \leq m.$$

Indeed, if $\eta(\sigma^{-1}(i)) \neq i$ for some smallest i, then $j_{i-1} = k_{i-1}$ but either $j_i = k_{\eta(\sigma^{-1}(i))} \geq k_{i+1}$ or $j_i = k_{\eta(\sigma^{-1}(i))} \leq k_{i-1}$. In the former case, all of $j_i, j_{i+1}, ..., j_m > k_i$, and k_i is omitted from the equalities in (3.2), a contradiction. In the latter case, we obtain $j_i \leq j_{i-1}$, contradicting the strict monotonicity of the $j's$. Thus necessarily $\eta = \sigma$, so (3.1) becomes, under (3.2),

$$I = \sum_\sigma \varepsilon_\sigma^2 = m!.$$

■

Proof of Theorem 1.3(b)

We first show that every $P \in \mathcal{AL}_{n-1}^m$ is a linear combination of the T polynomials. We can write

$$P(x_1, x_2, ...x_m) = \sum_{0 \leq j_1, j_2, ..., j_m < n} c_{j_1 j_2 ... j_m} p_{j_1}(x_1)\, p_{j_2}(x_2) ... p_{j_m}(x_m).$$

Because of the alternating property (1.6), and the linear independence of

$\{p_{j_1}(x_1)\, p_{j_2}(x_2) \ldots p_{j_m}(x_m)\}_{1 \leq j_1, j_2, \ldots, j_m \leq n}$, necessarily, when we swap indices j_k and j_ℓ, the coefficients change sign, that is,

$$c_{j_1 \ldots j_k \ldots j_\ell \ldots j_m} = -c_{j_1 \ldots j_\ell \ldots j_k \ldots j_m}.$$

In particular, coefficients vanish if any two subscripts coincide. More generally, this implies that if σ is a permutation of $\{1, 2, \ldots, m\}$ with sign ε_σ, then

$$c_{j_{\sigma(1)} j_{\sigma(2)} \ldots j_{\sigma(m)}} = \varepsilon_\sigma c_{j_1 j_2 \ldots j_m}.$$

Next, given distinct $0 \leq j_1, j_2, \ldots, j_m < n$, let $\tilde{j}_1 < \tilde{j}_2 < \ldots < \tilde{j}_m$ denote these indices in increasing order. We can write for some permutation σ,

$$j_i = \tilde{j}_{\sigma(i)},\ 1 \leq i \leq m.$$

Conversely, for the the given $\{\tilde{j}_i\}$, every such permutation σ defines a set of indices $\{j_i\}$ with $0 \leq j_1, j_2, \ldots, j_m < n$. Thus

$$\begin{aligned} P(x_1, x_2, \ldots x_m) &= \sum_{0 \leq \tilde{j}_1 < \tilde{j}_2 < \ldots < \tilde{j}_m < n} c_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m} \sum_\sigma \varepsilon_\sigma p_{\tilde{j}_{\sigma(1)}}(x_1)\, p_{\tilde{j}_{\sigma(2)}}(x_2) \ldots p_{\tilde{j}_{\sigma(m)}}(x_m) \\ &= \sum_{0 \leq \tilde{j}_1 < \tilde{j}_2 < \ldots < \tilde{j}_m < n} c_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m} \det \left[p_{\tilde{j}_i}(x_k) \right]_{1 \leq i,k \leq m} \\ &= \sum_{0 \leq \tilde{j}_1 < \tilde{j}_2 < \ldots < \tilde{j}_m < n} c_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m} T_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m}(x_1, x_2, \ldots, x_m). \end{aligned}$$

(3.3)

Inasmuch as each $T_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m} \in \mathcal{AL}^m_{n-1}$, we have shown that $\mathcal{AL}^m_{n-1}$ is the linear span of the T polynomials, and (3.3) is an orthogonal expansion. Orthogonality in the form (1.13) gives

$$c_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m} = \frac{1}{m!} \int P(\underline{t})\, T_{\tilde{j}_1 \tilde{j}_2 \ldots \tilde{j}_m}(\underline{t})\, d\mu^{\times m}(\underline{t}).$$

Now our definition (1.12) of the reproducing kernel gives (1.14). ■

Proof of Theorem 1.3 (c)

Fix $\underline{x} = (x_1, x_2, \ldots, x_m)$. Let

(3.4) $$P(\underline{t}) = P(t_1, t_2, \ldots, t_m) = \det [K_n(\mu, x_i, t_j)]_{1 \leq i,j \leq m}.$$

By successively extracting the sums from the 1st, 2nd, ..., mth rows, we see that

$$P(\underline{t}) = \det \begin{bmatrix} \sum_{j_1=0}^{n-1} p_{j_1}(x_1)\, p_{j_1}(t_1) & \ldots & \sum_{j_1=0}^{n-1} p_{j_1}(x_1)\, p_{j_1}(t_m) \\ \vdots & \ddots & \vdots \\ \sum_{j_m=0}^{n-1} p_{j_m}(x_m)\, p_{j_m}(t_1) & \ldots & \sum_{j_m=0}^{n-1} p_{j_m}(x_m)\, p_{j_1}(t_m) \end{bmatrix}$$

$$= \sum_{j_1=0}^{n-1} \dots \sum_{j_m=0}^{n-1} \left(p_{j_1}(x_1) \dots p_{j_m}(x_m)\right) T_{j_1 j_2 \dots j_m}(t_1, t_2, \dots, t_m).$$

