HOW POLES OF ORTHOGONAL RATIONAL FUNCTIONS AFFECT THEIR CHRISTOFFEL FUNCTIONS

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Abstract. We show that even a relatively small number of poles of a sequence of orthogonal rational functions approaching the interval of orthogonality, can prevent their Christoffel functions from having the expected asymptotics. We also establish a sufficient condition on the rate for such asymptotics, provided the rate of approach of the poles is sufficiently slow. This provides a supplement to recent results of the authors where poles were assumed to stay away from the interval of orthogonality.

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1. Introduction

Let μ be a finite positive Borel measure on $[-1,1]$, with infinitely many points in its support. Then we can define orthonormal polynomials $p_n(x) = p_n(d\mu, x) = \gamma_n x^n + ..., n \geq 0$, satisfying

$$\int_{-1}^{1} p_n p_m d\mu = \delta_{mn}.$$

We say the measure μ is *regular on* $[-1,1]$ *in the sense of Stahl, Totik, and Ullmann*, or just *regular* [5], if

$$\lim_{n\to\infty} \gamma_n^{1/n} = 2.$$

An equivalent definition involves norms of polynomials of degree $\leq n$:

$$\lim_{n\to\infty} \left[\sup_{\deg(P)\leq n} \frac{\|P\|^2_{L_\infty[-1,1]}}{\int_{-1}^{1} |P|^2 d\mu} \right]^{1/n} = 1.$$

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Regularity of a measure is useful in studying asymptotics of orthogonal polynomials. One simple criterion for regularity is that $\mu' > 0$ a.e. on $[-1, 1]$, the so-called Erdős-Turán condition. However, there are pure jump measures, and pure singularly continuous measures that are regular.

We define the nth *Christoffel function* for μ

$$\lambda_n (d\mu, x) = 1 / \sum_{j=0}^{n-1} p_j^2 (d\mu, x) ,$$

which satisfies the extremal property

$$\lambda_n (d\mu, x) = \inf_{\deg(P) \leq n-1} \frac{\int |P|^2 \, d\mu}{|P(x)|^2}.$$

A classical result of Maté, Nevai, and Totik [4] (see also [6]) asserts that if μ is regular on $[-1, 1]$, and in some subinterval $[a, b]$

$$\int_a^b \log \mu' > -\infty,$$

then for a.e. $x \in [a, b]$,

$$\lim_{n \to \infty} n\lambda_n (d\mu, x) = \pi\mu' (x) \sqrt{1 - x^2}.$$

If instead we assume that μ is regular in $[-1, 1]$, while μ is absolutely continuous in a neighborhood of some $x \in (-1, 1)$, and μ' is continuous at x, then this last limit holds at x.

The aim of this paper is to further investigate asymptotic behavior of Christoffel functions, associated with orthogonal rational functions. The monograph [2] provides a comprehensive study of the theory of orthogonal rational functions.

We shall assume that we are given a sequence of extended complex numbers that will serve as our poles

$$A = \{\alpha_1, \alpha_2, \alpha_3, ...\} \subset \bar{\mathbb{C}} \backslash [-1, 1] .$$

We let $\pi_0 (x) = 1$, and for $k \geq 1$,

$$\pi_k (x) = \prod_{j=1}^{k} (1 - x/\alpha_j) .$$

We let $\mathcal{P}_k$ denote the set of polynomials of degree $\leq k$, and define nested spaces of rational functions by $\mathcal{L}_{-1} = \{0\} ; \mathcal{L}_0 = \mathbb{C}$; and for $k \geq 1$,

$$\mathcal{L}_k = \mathcal{L}_k \{\alpha_1, \alpha_2, ..., \alpha_k\} = \left\{ \frac{P}{\pi_k} : \deg (P) \leq k \right\}.$$

Note that if all $\alpha_j = \infty$, then $\mathcal{L}_k = \mathcal{P}_k$. Moreover, $\mathcal{L}_{k-1} \subset \mathcal{L}_k$ for $k \geq 1$.

We define orthonormal rational functions φ_0, φ_1, φ_2, ... corresponding to the measure μ, such that $\varphi_k \in \mathcal{L}_k \backslash \mathcal{L}_{k-1}$, and

$$\int_{-1}^{1} \varphi_j \ \overline{\varphi_k} \ d\mu = \delta_{jk}.$$

We define the rational Christoffel functions

$$\lambda_n^r (d\mu, x) = 1 / \sum_{j=0}^{n-1} \left| \varphi_j (x) \right|^2 .$$

They admit an extremal property analagous to that for orthogonal polyonomials, namely

$$\lambda_n^r (d\mu, x) = \inf_{R \in \mathcal{L}_{n-1}} \frac{\int_{-1}^{1} |R|^2 \, d\mu}{|R(x)|^2}.$$

We shall often use the abbreviation $\lambda_n^r (x)$, when it is clear that the measure involved is μ.