When $j_i = j_k$ for distinct i, k, then $T_{j_1 j_2 \dots j_m} = 0$. Thus only terms with $j_1, j_2, \dots, j_m$ distinct are non-zero. As in the proof of Theorem 1.3(b), given distinct $0 \le j_1, j_2, \dots, j_m < n$, we can write for some permutation σ uniquely determined by these indices

$$j_i = \tilde{j}_{\sigma(i)}$$

where $0 \le \tilde{j}_1 < \tilde{j}_2 < \dots < \tilde{j}_m < n$. As there, this yields

$$\begin{aligned} P(\underline{t}) &= \sum_{0 \le \tilde{j}_1 < \tilde{j}_2 < \dots < \tilde{j}_m < n} \sum_{\sigma} \varepsilon_\sigma \left(p_{\tilde{j}_{\sigma(1)}}(x_1) \dots p_{\tilde{j}_{\sigma(m)}}(x_m)\right) T_{\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_m}(t_1, t_2, \dots, t_m) \\ &= \sum_{0 \le \tilde{j}_1 < \tilde{j}_2 < \dots < \tilde{j}_m < n} T_{\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_m}(x_1, x_2, \dots, x) \, T_{\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_m}(t_1, t_2, \dots, t_m). \end{aligned}$$

So

$$\det \left[K_n(\mu, x_i, t_j)\right]_{1 \le i,j \le m} = P(\underline{t}) = m! K_n^m(\mu, \underline{x}, \underline{t}),$$

and we have (1.15). Then (1.16) follows from (1.12). ■

Proof of Theorem 1.1

By the reproducing kernel relation (1.14), and Cauchy-Schwarz, for all $P \in \mathcal{AL}_{n-1}^m$,

$$\begin{aligned} P(\underline{x})^2 &\le \left(\int P(\underline{t})^2 d\mu^{\times m}(\underline{t})\right) \left(\int K_n^m(\mu, \underline{x}, \underline{t})^2 d\mu^{\times m}(\underline{t})\right) \\ &= \left(\int P(\underline{t})^2 d\mu^{\times m}(\underline{t})\right) K_n^m(\mu, \underline{x}, \underline{x}). \end{aligned}$$

Thus

$$K_n^m(\mu, \underline{x}, \underline{x}) \ge \sup_{P \in \mathcal{AL}_{n-1}^m} \frac{(P(\underline{x}))^2}{\int (P(\underline{t}))^2 d\mu^{\times m}(\underline{t})}. \tag{3.5}$$

By choosing P as in (3.4), we obtain equality in (3.5). Now (1.9) follows from (1.15). ■

Proof of Corollary 1.2

This follows immediately from (1.9) and the positivity of all the terms there. ■

Proof of Theorem 1.4

The upper bound in (1.18) is a standard inequality for determinants involving symmetric positive definite matrices. See, for example, [3, Thm. 7, p. 63]. For the lower bound, let $R(t_2, t_3, ..., t_m) \in \mathcal{AL}_{m-1}^{n-1}$. Let P be a univariate polynomial of degree $\leq n-1$ satisfying $P(x_j) = 0$, $2 \leq j \leq m$. Let

$$S(t_1, t_2, ..., t_m) = \sum_{j=1}^{m} P(t_j)(-1)^j R(t_1, t_2, ..., t_{j-1}, t_{j+1}, ..., t_m).$$

We claim that $S \in \mathcal{AL}_m^{n-1}$. Suppose we swap the variables t_k and t_ℓ, where $1 \leq k < \ell \leq m$. The terms involving $P(t_k)$ and $P(t_\ell)$ before the variable swap are

$$\begin{aligned} &P(t_k)(-1)^k R(t_1, ..., t_{k-1}, t_{k+1}, ..., t_{\ell-1}, t_\ell, t_{\ell+1}, ..., t_m) \\ &+P(t_\ell)(-1)^\ell R(t_1, ..., t_{k-1}, t_k, t_{k+1}, ..., t_{\ell-1}, t_{\ell+1}, ..., t_m) \end{aligned}$$

and become, after swapping t_k, t_ℓ,

$$\begin{aligned} &P(t_\ell)(-1)^k R(t_1, ..., t_{k-1}, t_{k+1}, ..., t_{\ell-1}, t_k, t_{\ell+1}, ..., t_m) \\ &+P(t_k)(-1)^\ell R(t_1, ..., t_{k-1}, t_\ell, t_{k+1}, ..., t_{\ell-1}, t_{\ell+1}, ..., t_m). \end{aligned}$$

Using $\ell - k - 1$ swaps of adjacent variables in each R term, the alternating property of R gives

$$\begin{aligned} &-\{P(t_\ell)(-1)^\ell R(t_1, ..., t_{k-1}, t_k, t_{k+1}, ..., t_{\ell-1}, t_{\ell+1}, ..., t_m) \\ &+P(t_k)(-1)^k R(t_1, ..., t_{k-1}, t_{k+1}, ..., t_{\ell-1}, t_\ell, t_{\ell+1}, ..., t_m)\}. \end{aligned}$$

In the remaining terms $P(t_j)(-1)^j R(t_1, t_2, ..., t_{j-1}, t_{j+1}, ..., t_m)$ with $j \neq k, \ell$, we swap t_k and t_ℓ, and use the alternating property to obtain $-P(t_j)(-1)^j R(t_1, t_2, ..., t_{j-1}, t_{j+1}, ..., t_m)$. So we have proved that $S \in \mathcal{AL}_m^n$. Moreover, as P has zeros at $x_2, x_3, ..., x_m$, we have

$$S(x_1, x_2, ..., x_m) = -P(x_1) R(x_2, x_3, ..., x_m).$$

Next, by Cauchy-Schwarz,

$$\begin{aligned} &\int S^2 d\mu^{\times m} \\ \leq \quad & m \int \sum_{j=1}^{m} P^2(t_j) R^2(t_1, ..., t_{j-1}, t_{j+1}, ..., t_m) d\mu(t_1) ... d\mu(t_m) \\ = \quad & m^2 \left(\int P^2 d\mu \right) \left(\int R^2 d\mu^{\times(m-1)} \right). \end{aligned}$$

Then (1.9) gives

$$\begin{aligned} & \det\left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m} \\ \geq\ & m! \frac{S^2\left(x_1, x_2, ..., x_m\right)}{\int S^2 d\mu^{\times m}} \\ \geq\ & \frac{m!}{m^2} \frac{P^2\left(x_1\right)}{\int P^2 d\mu} \frac{R^2\left(x_2, ..., x_m\right)}{\int R^2 d\mu^{\times(m-1)}}. \end{aligned}$$

Write

$$P\left(t\right) = P_1\left(t\right) \prod_{j=2}^{m} \left(t - x_j\right),$$

where P_1 is any polynomial of degree $\leq n-m$. Next, take sup's over P_1 of degree $\leq n-m$ and $R \in \mathcal{AL}_{m-1}^{n-1}$. Recalling the definition of ν, and (1.2) gives

$$\begin{aligned} & \det\left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m} \\ \geq\ & \frac{m!}{m^2} K_{n-m+1}\left(\nu, x_1, x_1\right) \left(\prod_{j=2}^{m} \left(x_1 - x_j\right)^2\right) \frac{1}{(m-1)!} \det\left[K_n\left(\mu, x_i, x_j\right)\right]_{2\leq i,j\leq m}. \end{aligned}$$

This gives the lower bound in (1.18). ■

4. Proof of Theorems 2.1, 2.2, and 2.3

We first prove:

Lemma 4.1
Let μ have compact support J, let μ be regular, and assume that I is a subset of the support consisting of finitely many intervals in which (2.3) holds. Let $m \geq 2$. Then for Lebesgue a.e. $(x_1, x_2, ..., x_m) \in I^m$,

$$\lim_{n\to\infty} \frac{1}{n^m} \det\left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m} = \prod_{j=1}^{m} \frac{\omega_J\left(x_j\right)}{\mu'\left(x_j\right)}. \tag{4.1}$$

Proof
We already know that for a.e. $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(\mu, x, x\right) \frac{\mu'\left(x\right)}{\omega_J\left(x\right)} = 1, \tag{4.2}$$

by Totik's result (2.5). (Formally, the integral condition (2.3) follows in each of the intervals whose union is I, and hence (2.5) does.) We next show that there is a set $\mathcal{E}$ of Lebesgue measure 0 such that for distinct $x, y \in I \backslash \mathcal{E}$, both (4.2) holds, and

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, y) \left(\frac{\mu'(x)\mu'(y)}{\omega_J(x)\omega_J(y)} \right)^{1/2} = 0. \tag{4.3}$$

These last two assertions give the result. Indeed for distinct $x_1, x_2 ... x_m \in I \backslash \mathcal{E}$, we have

$$\begin{aligned} & \frac{1}{n^m} \det [K_n(\mu, x_i, x_j)]_{1\le i,j\le m} \prod_{j=1}^m \frac{\mu'(x_j)}{\omega_J(x_j)} \\ = & \sum_\sigma \varepsilon_\sigma \prod_{i=1}^m \left(\frac{1}{n} K_n\left(\mu, x_i, x_{\sigma(i)}\right) \left(\frac{\mu'(x_i)\mu'\left(x_{\sigma(i)}\right)}{\omega_J(x_i)\omega_J\left(x_{\sigma(i)}\right)} \right)^{1/2} \right) \\ = & \prod_{i=1}^m \left(\frac{1}{n} K_n(\mu, x_i, x_i) \frac{\mu'(x_i)}{\omega_J(x_i)} \right) + o(1) = 1 + o(1), \end{aligned}$$

by (4.2) and (4.3). Of course the set of $x_1, x_2, ..., x_m$ where any two $x_i = x_j$ with $i \neq j$ has Lebesgue measure 0 in I^m.