In a recent paper [3], we proved the following asymptotics of rational Christoffel functions:

Theorem 1.1
Let μ be a regular measure on $[-1,1]$. Let I be an open subinterval of $(-1,1)$ in which μ is absolutely continuous. Assume that μ' is positive and continuous at a given $x \in I$. Let $A = \{\alpha_1, \alpha_2, \alpha_3, ...\} \subset \bar{\mathbb{C}} \backslash [-1,1]$. Assume that for some $\eta > 0$, the poles $\{\alpha_j\}$ satisfy for all $j \geq 1$,

$$\operatorname{dist}(\alpha_j, [-1,1]) \geq \eta. \tag{1.1}$$

Assume moreover, that the poles have an asymptotic distribution ν with support in $\bar{\mathbb{C}} \backslash [-1,1]$, so that the pole counting measures

$$\nu_n = \frac{1}{n} \left(\delta_\infty + \sum_{j=1}^{n-1} \delta_{\alpha_j} \right) \tag{1.2}$$

satisfy

$$\nu_n \xrightarrow{*} \nu \text{ as } n \to \infty. \tag{1.3}$$

Then

$$\lim_{n\to\infty} n \lambda_n^r (x) = \mu'(x) \, \pi \sqrt{1-x^2} / \int \operatorname{Re} \left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t). \tag{1.4}$$

Here the branch of the square root is chosen so that $\sqrt{t^2-1} > 0$ for $t \in (1,\infty)$. If μ' is positive and continuous in I, then this last limit also holds uniformly for x in compact subsets of I.

The weak convergence (1.3) is assumed to mean that

$$\lim_{n\to\infty}\int h\ d\nu_n = \int h\ d\nu$$

for all functions h that are continuous in $\overline{\mathbb{C}}$. In [3], (1.3) was assumed in the equivalent form

$$\lim_{k\to\infty}\log\left|\pi_{k-1}(y)\right|^{1/k} = \int \log\left|1-y/t\right| d\nu(t),$$

for $y \in [-1,1]$.

In the special case when all poles are at ∞ (so, $\nu = \delta_\infty$), (1.4) reduces to the classical limits for Christoffel functions for orthogonal polynomials. For varying weights, (1.4) would contain an appropriate equilibrium density.

Note the key restriction that the poles stay away from $[-1,1]$. In some results on asymptotics of orthogonal rational functions [1], such a restriction has been replaced by a Blaschke type assumption that

$$\sum_{j=1}^{\infty}\left(1-\left|\beta_j\right|\right) = \infty, \tag{1.5}$$

where $\left|\beta_j\right| < 1$ is determined by the equation

$$\alpha_j = \frac{1}{2}\left(\beta_j + \beta_j^{-1}\right).$$

So (1.5) may also be formulated as

$$\sum_{j=1}^{\infty}\left(1-\left|\alpha_j-\sqrt{\alpha_j^2-1}\right|\right) = \infty.$$

One of the lessons of this paper, is that even such a restriction is not enough to guarantee the expected asymptotics for Christoffel functions. Our first result shows that even a negligible proportion of poles, located sufficiently close to $[-1,1]$, can destroy (1.4) at every point of $(-1,1)$. We use the Chebyshev weight of the second kind because of the explicit formulae available for Christoffel functions for Bernstein-Szegő weights.

Theorem 1.2
Let μ be the Chebyshev measure of the second kind,

$$\mu'(x) = \sqrt{1-x^2}, x \in (-1,1). \tag{1.6}$$

Let ν be a measure with support in $\bar{\mathbb{C}}$. Then we may choose a sequence of poles $\{\alpha_j\}$ in $\bar{\mathbb{C}}\backslash[-1,1]$, that have asymptotic distribution ν, but such that for all $x \in (-1,1)$,

$$\liminf_{n\to\infty} n\lambda_n^r(x) = 0. \tag{1.7}$$

Remarks

(a) Most of the poles in the proof are chosen only to satisfy the distribution (1.3). We choose an increasing sequence $\{k_n\}$ of positive integers, with

$$\lim_{n\to\infty} k_n/n = \infty$$

but

$$\sum_{n=1}^{\infty} \frac{1}{k_n} = \infty$$

and then choose the real part of α_{k_n} in a suitable way to traverse $[-1,1]$, while

$$\lim_{n\to\infty} k_n \left(\operatorname{Im} \alpha_{k_n}\right) = 0.$$

The remaining $\{\alpha_j\}$ are chosen to satisfy (1.3). In particular, if $k_n = [n \log n]$, the poles α_{k_n} may approach $[-1,1]$ with rate scarcely faster than $O\left(\frac{1}{k_n}\right)$.

(b) For poles that approach $[-1,1]$ arbitrarily slowly, we can still ensure that (1.4) is violated on a dense sequence of points:

Theorem 1.3

Let μ be the Chebyshev measure of the second kind, given by (1.6). Let $\{\eta_j\}$ be a sequence of positive numbers with limit 0, and $\mathcal{S}$ be a countable set in $(-1,1)$. Let ν be a measure with support in $\bar{\mathbb{C}} \backslash [-1,1]$. Then we may choose a sequence of poles $\{\alpha_j\}$ in $\bar{\mathbb{C}} \backslash [-1,1]$, that have asymptotic distribution ν, such that

$$\mathit{dist}\left(\alpha_j, [-1,1]\right) \geq \eta_j \text{ for all } j \geq 1 \tag{1.8}$$

and such that for all $x \in \mathcal{S}$,

$$\liminf_{n\to\infty} n\lambda_n^r(x) = 0.$$

It seems unlikely that the result in Theorem 1.3 can hold for all $x \in (-1,1)$ without assuming more on $\{\eta_j\}$.

We now present a technical sufficient condition for convergence of the Christoffel functions when the poles are allowed to approach $[-1,1]$:

Theorem 1.4

Let μ be the Chebyshev measure of the second kind, and let ν be a measure with support in $\bar{\mathbb{C}}$ such that $\nu\left(\bar{\mathbb{C}} \backslash [-1,1]\right) > 0$. Assume that the poles $\{\alpha_j\}$ have asymptotic distribution ν. Fix $x \in (-1,1)$, and assume that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j\leq n: |\alpha_j - x| \leq \delta} \frac{|\operatorname{Im} \alpha_j|}{|x-\alpha_j|^2} < \varepsilon. \tag{1.9}$$

Then (1.4) holds at x.