We turn to the proof of (4.3). It follows from (4.2) that there is a set $\mathcal{E}$ of measure 0 such that for $x \in I \backslash \mathcal{E}$, we have

$$\lim_{n\to\infty} \frac{1}{n} p_n^2(x) = \lim_{n\to\infty} \frac{1}{n} \left(K_{n+1}(\mu, x, x) - K_n(\mu, x, x) \right) = 0.$$

Then for distinct x, y, the Christoffel-Darboux formula gives for $x, y \in I \backslash \mathcal{E}$,

$$\begin{aligned} & \frac{1}{n} K_n(\mu, x, y) \\ = & \frac{1}{n} \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x - y} = o(1). \end{aligned}$$

Here we are also using the fact that $\left\{ \frac{\gamma_{n-1}}{\gamma_n} \right\}$ is bounded as μ has compact support. ■

Proof of Theorem 2.1(a)

Since $J = \text{supp}[\mu]$ is compact, we can find a decreasing sequence of compact sets $\{J_\ell\}_{\ell=1}^\infty$ such that each J_ℓ consists of finitely many disjoint closed intervals, and

$$J = \bigcap_{\ell=1}^\infty J_\ell.$$

(This follows by a straightforward covering of J by open intervals, and using compactness, then closing them up; at the $(\ell+1)$st stage, we ensure that $J_{\ell+1} \subset J_\ell$ by intersecting those intervals in $J_{\ell+1}$ with those in J_ℓ). For $\ell \geq 1$, let

$$d\mu_\ell(x) = d\mu(x) + \frac{1}{\ell}\omega_{J_\ell}(x)\, dx, \tag{4.4}$$

so that we are adding a (small) multiple of the equilibrium measure for J_ℓ to μ. Because $\omega_{J_\ell} > 0$ in the interior of each J_ℓ, $\mu'_\ell > 0$ a.e. in J_ℓ, and so μ_ℓ is a regular measure [30, p. 102]. Moreover, in each compact subinterval I of the interior of J_ℓ, ω_{J_ℓ} is positive and continuous, so we have

$$\int_I \log \mu'_\ell > -\infty. \tag{4.5}$$

By Lemma 4.1, for a.e. $(x_1, x_2, ..., x_m) \in I^m$,

$$\lim_{n\to\infty} \frac{1}{n^m} \det\left[K_n\left(\mu_\ell, x_i, x_j\right)\right]_{1\leq i,j\leq m} = \prod_{j=1}^m \frac{\omega_{J_\ell}(x_j)}{\mu'_\ell(x_j)}.$$

As $\mu_\ell \geq \mu$, Corollary 1.2 gives

$$\liminf_{n\to\infty} \frac{1}{n^m} \det\left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m} \geq \prod_{j=1}^m \frac{\omega_{J_\ell}(x_j)}{\mu'_\ell(x_j)}. \tag{4.6}$$

Since a countable union of sets of the form I^m exhausts J_ℓ^m, this last relation actually holds for a.e. $(x_1, x_2, ..., x_m) \in J_\ell^m$. Now [33, Lemma 4.2] uniformly for x in compact subsets of an open set contained in J,

$$\lim_{\ell\to\infty} \omega_{J_\ell}(x) = \omega_J(x). \tag{4.7}$$

Moreover, ω_J is positive and continuous in that open set. We can now let $\ell \to \infty$ in (4.6), and use the fact that the left-hand side in (4.6) is independent of ℓ to obtain (2.8). ■

Proof of Theorem 2.1(b)

Let L be a compact subset of supp$[\mu]$ such that $\mu_{|L}$ is regular. $L = I$ is one such choice, because of the Szegő condition (2.3). We may assume that $I \subset L$, since ω_L decreases as L increases. Let

$$d\nu(x) = \mu'(x)_{|L}\, dx, \tag{4.8}$$

so that $d\nu$ is the restriction to L of the absolutely continuous part of μ. Here $\int_I \log \nu' > -\infty$, so ν satisfies the hypotheses of Lemma 4.1, while $\mu \geq \nu$, so Corollary 1.2, followed by Lemma 4.1, gives for a.e. $(x_1, x_2, ..., x_m) \in I^m$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n^m} \det \left[K_n\left(\mu, x_i, x_j\right)\right]_{1\leq i,j\leq m} \\ \leq\ &\limsup_{n\to\infty} \frac{1}{n^m} \det \left[K_n\left(\nu, x_i, x_j\right)\right]_{1\leq i,j\leq m} \\ =\ &\prod_{j=1}^{m} \frac{\omega_L\left(x_j\right)}{\mu'\left(x_j\right)}, \end{aligned}$$

recall that $\nu' = \mu'$ in $I \subset L$. Now take inf's over all such L and use the fact that the left-hand side is independent of L. ■

We turn to the

Proof of Theorem 2.2(a)

Let μ_ℓ and J_ℓ be as in the proof of Theorem 2.1(a). It then follows from results of Totik [33, Theorem 2.3] and/ or Simon [29, Thm. 5.11.13, p. 344] that for a.e. $x \in J_\ell$, and all real $a_1, a_2, .., a_m$, and $1 \leq i, j \leq m$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} K_n\left(\mu_\ell, x + \frac{a_i}{n}, x + \frac{a_j}{n}\right) \\ =\ &\frac{\omega_{J_\ell}(x)}{\mu_\ell'(x)} S\left(\left(a_i - a_j\right)\omega_{J_\ell}(x)\right). \end{aligned}$$

Consequently,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^m} R_m^n\left(\mu_\ell; x + \frac{a_1}{n}, ..., x + \frac{a_m}{n}\right) \\ =\ &\left(\frac{\omega_{J_\ell}(x)}{\mu_\ell'(x)}\right)^m \det\left(S\left(\left(a_i - a_j\right)\omega_{J_\ell}(x)\right)\right)_{1\leq i,j\leq m}. \end{aligned}$$

Now we use the fact that $\mu \leq \mu_\ell$, and Corollary 1.2: for a.e. $x \in J$, and all $a_1, a_2, ..., a_m$,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n^m} R_m^n\left(\mu; x + \frac{a_1}{n}, ..., x + \frac{a_m}{n}\right) \\ \geq\ &\left(\frac{\omega_{J_\ell}(x)}{\mu_\ell'(x)}\right)^m \det\left(S\left(\left(a_i - a_j\right)\omega_{J_\ell}(x)\right)\right)_{1\leq i,j\leq m}. \end{aligned} \tag{4.9}$$

Moreover we have (4.7). We can now let $\ell \to \infty$ in (4.9), and use the fact that the left-hand side in (4.9) is independent of ℓ to obtain (2.11), with a scale change. ■

Proof of Theorem 2.2(b)

Let L and ν be, as in the proof of Theorem 2.1(b). We can use the aforementioned results of Totik applied to ν, to obtain for a.e. $x \in I$, and real $a_1, a_2, ..., a_m$,

$$\lim_{n\to\infty} \frac{1}{n^m} R_m^n \left(\nu; x + \frac{a_1}{n}, ..., x + \frac{a_m}{n}\right)$$

$$(4.10) \qquad = \left(\frac{\omega_L(x)}{\nu'(x)}\right)^m \det\left(S\left((a_i - a_j)\,\omega_L(x)\right)\right)_{1\le i,j\le m}.$$

Now we use the fact that $\mu \geq \nu$, and that $\mu' = \nu'$ in $I \subset L$ and Corollary 1.2: for a.e. $x \in I$, and real $a_1, a_2, ..., a_m$,

$$\limsup_{n\to\infty} \frac{1}{n^m} R_m^n \left(\mu; x + \frac{a_1}{n}, ..., x + \frac{a_m}{n}\right)$$

$$\leq \left(\frac{\omega_L(x)}{\mu'(x)}\right)^m \det\left(S\left((a_i - a_j)\,\omega_L(x)\right)\right)_{1\le i,j\le m}.$$

Now choose a sequence of compact subsets L of supp$[\mu]$ such that $\omega_L(x)$ converges to the infimum $\omega_\mu(x)$. ■

Proof of Theorem 2.3

Let $\eta \in (0, C_1)$, and choose $\delta > 0$ such that in $(x - \delta, x + \delta)$,

$$C_1 - \eta \leq \mu' \leq C_2 + \eta.$$

Here μ' denotes the derivative of the absolutely continuous component of μ. Define

$$d\nu = d\mu \text{ in } J\backslash (x - \delta, x + \delta)$$

and

$$d\nu(t) = d\mu_s(t) + (C_1 - \eta)\, dt \text{ in } (x - \delta, x + \delta).$$

Then $d\nu \leq d\mu$, and ν is regular on J (see [30, Thm. 5.3.3, p.148]). Moreover, the derivative ν' of the absolutely continuous part of ν exists and equals $C_1 - \eta$ in $(x - \delta, x + \delta)$, while (2.13) implies that

$$\lim_{h\to 0} \nu_s[x - h, x + h]/h = 0.$$

By a theorem of Totik [33, Theorem 2.3], we obtain for the given x and real $a_1, a_2, ..., a_m$, that

$$\lim_{n\to\infty} \frac{1}{n^m} R_m^n \left(\nu; x + \frac{a_1}{n}, ..., x + \frac{a_m}{n}\right)$$

$$(4.11) \qquad = \left(\frac{\omega_J(x)}{C_1 - \eta}\right)^m \det\left(S\left((a_i - a_j)\,\omega_J(x)\right)\right)_{1\le i,j\le m}.$$

Note that the Lebesgue condition for the local Szegő function required by Totik is satisfied because ν' is smooth (even constant) near x. Then

Corollary 1.2 gives

$$\limsup_{n\to\infty} \frac{1}{n^m} R_m^n\left(\mu; x+\frac{a_1}{n}, ..., x+\frac{a_m}{n}\right)$$
$$\leq \left(\frac{\omega_J(x)}{C_1-\eta}\right)^m \det\left(S\left(\left(a_i-a_j\right)\omega_J(x)\right)\right)_{1\leq i,j\leq m}.$$