Corollary 1.5

Assume the hypotheses of Theorem 1.4, except that instead of (1.9), we assume that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j\leq n:|\alpha_j - x|\leq\delta} \frac{1}{|\operatorname{Im}\alpha_j|} < \varepsilon. \tag{1.10}$$

Then (1.4) holds at x.

Remarks

(a) One can reformulate (1.9) as

$$\lim_{\delta\to 0+}\left(\limsup_{n\to\infty}\int_{\{t:|t-x|\leq\delta\}}\frac{|\operatorname{Im} t|}{|t-x|^2}d\nu_n(t)\right)=0,$$

where ν_n is the pole counting measure defined by (1.2).
(b) If $\nu\left(\bar{\mathbb{C}}\backslash[-1,1]\right)=0$, so that ν is supported on $[-1,1]$, it is possible that (1.4) holds in the form

$$\lim_{n\to\infty} n\lambda_n^r(x)=\infty, \text{ for } x\notin\operatorname{supp}[\nu].$$

(c) One can also allow the poles to change with n in Theorem 1.4, so that instead of a fixed sequence $\{\alpha_j\}$, at the nth stage, we have $\{\alpha_{n,j}\}_{j=1}^n$.

Theorem 1.4 admits an extension to a larger class of measures:

Theorem 1.6

Let $g:[-1,1]\to[0,\infty)$ be measurable. Assume there is a polynomial U such that gU and $g^{-1}U$ are bounded in $[-1,1]$. Let μ be the absolutely continuous measure with

$$\mu'(t)=g(t)\sqrt{1-t^2}, t\in(-1,1),$$

and assume that μ' is integrable. Assume that the poles $\{\alpha_j\}$ have asymptotic distribution ν. Fix $x\in(-1,1)$, and assume that given $\varepsilon>0$, there exists $\delta>0$ such that (1.9) holds, while g is positive and continuous at x. Assume, moreover, that there exists $\eta>0$ such that (1.1) holds for infinitely many j. Then (1.4) holds.

For example a generalized Jacobi weight

$$\mu'(t)=h(t)\prod_{j=1}^m|t-a_j|^{b_j},$$

where h is positive and continuous in $[-1,1]$, and $\{a_j\}$ are distinct points in $[-1,1]$, while all $b_j>-1$, satisfies the hypotheses of Theorem 1.6. Of course, this is far less general than the regular measures considered in Theorem 1.1, but there is a major technical problem when the poles are allowed to approach $[-1,1]$: it is no longer necessarily true that

$$\|R_n\|_{L_\infty[-1,1]}^{1/n}\Big/\left(\int|R_n|^2\,d\mu\right)^{1/2n}\to 1 \text{ as } n\to\infty,$$

for sequences $\{R_n\}$ with $R_n \in \mathcal{L}_n\{\alpha_1, \alpha_2, ..., \alpha_n\}$. It is in dealing with a weaker form of this condition, that we need that infinitely many poles $\{\alpha_j\}$ avoid $[-1,1]$, though it does not matter how sparse they are.

We prove Theorems 1.2 and 1.3 in Section 2, and Theorem 1.4 and Corollary 1.5 in Section 3. Theorem 1.6 is proved in Section 4.

2. Proof of Theorems 1.2, 1.3

We need from [3]:

Lemma 2.1
Assume that μ is the Chebyshev measure of the second kind, so that

$$\mu'(x) = \sqrt{1-x^2},\ x \in (-1,1).$$

Let $A = \{\alpha_1, \alpha_2, \alpha_3, ...\} \subset \bar{\mathbb{C}} \backslash [-1,1]$. Let $[a,b] \subset (-1,1)$. Then uniformly for x in $[a,b]$, as $n \to \infty$,

$$\frac{\pi}{n} \lambda_n^r(x)^{-1} \mu'(x) \sqrt{1-x^2} = \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu_n(t) + O\left(\frac{1}{n}\right). \tag{2.1}$$

Remarks
(a) This lemma does not require the poles to be a fixed distance away from $[-1,1]$, nor does it require weak convergence of $\{\nu_n\}$. Moreover, the order term does not depend on the particular choice of $\{\pi_n\}$. It depends only on the size of $\frac{1}{\sqrt{1-x^2}}$.
(b) Similarly as in Theorem 1.1, the branch of the square root in (2.1) is chosen so that $\sqrt{t^2-1} > 0$ for $t \in (0,\infty)$. Note that in this way, $\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} \geqslant 0$ for every $t \in \bar{\mathbb{C}}$ and every $x \in [-1,1]$.
Proof
In [3, Lemma 3.3], this lemma is stated for Christoffel functions associated with orthogonal polynomials, in the form

$$\frac{\pi}{n} \lambda_n^{-1}(d\mu_n, x) \mu_n'(x) \sqrt{1-x^2} = \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu_n(t) + O\left(\frac{1}{n}\right),$$

where

$$\mu_n'(t) = \mu'(t) / |\pi_{n-1}(t)|^2,\ t \in (-1,1).$$

Now apply Lemma 2.1 in [3], which asserts that

$$\lambda_n^r(x) = \lambda_n(d\mu_n, x) |\pi_{n-1}(x)|^2.$$

■

Proof of Theorem 1.2
Let $\{k_n\}$ be an increasing sequence of positive integers such that

$$\lim_{n\to\infty} k_n/n = \infty, \tag{2.2}$$

but still

$$\sum_{n=1}^{\infty} \frac{1}{k_n} = \infty.$$

For example, $k_n = [(n+1)\log(n+1)]$, $n \geq 1$, would do. Now choose a sequence of positive numbers $\{\delta_n\}$ such that

$$\lim_{n \to \infty} \delta_n = 0,$$

but still

$$\sum_{n=1}^{\infty} \frac{\delta_n}{k_n} = \infty.$$

We shall choose

$$\alpha_{k_n} = t_n + i\frac{\delta_n}{k_n},\ n \geq 1, \tag{2.3}$$

where the $\{t_n\}$ will be chosen inductively below. First, we show that the $\{\alpha_{k_n}\}$ are so sparse in the set of poles that they do not affect the asymptotic distribution ν of $\{\alpha_j\}$. Indeed

$$\ell_n = \#\{j : k_j \leq n\}$$

satisfies

$$\ell_n = o(n) \text{ as } n \to \infty.$$

To see this, observe that

$$\frac{k_{\ell_n}}{\ell_n} \leq \frac{n}{\ell_n}$$

and now use (2.2). We choose $\{\alpha_j : j \notin \{k_n\}\}$ in any way that satisfies the weak convergence of $\{\nu_n\}$ to ν. This can be done by a fairly standard discretisation of ν.

Now we proceed to choose $\{t_n\}$. We let I_n denote a half-open interval of the form $[a, c)$, with length $\frac{\delta_n}{k_n}$, and center t_n (which still has to be chosen). The essential feature is that for any N,

$$\sum_{j=N}^{\infty} \frac{\delta_j}{k_j} = \infty, \tag{2.4}$$

so we can choose finitely many disjoint $\{I_j\}$ with $j \geq N$, whose sum of lengths exceed 2, and hence can be used to cover $[-1, 1)$.

Let us now describe this in more detail. Let I_1 have left endpoint -1, I_2 have left endpoint that is the right endpoint of I_1, and so on, until we reach the right endpoint 1 of $[-1, 1)$. This will be possible because of (2.4). Thus for some N_1, we are choosing adjacent disjoint intervals $\{I_j\}_{j=1}^{N_1}$ that cover $[-1, 1)$. Now we start again, choosing I_{N_1+1} with left endpoint -1, I_{N_1+2} with left endpoint that is the right endpoint of I_{N_1+1}, and so on, until we reach the right endpoint 1 of $[-1, 1)$. Thus for some N_2, we are choosing adjacent disjoint intervals $\{I_j\}_{j=N_1+1}^{N_2}$ that cover $[-1, 1)$. We continue this inductively, obtaining a sequence of intervals $\{I_j\}_{j=1}^{\infty} = \bigcup_{k=0}^{\infty} \{I_j\}_{j=N_k+1}^{N_{k+1}}$ where $N_0 = 0$ and $\{I_j\}_{j=N_k+1}^{N_{k+1}}$ are disjoint intervals covering $[-1, 1)$.

Now fix $x \in (-1, 1)$. Then for infinitely many n, say for $n \in \mathcal{N}$, we have $x \in I_n$, so for such n

$$|x - t_n| \leq \frac{1}{2} \frac{\delta_n}{k_n}. \tag{2.5}$$

Because $\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} \geq 0$ in the integral below, Lemma 2.1 yields

$$\begin{aligned} \frac{\pi}{k_n} \lambda_{k_n}^r (x)^{-1} \mu'(x) \sqrt{1 - x^2} &= \int \operatorname{Re}\left\{\frac{\sqrt{t^2 - 1}}{t - x}\right\} d\nu_{k_n}(t) + O\left(\frac{1}{k_n}\right) \\ &\geq \frac{1}{k_n} \operatorname{Re}\left\{\frac{\sqrt{\alpha_{k_n}^2 - 1}}{\alpha_{k_n} - x}\right\} + O\left(\frac{1}{k_n}\right). \end{aligned}$$

(2.6)

Now as $n \to \infty$ through $\mathcal{N}$, $t_n \to x$. Then, recalling (2.3),

$$\begin{aligned} &\alpha_{k_n}^2 - 1 \\ = \; & t_n^2 - 1 - \left(\frac{\delta_n}{k_n}\right)^2 + 2it_n \frac{\delta_n}{k_n} \\ = \; & -\left(1 - t_n^2\right)\left(1 - \frac{2it_n}{1 - t_n^2} \frac{\delta_n}{k_n} + O\left(\left(\frac{\delta_n}{k_n}\right)^2\right)\right), \end{aligned}$$

so

$$\sqrt{\alpha_{k_n}^2 - 1} = i\sqrt{1 - t_n^2}\left(1 - \frac{it_n}{1 - t_n^2} \frac{\delta_n}{k_n} + O\left(\left(\frac{\delta_n}{k_n}\right)^2\right)\right).$$

Then

$$\begin{aligned} &\operatorname{Re}\left\{\frac{\sqrt{\alpha_{k_n}^2 - 1}}{\alpha_{k_n} - x}\right\} \\ = \; & \frac{1}{|\alpha_{k_n} - x|^2} \operatorname{Re}\left\{i\sqrt{1 - t_n^2}\left(1 - \frac{it_n}{1 - t_n^2} \frac{\delta_n}{k_n} + O\left(\left(\frac{\delta_n}{k_n}\right)^2\right)\right)\left(t_n - x - i\frac{\delta_n}{k_n}\right)\right\} \\ = \; & \frac{1}{(t_n - x)^2 + \left(\frac{\delta_n}{k_n}\right)^2}\left\{\sqrt{1 - t_n^2} \frac{\delta_n}{k_n} + O\left(\left(\frac{\delta_n}{k_n}\right)^2\right)\right\}, \end{aligned}$$

by (2.5). We continue this as

$$\begin{aligned}&\operatorname{Re}\left\{\frac{\sqrt{\alpha_{k_n}^2-1}}{\alpha_{k_n}-x}\right\}\\ \geq\ &\frac{4}{5\left(\frac{\delta_n}{k_n}\right)^2}\sqrt{1-x^2}\frac{\delta_n}{k_n}\left(1+o\left(1\right)\right)\geq C\frac{k_n}{\delta_n},\end{aligned}$$

where C depends on x, but not on $n\in\mathcal{N}$. Substituting this into (2.6) gives

$$\frac{\pi}{k_n}\lambda_{k_n}^r\left(x\right)^{-1}\mu'\left(x\right)\sqrt{1-x^2}\geq\frac{C}{\delta_n},$$

so that

$$\lim_{n\to\infty,n\in\mathcal{N}}\frac{1}{k_n}\lambda_{k_n}^r\left(x\right)^{-1}=\infty.$$

This yields (1.7). ■

Proof of Theorem 1.3

Let us choose a sequence $\{\tau_n\}$ in which each element of $\mathcal{S}$ is repeated infinitely often. We shall place multiple poles at each τ_n in such a way that the pole distribution ν of $\{\alpha_j\}$ is not affected. To this end, we shall choose a rapidly increasing sequence of integers $\{k_n\}$, and corresponding quantities

$$\eta_n^*=\max\left\{\eta_j:k_n/2\leq j\leq k_n\right\},\tag{2.7}$$

and

$$\ell_n=\left[k_n\left(1-\sqrt{\eta_n^*}\right)\right].\tag{2.8}$$

Also, we set

$$\alpha_j=\tau_n+i\eta_n^*,\text{ for }\ell_n+1\leq j\leq k_n,\tag{2.9}$$

so that we are placing $k_n-\ell_n$ poles at $\tau_n+i\eta_n^*$. The remaining poles are chosen only to ensure the asymptotic distribution ν.

We turn to the choice of $\{k_n\}$. Choose $k_1\geq 4$ so large that

$$\sqrt{\eta_1^*}\leq 1/2.$$

Having chosen $k_1,k_2,...,k_{n-1}$, and having defined $\eta_1^*,\eta_2^*,..,\eta_{n-1}^*$ and $\ell_1,\ell_2,...,\ell_{n-1}$ as above, we choose k_n so large that

$$\sum_{j=1}^{n-1}\left(k_j-\ell_j\right)\leq\frac{1}{\log n}k_n.\tag{2.10}$$

This condition is designed to ensure that the proportion of poles assigned by (2.9) does not affect the asymptotic distribution of poles. In this way, we can choose the sequence $k_1,k_2,k_3,...$.

Now we verify that (2.10) fulfils its stated role. Let $N\geq k_1$ and choose n such that

$$k_n\leq N<k_{n+1}.$$

The total number of poles $\alpha_j, j \leq N$, chosen according to (2.9), is at most

$$\begin{aligned} T_n &= \sum_{j=1}^{n} (k_j - \ell_j) + \max\{0, N - \ell_{n+1}\} \\ &\leq \frac{1}{\log n} k_n + k_n \sqrt{\eta_n^*} + 1 + \max\{0, N - \ell_{n+1}\} \\ &\leq N \left(\frac{1}{\log n} + \sqrt{\eta_n^*} + \frac{1}{N} \right) + \max\{0, N - \ell_{n+1}\}. \end{aligned}$$

Here if $N \leq \ell_{n+1}$, we already have $o(N)$ such poles. If $N \geq \ell_{n+1}$, then

$$k_{n+1} > N \geq k_{n+1} \left(1 - \sqrt{\eta_{n+1}^*} \right) - 1 \geq k_{n+1}/2,$$

for large n, so

$$\begin{aligned} & N - \ell_{n+1} \\ \leq\ & k_{n+1} - \ell_{n+1} \\ \leq\ & \sqrt{\eta_{n+1}^*} k_{n+1} + 1 \\ \leq\ & \sqrt{\eta_{n+1}^*} 2N + 1 = o(N). \end{aligned}$$

Thus in all cases, the total number of poles $\alpha_j, j \leq N$, chosen according to (2.9), is $o(N)$.

Now fix some $x \in \mathcal{S}$. We have $x = \tau_n$ for infinitely many n, say for $n \in \mathcal{N}$. By Lemma 2.1, we see that for such n,

$$\begin{aligned} \frac{\pi}{k_n} \lambda_{k_n}^r (x)^{-1} \mu'(x) \sqrt{1 - x^2} &= \int \operatorname{Re} \left\{ \frac{\sqrt{t^2 - 1}}{t - x} \right\} d\nu_{k_n}(t) + O\left(\frac{1}{k_n} \right) \\ &\geq \frac{k_n - \ell_n}{k_n} \operatorname{Re} \left\{ \frac{\sqrt{\alpha_{k_n}^2 - 1}}{i \eta_n^*} \right\} + O\left(\frac{1}{k_n} \right) \\ &\geq \frac{k_n \sqrt{\eta_n^*}}{k_n} \frac{\sqrt{1 - x^2}}{\eta_n^*} (1 + o(1)) + O\left(\frac{1}{k_n} \right) \\ &\geq \frac{C}{\sqrt{\eta_n^*}}, \end{aligned}$$

with C depending on x. In particular, then,

$$\lim_{n \to \infty, n \in \mathcal{N}} \frac{1}{k_n} \lambda_{k_n}^r (x)^{-1} = \infty.$$

■

3. Proof of Theorem 1.4

Proof of Theorem 1.4
Recall that $x \in (-1,1)$ is fixed. Fix $\varepsilon > 0$. By hypothesis, ν_n converges weakly to ν as $n \to \infty$, and there exists $\delta > 0$ and n_0 such that for $n \geq n_0$,

$$\frac{1}{n} \sum_{j \leq n : |\alpha_j - x| \leq \delta} \frac{|\operatorname{Im} \alpha_j|}{|x - \alpha_j|^2} < \varepsilon. \tag{3.1}$$

We claim that we can recast this as

$$\int_{\{t : |t-x| \leq \delta\}} \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) \leq C\varepsilon, \; n \geq n_0, \tag{3.2}$$

where C is independent of n, ε, δ. Indeed, writing $\alpha_j = t_j + i s_j$, we have as in the proof of Theorem 1.2, for $|\alpha_j - x| \leq \delta$

$$\begin{aligned} & \operatorname{Re}\left\{ \frac{\sqrt{\alpha_j^2 - 1}}{\alpha_j - x} \right\} \\ = \; & \frac{1}{|\alpha_j - x|^2} \operatorname{Re}\left\{ \sqrt{1 - t_j^2}\, |s_j| + O\left(s_j^2\right) \right\} \\ = \; & \frac{|\operatorname{Im} \alpha_j| \sqrt{1-x^2}\,(1 + O(\delta))}{|\alpha_j - x|^2}, \end{aligned}$$

so

$$\begin{aligned} & \int_{\{t : |t-x| \leq \delta\}} \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) \\ = \; & \sqrt{1-x^2} \left(\frac{1}{n} \sum_{j \leq n : |\alpha_j - x| \leq \delta} \frac{|\operatorname{Im} \alpha_j|}{|x - \alpha_j|^2} (1 + O(\delta)) \right), \end{aligned}$$

and (3.2) follows from (3.1).

Next, let h be a non-negative function that is continuous in $\bar{\mathbb{C}}$ (so that it has a finite limit at ∞ and is bounded). Let $\rho > 0$. Since

$$\frac{1}{|t-x|^2 + \rho} \operatorname{Re}\left\{ \sqrt{t^2-1}\,(\bar{t} - x) \right\}$$

is a bounded continuous function of $t \in \bar{\mathbb{C}}$, we have

$$\begin{aligned} & \liminf_{n \to \infty} \int h(t) \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) \\ \geq \; & \liminf_{n \to \infty} \int h(t) \frac{1}{|t-x|^2 + \rho} \operatorname{Re}\left\{ \sqrt{t^2-1}\,(\bar{t} - x) \right\} d\nu_n(t) \\ = \; & \int h(t) \frac{1}{|t-x|^2 + \rho} \operatorname{Re}\left\{ \sqrt{t^2-1}\,(\bar{t} - x) \right\} d\nu(t), \end{aligned}$$

by weak convergence. We can now let ρ decrease to 0 and use Lebesgue's monotone convergence theorem, to obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int h(t) \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) \\
(3.3) \qquad &\geq \int h(t) \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t).
\end{aligned}$$

Next let $0 < \eta < \delta$, and choose h to be a continuous function that equals 1 in $\{t : |t-x| \leq \eta\}$, and equals 0 for $|t-x| \geq \delta$, and such that $0 \leq h \leq 1$ everywhere. Then (3.3) gives

$$\begin{aligned}
&\int_{\{t:|t-x|\leq\eta\}} \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t) \\
\leq &\int h(t) \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t) \\
\leq &\liminf_{n\to\infty} \int h(t) \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) \leq C\varepsilon,
\end{aligned}$$

by (3.2). Thus we have shown, that given $\varepsilon > 0$, there exists $\eta > 0$ with

$$\int_{\{t:|t-x|\leq\eta\}} \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t) \leq C\varepsilon. \tag{3.4}$$

Next, let H be a non-negative function that is continuous in $\bar{\mathbb{C}}$ with $0 \leq H \leq 1$, and $H = 1$ in $|t-x| \geq \eta$, while $H = 0$ in $|t-x| \leq \eta/2$. We have

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d(\nu_n(t) - d\nu(t)) \right| \\
\leq &\limsup_{n\to\infty} \left| \int H(t) \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d(\nu_n(t) - d\nu(t)) \right| \\
&+ \limsup_{n\to\infty} \left| \int |1-H(t)| \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d(\nu_n(t) + d\nu(t)) \right| \\
\leq &\; 0 + 2 \limsup_{n\to\infty} \left| \int_{\{t:|t-x|\leq\eta\}} \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d(\nu_n(t) + d\nu(t)) \right| \\
\leq &\; 2C\varepsilon,
\end{aligned}$$

by weak convergence, (3.2) and (3.4). Thus, as ε is arbitrary,

$$\lim_{n\to\infty} \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) = \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t).$$

Here as $\nu\left(\bar{\mathbb{C}}\backslash[-1,1]\right)>0$, the right-hand side is positive. The result now follows from Lemma 2.1. ■

Proof of Corollary 1.5
This follows directly as

$$\frac{|\operatorname{Im}\alpha_j|}{|x-\alpha_j|^2}\leq\frac{1}{|\operatorname{Im}\alpha_j|}.$$

■

4. Proof of Theorem 1.6

We need:

Lemma 4.1
Let $\eta\in(0,1)$. There exists $\tau>0$ with the following property: given any $x\in[-1,1]$ and any 3 points α,β,Δ all at a distance at least η from $[-1,1]$, there exists a rational function $R\in\mathcal{L}_3\{\alpha,\beta,\Delta\}$ such that $R(x)=1$ and

$$|R(t)|^2\leq 1-\tau(t-x)^2\,,\ t\in[-1,1]\,. \tag{4.1}$$

Proof
See Lemma 2.3 in [3]. ■

Remark
We emphasize that τ is independent of x and α,β,Δ, depending only on η.

Our hypothesis allows us to choose, for $k\geq1$, $j_k\geq 2^k$ such that

$$dist\left(\alpha_{j_k},[-1,1]\right)\geq\eta.$$

We let

$$\begin{aligned} A^* &= \{\alpha_1,\alpha_2,...\}\backslash\{\alpha_{j_1},\alpha_{j_2},\alpha_{j_3}...\} \\ &= \{\alpha_1^*,\alpha_2^*,\alpha_3^*,...\}\,, \end{aligned}$$

say, and

$$\mathcal{L}_n^*=\mathcal{L}_n\{\alpha_1^*,\alpha_2^*,\alpha_3^*,...,\alpha_n^*\}\,.$$

Also, let

$$\omega'(t)=\sqrt{1-t^2},t\in[-1,1]$$

and

$$\lambda_n^{*r}(d\omega,x)=\inf_{R\in\mathcal{L}_{n-1}^*}\frac{\int_{-1}^1|R|^2\,d\omega}{|R(x)|^2}.$$

In addition to removing poles, we also need to add poles for later use. Assume that $\{\beta_j\}$ are complex numbers satisfying for $j\geq1$,

$$dist\left(\beta_j,[-1,1]\right)\geq\eta.$$

Let

$$\begin{aligned} A^{\#} &= \{\alpha_1, \alpha_2, ...\} \cup \{\beta_1, \beta_2, \beta_3, ...\} \\ &= \left\{\alpha_1^{\#}, \alpha_2^{\#}, \alpha_3^{\#}, ...\right\}, \end{aligned}$$

say, and

$$\mathcal{L}_n^{\#} = \mathcal{L}_n \left\{\alpha_1^{\#}, \alpha_2^{\#}, \alpha_3^{\#}, ..., \alpha_n^{\#}\right\}.$$

Here we insert β_j into $A^{\#}$ so sparsely that

$$\beta_j = \alpha_{2^j}^{\#},$$

so that some α_j are "shifted further down". Let

$$\lambda_n^{\#r}(d\omega, x) = \inf_{R \in \mathcal{L}_{n-1}^{\#}} \frac{\int_{-1}^{1} |R|^2 \, d\omega}{|R(x)|^2}.$$

We shall denote $\lambda_n^r(x)$ by $\lambda_n^r(d\mu, x)$ to emphasize its dependence on μ. Note that because the $\{\alpha_{j_k}\}$ and $\{\beta_j\}$ are removed or added so sparsely, the sequences A^* and $A^{\#}$ fulfil the hypotheses of Theorem 1.4. In particular, both

$$\lim_{n\to\infty} n\lambda_n^{*r}(d\omega, x) = \omega'(x)\,\pi\sqrt{1-x^2} \Big/ \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t); \tag{4.2}$$

$$\lim_{n\to\infty} n\lambda_n^{\#r}(d\omega, x) = \omega'(x)\,\pi\sqrt{1-x^2} \Big/ \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t). \tag{4.3}$$

We turn to

The Proof of Theorem 1.6

Let U be the polynomial of degree ℓ, say, such that gU is bounded. Let

$$S_0(t) = U(t) \Big/ \prod_{k=1}^{\ell} \left(1 - \frac{t}{\alpha_{j_k}}\right).$$

We may assume that $S_0(x) = 1$, by multiplying U by a constant. Then still gS_0 is bounded in absolute value on $[-1, 1]$. It will be important below that S_0 is fixed and does not change as n increases. Next, given n large enough, we can choose $m = m(n)$ such that

$$j_{\ell+m} \le n - 1 < j_{\ell+m+1}. \tag{4.4}$$

There are m points in the set $\left\{\alpha_{j_{\ell+1}}, \alpha_{j_{\ell+2}}, ..., \alpha_{j_{\ell+m}}\right\}$, all lying a distance at least η from $[-1, 1]$, so we can choose $[m/3]$ different functions R as in Lemma 4.1. Multiplying these together, yields a rational function $R_0 \in \mathcal{L}_m\left\{\alpha_{j_{\ell+1}}, \alpha_{j_{\ell+2}}, ..., \alpha_{j_{\ell+m}}\right\}$ such that $R_0(x) = 1$ and

$$|R_0(t)| \le \left(1 - \tau\left(t - x^2\right)\right)^{[m/3]}, t \in [-1, 1]. \tag{4.5}$$

Next, let $\varepsilon > 0$, and choose an interval J containing x in its interior, such that for $t \in J$

$$\begin{aligned} (1+\varepsilon)^{-1} &\leq g(t)/g(x) \leq 1+\varepsilon \\ (1+\varepsilon)^{-1} &\leq |S_0(t)| \leq 1+\varepsilon. \end{aligned} \tag{4.6}$$

(Recall that $S_0(x) = 1$). There exists $\kappa \in (0,1)$ depending only on ε, but not on m nor n, such that

$$|R_0(t)| \leq \kappa^m, \ t \in [-1,1] \backslash J. \tag{4.7}$$

Next, choose $P_0 \in \mathcal{L}_{n-1-\ell-m}\left\{\alpha_1^*, \alpha_2^*, \alpha_3^*, ..., \alpha_{n-1-\ell-m}^*\right\}$ such that $P_0(x) = 1$ and

$$\lambda_{n-\ell-m}^{*r}(d\omega, x) = \int_{-1}^{1} |P_0|^2 \, d\omega.$$

Set

$$P = P_0 R_0 S_0.$$

We claim that

$$P \in \mathcal{L}_{n-1}\left\{\alpha_1, \alpha_2, ..., \alpha_{n-1}\right\}.$$

Indeed $R_0 S_0$ have poles in $\left\{\alpha_{j_1}, \alpha_{j_2}, ..., \alpha_{j_{\ell+m}}\right\}$, and by (4.4), $j_{\ell+m} \leq n-1$. Then, using (4.5), (4.6), and (4.7),

$$\begin{aligned} \lambda_n^r(d\mu, x) &\leq \int_{-1}^{1} |P_0 R_0 S_0|(t)^2 g(t) \sqrt{1-t^2} dt \\ &\leq g(x)(1+\varepsilon)^3 \int_J |P_0(t)|^2 \sqrt{1-t^2} dt \\ &\quad + \left\|g S_0^2\right\|_{L_\infty[-1,1]} \kappa^{2m} \int_{[-1,1]\backslash J} |P_0(t)|^2 \sqrt{1-t^2} dt \\ &\leq \lambda_{n-\ell-m}^{*r}(d\omega, x)\left\{g(x)(1+\varepsilon)^3 + o(1)\right\}, \end{aligned}$$

by choice of P_0, and as $m = m(n) \to \infty$ as $n \to \infty$. Note that S_0 is itself bounded in absolute value on $[-1,1]$, so it does not matter that S_0^2, rather than S_0, multiplies g. Now the sparsity condition $j_k \geq 2^k$, ensures that $2^{\ell+m} \leq n-1$, so $m = O(\log n)$. Then the asymptotic (4.2) gives

$$\limsup_{n\to\infty} n\lambda_n^{*r}(d\mu, x) \leq g(x)(1+\varepsilon)^3 \omega'(x) \pi\sqrt{1-x^2} / \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t).$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{n\to\infty} n\lambda_n^{*r}(d\mu, x) \leq \mu'(x) \pi\sqrt{1-x^2} / \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t). \tag{4.8}$$

For the converse direction, we use the set of poles $A^{\#}$. Much as above, we now choose

$$S_0(x) = U(x) / \prod_{k=1}^{\ell}\left(1 - \frac{x}{\beta_k}\right).$$

Much as above, we choose $R_0 \in \mathcal{L}_m\left\{\beta_{\ell+1}, \beta_{\ell+2}, ..., \beta_{\ell+m}\right\}$ satisfying (4.5) and (4.7), where now $m = m(n)$ is chosen so that

$$2^{\ell+m} \leq n-1 < 2^{\ell+m+1}.$$

Next, choose $P_0 \in \mathcal{L}_{n-1-\ell-m}\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1-\ell-m}\}$ such that $P_0(x) = 1$ and

$$\lambda^r_{n-\ell-m}(d\mu, x) = \int_{-1}^{1} |P_0|^2 \, d\mu.$$

Set

$$P = P_0 R_0 S_0.$$

As above, (4.5), (4.6), (4.7) give

$$P \in \mathcal{L}^{\#}_{n-1}\left\{\alpha^{\#}_1, \alpha^{\#}_2, ..., \alpha^{\#}_{n-1}\right\}.$$

As above,

$$\begin{aligned} \lambda^{\#r}_n(d\omega, x) &\leq \int_{-1}^{1} |P_0 R_0 S_0|(t)^2 \sqrt{1-t^2}dt \\ &\leq g(x)^{-1}(1+\varepsilon)^3 \int_J |P_0(t)|^2 g(t)\sqrt{1-t^2}dt \\ &\quad + \left\|g^{-1}S_0^2\right\|_{L_\infty[-1,1]} \kappa^{2m} \int_{[-1,1]\setminus J} |P_0(t)|^2 g(t)\sqrt{1-t^2}dt \\ &\leq \lambda^r_{n-\ell-m}(d\mu, x)\left\{g(x)^{-1}(1+\varepsilon)^3 + o(1)\right\}. \end{aligned}$$

Now we use (4.3), and obtain

$$\liminf_{n\to\infty} n\lambda^r_{n-\ell-m}(d\mu, x) \geq g(x)(1+\varepsilon)^{-3}\omega'(x)\pi\sqrt{1-x^2} / \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t).$$

As $\varepsilon > 0$ is arbitrary,

$$\liminf_{n\to\infty} n\lambda^r_{n-\ell-m}(d\mu, x) \geq \mu'(x)\pi\sqrt{1-x^2} / \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t).$$

Of course, we need to replace $n-\ell-m$ by n. For this purpose, we use that $m = m(n) = O(\log n)$. If k lies in the set of positive integers whose extreme points are $n-1-\ell-m(n-1)$ and $n-\ell-m(n)$, then $\lambda^r_k(d\mu, x)$ is bounded below by either $\lambda^r_{n-1-\ell-m(n-1)}(d\mu, x)$ or $\lambda^r_{n-\ell-m(n)}(d\mu, x)$, and as $n/k \to 1$ as $n \to \infty$, we obtain

$$\liminf_{n\to\infty} n\lambda^r_n(d\mu, x) \geq \mu'(x)\pi\sqrt{1-x^2} / \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t).$$

Together with (4.8), this gives the result. ■

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