As the left-hand side is independent of η, we obtain

$$\limsup_{n\to\infty} \frac{1}{n^m} R_m^n\left(\mu; x+\frac{a_1}{n}, ..., x+\frac{a_m}{n}\right)$$
$$\leq \left(\frac{\omega_J(x)}{C_1}\right)^m \det\left(S\left(\left(a_i-a_j\right)\omega_J(x)\right)\right)_{1\leq i,j\leq m}.$$

The lower bound is similar. ■

5. Proof of Theorem 2.4

Let

$$w(t) = (1-t)^\alpha, t \in (-1,1).$$

Choose $\delta > 0$ such that μ is absolutely continuous in $(1-\delta, 1)$, satisfying there

$$(C_1-\delta)\, w(t) \leq \mu'(t) \leq (C_2+\delta)\, w(t).$$

Here C_1, C_2 are as in (2.18). Let

$$d\nu(t) = d\mu(t) + (C_2+\delta)\, w(t)\, dt, \text{ in } (-1, 1-\delta]$$

and

$$d\nu(t) = (C_2+\delta)\, w(t)\, dt \text{ in } (1-\delta, 1].$$

Then

$$d\nu \geq d\mu \text{ in } [-1,1].$$

Note too that in $(1-\delta, 1)$, the derivative μ' of the absolutely continuous component of μ satisfies

$$\frac{\mu'(t)}{\nu'(t)} \geq \frac{C_1-\delta}{C_2+\delta}. \tag{5.1}$$

Inasmuch as $w > 0$ in $(-1,1)$, ν is a regular measure in the sense of Stahl, Totik and Ullman, while $\nu'(t)(1-t)^{-\alpha}$ is continuous and positive at 1. By a result of the author [17, Theorem 1.2],

$$\lim_{n\to\infty} \frac{1}{2n^2} \tilde{K}_n\left(\nu, 1-\frac{a}{2n^2}, 1-\frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a,b),$$

uniformly for a, b in compact subsets of $(0, \infty)$. If $\alpha \geq 0$, we may also allow a, b to lie in compact subsets of $[0, \infty)$. Then for $m \geq 2$, Corollary 1.2 and (5.1) give for $a_1, a_2, ..., a_m > 0$,

$$\begin{aligned}
&\liminf_{n\to\infty} \left(\frac{1}{2n^2}\right)^m R_m^n \left(\mu; 1 - \frac{a_1}{2n^2}, ..., 1 - \frac{a_m}{2n^2}\right) \prod_{j=1}^m \mu' \left(1 - \frac{a_j}{2n^2}\right) \\
\geq\ & \left(\frac{C_1 - \delta}{C_2 + \delta}\right)^m \liminf_{n\to\infty} \left(\frac{1}{2n^2}\right)^m R_m^n \left(\nu; 1 - \frac{a_1}{2n^2}, ..., 1 - \frac{a_m}{2n^2}\right) \prod_{j=1}^m \nu' \left(1 - \frac{a_j}{2n^2}\right) \\
=\ & \left(\frac{C_1 - \delta}{C_2 + \delta}\right)^m \det \left(\mathbb{J}_\alpha \left(a_i, a_j\right)\right)_{1 \leq i,j \leq m}.
\end{aligned}$$

Now let $\delta \to 0+$. ■

6. Proof of Theorem 2.5 and Corollary 2.6

We begin with a lemma that uses the by now classical technique of Totik involving fast decreasing polynomials:

Lemma 6.1

Assume the hypotheses of Theorem 2.5, except that we do not assume (2.22), nor that μ is regular. Let $\varepsilon \in (0, 1)$. Then

$$\liminf_{n\to\infty} \frac{K_n^m \left(\mu, \underline{y}_n, \underline{y}_n\right)}{K_{[n(1-\varepsilon)]}^m \left(\nu, \underline{y}_n, \underline{y}_n\right)} \geq \left(\frac{\nu'(\xi)}{\mu'(\xi)}\right)^m. \tag{6.1}$$

Proof

We may assume that the common support J of μ and ν is contained in $[-1, 1]$, as a linear transformation of the variable changes the limits in a trivial way. Let $\eta > 0$, and

$$c = \frac{\mu'(\xi)}{\nu'(\xi)}.$$

Our hypothesis (2.20) ensures that we can choose $\delta > 0$ such that

$$\frac{\mu(I)}{\nu(I)} \leq (c + \eta) \text{ for } I \subset [\xi - \delta, \xi + \delta]. \tag{6.2}$$

Let $n \geq \frac{4}{\varepsilon}$ and $\ell = \ell(n) = \left[\frac{\varepsilon}{2} n\right]$, so that $n - \ell \geq [n(1 - \varepsilon)]$. We may choose a polynomial R_ℓ of degree $\leq \ell$ and $\kappa \in (0, 1)$ such that

$$0 \leq R_\ell \leq 1 \text{ in } [-2, 2];$$

$$|R_\ell(t) - 1| \leq \kappa^\ell \text{ in } \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]; \tag{6.3}$$

$$|R_\ell(t)| \le \kappa^\ell \text{ in } [-2,-\delta] \cup [\delta, 2]. \tag{6.4}$$

The crucial thing here is that κ is independent of ℓ, depending only on δ. These polynomials are easily constructed from the approximations to the sign function of Ivanov and Totik [14, Theorem 3, p. 3]. For the given ξ and n, we let

$$\Psi_n(\underline{t}) = \Psi_n(t_1, t_2, ..., t_m) = \prod_{j=1}^{m} R_\ell(\xi - t_j).$$

Observe that this is a symmetric polynomial in $t_1, t_2, ..., t_m$. Moreover, for large enough n, we have from (2.21), (6.3), and (6.4),

$$\Psi_n\left(\underline{y}_n\right) \ge \left(1-\kappa^\ell\right)^m; \tag{6.5}$$

$$|\Psi_n(\underline{t})| \le \kappa^l \text{ in } [-1,1]^m \backslash \mathbb{Q}, \tag{6.6}$$

where

$$\mathbb{Q} = \left\{(t_1, t_2, ..., t_m) : \max_{1\le j\le m} |\xi - t_j| \le \delta\right\}.$$

Next, let $P_1 \in \mathcal{AL}^m_{n-\ell-1}$, and set $P = P_1\Psi_n$. We see that $P \in \mathcal{AL}^m_{n-1}$. Using (6.2), (6.6), we see that

$$\begin{aligned} &\int P^2 d\mu^{\times m} \\ \le\ & (c+\eta)^m \int_{\mathbb{Q}} P_1^2 d\nu^{\times m} + \|P_1\|^2_{L_\infty(J^m)} \kappa^{2\ell} \int_{J^m\backslash\mathbb{Q}} d\mu^{\times m}. \end{aligned} \tag{6.7}$$

Now we use the regularity of ν, and the fact that J is regular for the Dirichlet problem. These properties imply that [30, Thm. 3.2.3(v), p. 68]

$$\lim_{n\to\infty} \left(\sup_{\deg(T)\le n} \frac{\|T\|^2_{L_\infty(J)}}{\int |T^2|\, d\nu}\right)^{1/n} = 1.$$

The sup is taken over all univariate polynomials T of degree at most n. By successively applying this in each of the m variables, we see that

$$\|P_1\|^2_{L_\infty(J^m)} \le (1+o(1))^n \int P_1^2 d\nu^{\times m},$$

where the $o(1)$ term is crucially independent of P_1. Thus we may continue (6.7) as

$$\begin{aligned} &\int P^2 d\mu^{\times m} \\ \le\ & (c+\eta)^m \left(\int P_1^2 d\nu^{\times m}\right)\left(1 + (1+o(1))^n \kappa^{n\varepsilon}\right). \end{aligned}$$

Since also

$$P^2\left(\underline{y}_n\right) \geq P_1^2\left(\underline{y}_n\right)\left(1+O\left(\kappa^{\varepsilon n}\right)\right),$$

we see from (3.5), with an appropriate choice of P_1, that

$$\begin{aligned}
&K_n^m\left(\mu, \underline{y}_n, \underline{y}_n\right) \\
\geq\ & \frac{P^2\left(\underline{y}_n\right)}{\int P^2 d\mu^{\times m}} \\
\geq\ & \sup_{P_1 \in \mathcal{AL}^m_{n-\ell-1}} \frac{P_1^2\left(\underline{y}_n\right)\left(1+O\left(\kappa^{\varepsilon n}\right)\right)}{(c+\eta)^m\left(\int P_1^2 d\nu^{\times m}\right)\left(1+(1+o(1))^n \kappa^{n\varepsilon}\right)} \\
=\ & \frac{1+o(1)}{(c+\eta)^m} K^m_{n-\ell}\left(\nu, \underline{y}_n, \underline{y}_n\right).
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} \frac{K_n^m\left(\mu, \underline{y}_n, \underline{y}_n\right)}{K^m_{[n(1-\varepsilon)]}\left(\nu, \underline{y}_n, \underline{y}_n\right)} \geq (c+\eta)^{-m}.$$

As the left-hand side is independent of η, we obtain (6.1). ■

Proof of Theorem 2.5

Lemma 6.1 asserts that

$$\liminf_{n\to\infty} \frac{K_n^m\left(\mu, \underline{y}_n, \underline{y}_n\right)}{K^m_{[n(1-\varepsilon)]}\left(\nu, \underline{y}_n, \underline{y}_n\right)} \geq \left(\frac{\nu'(\xi)}{\mu'(\xi)}\right)^m.$$

Swapping the roles of μ and ν, Lemma 6.1 also gives

$$\liminf_{n\to\infty} \frac{K^m_{[n(1+\varepsilon)]}\left(\nu, \underline{y}_n, \underline{y}_n\right)}{K_n^m\left(\mu, \underline{y}_n, \underline{y}_n\right)} \geq \left(\frac{\mu'(\xi)}{\nu'(\xi)}\right)^m.$$

Now we apply our hypothesis (2.22) and let $\varepsilon \to 0+$. ■

Proof of Corollary 2.6

We apply Theorem 2.5 with $\xi = x$ and for $n \geq 1$,

$$\underline{y}_n = \left(x + \frac{a_1}{n\omega_J(x)}, ..., x + \frac{a_m}{n\omega_J(x)}\right).$$

This satisfies (2.21) with $\xi = x$. Now $\det\left[S\left(a_i - a_j\right)\right]_{1\leq 1, j\leq m} > 0$, so our hypothesis (2.24) easily implies (2.22). Then Theorem 2.5 gives the result. ■

References

[1] J. Baik, T.Kriecherbauer, K.McLaughlin and P.Miller, *Uniform asymptotics for polynomials orthogonal with respect to a general class of discrete weights and universality results for associated Ensembles,* Annals of Math. Studies No. 164, Princeton University Press, Princeton, 2007.

[2] J. Baik, L. Li, T. Kriecherbauer, K. McLaughlin, C. Tomei, *Proceedings of the Conference on Integrable Systems, Random Matrices and Applications,* Contemporary Mathematics, Vol. 458, American Mathematical Society, 2008.

[3] E. Beckenbach, R. Bellman, *Inequalities,* Springer, Berlin, 1961.

[4] J. Breuer, *Sine kernel asymptotics for a class of singular measures*, J. Approx. Theory 163 (2011), 1478–1491.

[5] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[6] P. Deift, D. Gioev, *Random Matrix Theory: Invariant Ensembles and Universality,* Courant Institute Lecture Notes, Vol. 18, New York University Pres, New York, 2009.

[7] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Uniform Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory,* Communications in Pure and Applied Mathematics, 52(1999), 1335-1425.

[8] L. Erdős, *Universality of Wigner Random Matrices: A Survey of Recent Results*, Russian Math. Surveys, 66 (3) 67–198.

[9] L. Erdős, J. Ramirez, B. Schlein, T. Tao, Van Vu, H-T. Yau, *Bulk universality for Wigner Hermitian Matrices with Subexponential Decay*, Math. Res. Lett. 17 (2010), 667–674.

[10] L. Erdős, B. Schlein, H-T. Yau, *Universality of Random Matrices and Local Relaxation Flow*, Invent. Math. 185 (2011), 75–119.

[11] E. Findley, *Universality for Regular Measures satisfying Szegő's Condition,* J. Approx. Theory, 155 (2008), 136–154.

[12] P. Forrester, *Log-Gases and Random Matrices*, Princeton University Press, Princeton, 2010.

[13] A. Foulquie Moreno, A. Martínez-Finkelshtein, V.L. Sousa, *Asymptotics of orthogonal polynomials for a weight with a jump on* $[-1,1]$, Constr. Approx., 33(2011), 219-263.

[14] K. Ivanov and V. Totik, *Fast Decreasing Polynomials,* Constr. Approx., 6(1990), 1-20.

[15] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations from the Modified Jacobi Unitary Ensemble*, International Maths. Research Notices, 30(2002), 1575-1600.

[16] Eli Levin and D.S. Lubinsky, *Universality Limits in the Bulk for Varying Measures,* Advances in Mathematics, 219(2008), 743-779.

[17] D.S. Lubinsky, *A New Approach to Universality Limits at the Edge of the Spectrum*, Contemporary Mathematics, 458(2008), 281-290.

[18] D.S. Lubinsky, *Universality Limits at the Hard Edge of the Spectrum for Measures with Compact Support,* International Mathematics Research Notices, (2008) 2008: rnn099-39.

[19] D.S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, Annals of Mathematics, 170(2009), 915-939.

[20] D.S. Lubinsky, *Universality Limits for Random Matrices and de Branges Spaces of Entire Functions*, Journal of Functional Analysis, 256(2009), 3688-3729.

[21] D.S. Lubinsky, *Bulk Universality Holds in Measure for Compactly Supported Measures*, to appear in J. d' Analyse de Mathematique.

[22] A. Maté, P. Nevai, and V. Totik, *Szegő's Extremum Problem on the Unit Circle*, Annals of Mathematics, 134(1991), 433-453.

[23] M.L. Mehta, *Random Matrices*, 2nd edn., Academic Press, Boston, 1991.

[24] P. Nevai, *Geza Freud, Orthogonal Polynomials and Christoffel Functions. A Case Study*, J. Approx. Theory, 48(1986), 3-167.

[25] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[26] E.B. Saff, V. Totik, *Logarithmic Potentials with External Fields*, Springer, New York, 1997.

[27] B. Simon, *Two Extensions of Lubinsky's Universality Theorem*, J. d'Analyse Mathematique, 105 (2008), 345-362.

[28] B. Simon, *The Christoffel-Darboux Kernel*, (in) Perspectives in PDE, Harmonic Analysis and Applications, a volume in honor of V.G. Maz'ya's 70th birthday, Proceedings of Symposia in Pure Mathematics, 79 (2008), 295-335.

[29] B. Simon, *Szegő's Theorem and its Descendants*, Princeton University Press, Princeton, 2011.

[30] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.

[31] T.Tao, V. Vu, *The Wigner-Dyson-Mehta bulk universality conjecture for Wigner matrices*, manuscript.

[32] V. Totik, *Asymptotics for Christoffel Functions for General Measures on the Real Line*, J. d' Analyse de Mathematique, 81(2000), 283-303.

[33] V. Totik, *Universality and fine zero spacing on general sets*, Arkiv för Matematik, 47(2009), 361-391.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu