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ASYMPTOTICS OF DERIVATIVES OF ORTHOGONAL POLYNOMIALS ON THE REAL LINE

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Abstract. We show that uniform asymptotics of orthogonal polynomials on the real line imply uniform asymptotics for all their derivatives. This is more technically challenging than the corresponding problem on the unit circle. We also examine asymptotics in the L_2 norm.

1. Results

Let μ be a finite positive Borel measure on $[-1,1]$ and let $\{p_n\}_{n=0}^{\infty}$ denote the corresponding orthonormal polynomials, so that

$$\int_{-1}^{1} p_n p_m \, d\mu = \delta_{mn}.$$

Asymptotics for derivatives of p_n have been established under various hypotheses [1], [2], [9], [10], [13]. Many of these results deal with orthogonal polynomials on the unit circle. Recall that corresponding to μ, we may define

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a measure σ on the unit circle by

$$d\sigma(\theta) = d\big(\mu(\cos\theta)\big), \qquad \theta \in [-\pi, \pi].$$

The absolutely continuous components of the two measures are connected by

$$\sigma'(\theta) = \mu'(\cos\theta)|\sin\theta|. \tag{1.1}$$

Let $\{\varphi_n\}$ denote the orthonormal polynomials for σ, so that

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\,d\sigma(\theta) = \delta_{mn}.$$

The positive leading coefficient of φ_n is denoted κ_n. In analysing $\{\varphi_n\}$, their reversed cousins φ_n^* play a useful role:

$$\varphi_n^*(z) = z^n\overline{\varphi_n(1/\overline{z})}.$$

We also need the monic orthogonal polynomials

$$\Phi_n(z) = \varphi_n(z)/\kappa_n = z^n + \cdots .$$

In a recent paper [6], the second author proved that uniform asymptotics for φ_n imply uniform asymptotics for the derivatives of φ_n. More precisely, the following was proved, for general measures on the unit circle, that are not necessarily linked with some orthogonal polynomials on the real line:

THEOREM 1. *Let J be a subinterval of $[0, 2\pi]$, and assume that*

$$\lim_{n\to\infty} \varphi_n^*(e^{i\theta}) = g(\theta),$$

uniformly for $\theta \in J$, where $g(\theta) \neq 0$ for $\theta \in J$. Let $m \geqq 1$ and $I \subset J^0$ be a closed interval. Then uniformly for $z = e^{i\theta}$, $\theta \in I$,

$$\lim_{n\to\infty} z^m\varphi_n^{(m)}(z)/\big(n^m\varphi_n(z)\big) = 1.$$

The proof of this involves reworking ideas from a 1979 paper of Paul Nevai [10]. It was also proved that ratio asymptotics for $\{\Phi_n\}$ imply ratio asymptotics for their derivatives.

In this paper, we prove analogous results for orthogonal polynomials on the real line. However, the formulation is more complex, because of the more complicated form of the asymptotics. Assuming Szegő's condition on the real line

$$\int_{-1}^{1} \frac{\log\mu'(x)}{\sqrt{1-x^2}}\,dx > -\infty, \tag{1.2}$$

one can form the Szegő function

$$D(z) = \exp\left(-\frac{1}{4\pi}\int_{-\pi}^{\pi} \log \sigma'(\theta) \frac{z+e^{i\theta}}{z-e^{i\theta}}\, d\theta\right), \tag{1.3}$$

where σ' is given by (1.1). The standard Szegő asymptotic for p_n has the form

$$p_n(x) = \sqrt{\frac{2}{\pi}}\, \mathrm{Re}\left(z^n / D(z^{-1})\right) + o(1), \tag{1.4}$$

as $n \to \infty$. Here and throughout, x, θ and z are connected by the relation

$$x = \cos\theta; \quad z = e^{i\theta}. \tag{1.5}$$

The Szegő condition guarantees that (1.4) holds in an L_2 sense, but not necessarily pointwise. For pointwise or uniform asymptotics, one typically needs some smoothness on w, such as a local L_2 Lipschitz condition [4].

The relation (1.4) helps to explain the form of the hypothesis in the following theorem. In its formulation, and throughout the paper, we use the notation

$$\mathcal{I}_r f(z) = \begin{cases} \mathrm{Re}\, f(z), & \text{if } r \text{ is even} \\ \mathrm{Im}\, f(z), & \text{if } r \text{ is odd.} \end{cases} \tag{1.6}$$

We assume in the sequel that $\{p_n\}$ are the orthonormal polynomials corresponding to the measure μ, and that σ is the corresponding measure on the unit circle, with orthonormal polynomials $\{\varphi_n\}$ and monic orthogonal polynomials $\{\Phi_n\}$. We also let $[x]$ denote the greatest integer $\leqq x$. Thus for a positive integer r, we have

$$(-1)^{[r/2]} = \begin{cases} (-1)^{r/2}, & \text{if } r \text{ is even} \\ (-1)^{(r-1)/2}, & \text{if } r \text{ is odd.} \end{cases}$$

We use $\mathcal{D} = \frac{d}{d\theta}$, which should not be confused with the Szegő function $D(z)$. Finally, if I is a subinterval of $[-1,1]$, then $\tilde{I} = \left\{\theta \in [0,\pi] : \cos\theta \in I\right\}$ is the image of I under the function $\arccos$, while $\hat{I} = \left\{e^{i\theta} : \theta \in \tilde{I}\right\}$ is the projection of I onto the unit circle.

THEOREM 2. *Let μ be a positive Borel measure on $[-1,1]$. Assume that I is a closed subinterval of $(-1,1)$, and uniformly for $x = \cos\theta \in I$, we have as $n \to \infty$,*

$$p_n(x) = \mathrm{Re}\left(z^n f(z)\right) + o(1), \tag{1.7}$$

where f is bounded in $\hat{I}$. Assume moreover, that

$$\lim_{n\to\infty} \Phi_n(0) = 0. \tag{1.8}$$

Let $r \geqq 1$ and I_1 be a closed subinterval of I^0. Then uniformly for $x \in I_1$,

$$n^{-r}(1-x^2)^{\frac{r}{2}}(-1)^{\left[\frac{r}{2}\right]} p_n^{(r)}(x) = \mathcal{I}_r\bigl(z^n f(z)\bigr) + o(1). \tag{1.9}$$

Note that if μ' is positive a.e. in $(-1,1)$, then (1.8) is true [11, p. 467], so we have:

COROLLARY 3. *Assume the hypotheses of Theorem* 2, *except that instead of* (1.8), *we assume that μ' is positive a.e. in $(-1,1)$. Then the conclusion of Theorem* 2 *is true.*

Thus Theorem 2 asserts that once we have uniform asymptotics for orthogonal polynomials, we also obtain uniform asymptotics for their derivatives.

We shall also study mean asymptotics of derivatives of orthogonal polynomials. As far as the authors are aware, this has not been studied in general.

THEOREM 4. *Let μ be a positive absolutely continuous Borel measure on $[-1,1]$ satisfying Szegő's condition* (1.2). *Assume, moreover, that σ admits the following Markov–Bernstein inequality: for $n \geqq 1$, and all trigonometric polynomials R of degree $\leqq n$,*

$$\left[\int_{-\pi}^{\pi} |R'|^2\, d\sigma\right]^{1/2} \leqq Cn \left[\int_{-\pi}^{\pi} |R|^2\, d\sigma\right]^{1/2}. \tag{1.10}$$

Let $r \geqq 1$. Let D be the Szegő function defined by (1.3). *Then*

(1.11)
$$\lim_{n\to\infty} \int_0^{\pi} \left| n^{-r}(-1)^{\left[\frac{r}{2}\right]} \left(\frac{d}{d\theta}\right)^r \bigl[p_n(\cos\theta)\bigr] - \sqrt{\frac{2}{\pi}}\mathcal{I}_r\bigl(z^n/D(z^{-1})\bigr)\right|^2 d\sigma(\theta) = 0$$

and for each compact subinterval I of $(-1,1)$,

(1.12)
$$\lim_{n\to\infty} \int_I \left| n^{-r}(1-x^2)^{\frac{r}{2}}(-1)^{\left[\frac{r}{2}\right]} p_n^{(r)}(x) - \sqrt{\frac{2}{\pi}}\mathcal{I}_r\bigl(z^n/D(z^{-1})\bigr)\right|^2 d\mu(x) = 0.$$

COROLLARY 5. *Under the hypotheses of Theorem* 4,

$$\lim_{n\to\infty} \int_{-1}^{1} \left| n^{-1}(1-x^2)^{\frac{1}{2}} p_n'(x) - \sqrt{\frac{2}{\pi}}\,\mathrm{Im}\bigl(z^n/D(z^{-1})\bigr)\right|^2 d\mu(x) = 0. \tag{1.13}$$

It is not clear that (1.12) holds with $I=(-1,1)$ and $r \geqq 2$ without additional assumptions on w, such as further Markov–Bernstein or Schur inequalities. One can already observe some of the difficulties for $r=2$:

$$\mathcal{D}^2\big[p_n(\cos\theta)\big] - p_n''(\cos\theta)(\sin\theta)^2 = -p_n'(\cos\theta)\cos\theta = \cot\theta\, \mathcal{D}\big[p_n(\cos\theta)\big],$$

and the term $\cot\theta$ becomes unbounded near the endpoints of $[0,\pi]$. For Jacobi weights, one can verify that the requisite estimates hold.

The hypothesis (1.10) holds for Jacobi weights, generalized Jacobi weights, and still more generally, the doubling weights of Mastroianni and Totik [7]. It is likely that there are Szegő weights violating (1.10), but we do not have an explicit example.

We shall also state a local version of Theorem 4:

THEOREM 6. *Let μ be a positive absolutely continuous Borel measure on $[-1,1]$ satisfying Szegő's condition* (1.2). *Assume that L is a closed subinterval of $[-1,1]$ in which σ admits the following Markov–Bernstein inequality: for $n \geqq 1$, and all trigonometric polynomials R of degree $\leqq n$,*

$$\left[\int_{\tilde{L}} |R'|^2\, d\sigma\right]^{1/2} \leqq Cn\left[\int_{-\pi}^{\pi} |R|^2\, d\sigma\right]^{1/2}. \tag{1.14}$$

Then the conclusion (1.11) *holds if $(0,\pi)$ is replaced by $\tilde{L}$, while* (1.12) *holds for any closed subinterval I of L^0.*

In particular, if σ' is bounded above and below by positive constants in some closed interval L_1, then the hypothesis (1.14) of Theorem 6 is satisfied with L taken as any compact subinterval of the interior of L_1. In the sequel $C, C_1, C_2, \dots$ denote constants independent of n, x, θ. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C=C(\alpha)$ or $C \neq C(\alpha)$ to denote dependence on, or independence of, α, respectively.

2. Proof of Theorem 2

In the proof of Theorem 2, we need the polynomials $\{q_n\}$, orthonormal with respect to the weight $(1-x^2)w(x)$:

$$\int_{-1}^{1} q_n(x)q_m(x)(1-x^2)w(x)\,dx = \delta_{mn}.$$

We also set

$$\beta_m = \frac{1}{\sqrt{2\pi}}\big(1+\Phi_m(0)\big)^{-1/2}; \quad \lambda_m = \frac{1}{\sqrt{2\pi}}\big(1-\Phi_m(0)\big)^{-1/2}.$$

The most important idea is to represent φ_n^* in terms of p_n alone. The ideas to do this are contained in a paper of Máté, Nevai and Totik [8, p. 262 ff.], although the identity in (b) below is not stated in the form there.

LEMMA. (a)

$$p_n(x) = \beta_{2n} \left\{ z^{-n}\varphi_{2n}(z) + z^n \varphi_{2n}\left(\frac{1}{z}\right) \right\}; \tag{2.1}$$

$$q_n(x) = 2\lambda_{2n+2} \frac{z^{-n-1}\varphi_{2n+2}(z) - z^{n+1}\varphi_{2n+2}\left(\frac{1}{z}\right)}{z - z^{-1}}. \tag{2.2}$$

(b)

$$2z^n \varphi_{2n}^*\left(\frac{1}{z}\right) = \frac{i}{\sqrt{1-x^2}} \left[\frac{z^{-1}p_n(x)}{\beta_{2n}} - \frac{p_{n+1}(x)}{\beta_{2n+2}} + \operatorname{Re}\left(\eta_n(z)\right) \right], \tag{2.3}$$

where

$$\eta_n(z) = 2z^{-n-1}\varphi_{2n}(z) \left[\frac{\varphi_{2n+2}(z)}{\varphi_{2n}(z)} - z^2 \right]. \tag{2.4}$$

PROOF. (a) See [12, p. 294].

(b) Since $z - z^{-1} = 2i\sqrt{1-x^2}$, we can rewrite the identities of (a) as

$$\frac{p_n(x)}{\beta_{2n}} = z^{-n}\varphi_{2n}(z) + z^n\varphi_{2n}\left(\frac{1}{z}\right);$$

$$i\frac{q_{n-1}(x)}{\lambda_{2n}}\sqrt{1-x^2} = z^{-n}\varphi_{2n}(z) - z^n\varphi_{2n}\left(\frac{1}{z}\right).$$

We add these to obtain

$$\frac{p_n(x)}{\beta_{2n}} + i\frac{q_{n-1}(x)}{\lambda_{2n}}\sqrt{1-x^2} = 2z^{-n}\varphi_{2n}(z) \tag{2.5}$$

and hence

$$\frac{p_{n+1}(x)}{\beta_{2n+2}} + i\frac{q_n(x)}{\lambda_{2n+2}}\sqrt{1-x^2} = 2z^{-n-1}\varphi_{2n+2}(z).$$

We multiply the second last equation by z and subtract it from the last equation, to obtain

$$\left[\frac{p_{n+1}(x)}{\beta_{2n+2}} - z\frac{p_n(x)}{\beta_{2n}} \right] + i\sqrt{1-x^2} \left[\frac{q_n(x)}{\lambda_{2n+2}} - z\frac{q_{n-1}(x)}{\lambda_{2n}} \right]$$

$$= 2z^{-n-1}\varphi_{2n+2}(z) - 2z^{-n+1}\varphi_{2n}(z) = \eta_n(z).$$

Now take real parts:

$$\left[\frac{p_{n+1}(x)}{\beta_{2n+2}} - x\frac{p_n(x)}{\beta_{2n}}\right] + (1-x^2)\frac{q_{n-1}(x)}{\lambda_{2n}} = \operatorname{Re}\big(\eta_n(z)\big)$$

and hence

$$(1-x^2)\frac{q_{n-1}(x)}{\lambda_{2n}} = \operatorname{Re}\big(\eta_n(z)\big) - \left[\frac{p_{n+1}(x)}{\beta_{2n+2}} - x\frac{p_n(x)}{\beta_{2n}}\right].$$

Then (2.5) gives

(2.6)

$$2z^{-n}\varphi_{2n}(z) = \frac{p_n(x)}{\beta_{2n}} + \frac{i}{\sqrt{1-x^2}}\left[\operatorname{Re}\big(\eta_n(z)\big) - \left[\frac{p_{n+1}(x)}{\beta_{2n+2}} - x\frac{p_n(x)}{\beta_{2n}}\right]\right]$$

$$= \frac{i}{\sqrt{1-x^2}}\left[\frac{p_n(x)}{\beta_{2n}}z^{-1} - \frac{p_{n+1}(x)}{\beta_{2n+2}} + \operatorname{Re}\big(\eta_n(z)\big)\right].$$

As φ_{2n} has real coefficients, we see that

$$z^{-n}\varphi_{2n}(z) = z^n\varphi_{2n}^*\left(\frac{1}{z}\right),$$

and then the result follows (cf. [4], p. 189, Lemma 1.3).

PROOF OF THEOREM 2. We have by our hypothesis (1.8),

$$\lim_{n\to\infty}\beta_n = \frac{1}{\sqrt{2\pi}} = \lim_{n\to\infty}\lambda_n.$$

By standard results [11, pp. 91–92], (1.8) also ensures that

$$\lim_{n\to\infty}\frac{\varphi_{n+1}(z)}{z\varphi_n(z)} = 1,$$

uniformly in $\{z : |z| \geqq 1\}$, and hence uniformly in the same region,

$$\lim_{n\to\infty}\frac{\varphi_{2n+2}(z)}{z^2\varphi_{2n}(z)} = 1.$$

Then uniformly for $z \in \hat{I}$, $\eta_n(z) = o\big(|\varphi_{2n}(z)|\big)$.

Next, the boundedness of f, and our assumed asymptotic (1.7) for $\{p_n\}$ give for $n \geqq 1$,

$$\left|p_n(x)\right| \leqq C\left|f(z)\right| + C \leqq C_1.$$

Then (2.6) implies also that

$$\left|\varphi_{2n}(z)\right| \leqq C\left[1 + \left|\eta_n(z)\right|\right] \leqq C\left[1 + o\left(\left|\varphi_{2n}(z)\right|\right)\right].$$

Then $\{\varphi_{2n}\}_n$ must be uniformly bounded in $\hat{I}$, and so $\lim_{n\to\infty} \eta_n(z) = 0$, uniformly in $\hat{I}$. From (2.3), we now deduce that uniformly in $\hat{I}$,

$$(2.7) \qquad 2z^n \varphi^*_{2n}\left(\frac{1}{z}\right) = \frac{i}{\sqrt{1-x^2}}\left[\frac{z^{-1}p_n(x)}{\beta_{2n}} - \frac{p_{n+1}(x)}{\beta_{2n+2}} + o(1)\right]$$

$$= \sqrt{2\pi}\frac{i}{\sqrt{1-x^2}}\left[p_n(x)z^{-1} - p_{n+1}(x)\right] + o(1).$$

Now write, for a given x, and n, $z^n f(z) = a + ib$. Then

$$\operatorname{Re}\left(z^{n+1}f(z)\right) = \operatorname{Re}\left(z(a+ib)\right) = ax - b\sqrt{1-x^2},$$

Substituting our assumed asymptotics for p_n into (2.7), and using these last observations,

$$\sqrt{\frac{2}{\pi}(1-x^2)}z^n\varphi^*_{2n}\left(\frac{1}{z}\right) = i\left[az^{-1} - ax + b\sqrt{1-x^2}\right] + o(1)$$

$$= \sqrt{1-x^2}[a+ib] + o(1) = \sqrt{1-x^2}z^n f(z) + o(1).$$

Thus uniformly for $z \in \hat{I}$,

$$(2.8) \qquad \varphi^*_{2n}\left(\frac{1}{z}\right) = \sqrt{\frac{\pi}{2}}f(z) + o(1)$$

and hence uniformly for z such that $\bar{z} \in \hat{I}$,

$$(2.9) \qquad \varphi^*_{2n}(z) = \sqrt{\frac{\pi}{2}}f\left(\frac{1}{z}\right) + o(1).$$

From this, we deduce that uniformly for z such that $\bar{z} \in \hat{I}$,

$$\varphi^*_{2n}(z) - \varphi^*_{2[\sqrt{n}]}(z) = o(1).$$

Since φ_n has real coefficients, this also implies that uniformly for $z \in \hat{I}$,

$$\varphi_{2n}^*(z) - \varphi_{2[\sqrt{n}]}^*(z) = o(1).$$

Now let I_1 be a closed subinterval of I^0. Recall our notation $\mathcal{D} = \frac{d}{d\theta}$. Local Markov–Bernstein inequalities [3, pp. 242–243] give in $\hat{I}_1$,

$$\left\| \mathcal{D}^\ell \big(\varphi_{2n}^* - \varphi_{2[\sqrt{n}]}^* \big) \right\|_{L_\infty(\hat{I}_1)} = o(n^\ell)$$

and also

$$\left\| \mathcal{D}^\ell \varphi_{2[\sqrt{n}]}^* \right\|_{L_\infty(\hat{I}_1)} = O\big(\sqrt{n}\,\big)^\ell.$$

Combining these gives

$$\left\| \mathcal{D}^\ell \varphi_{2n}^* \right\|_{L_\infty(\hat{I}_1)} = o(n^\ell), \qquad \ell \geqq 1.$$

Differentiating the relation

$$p_n(\cos\theta) = 2\beta_{2n} \operatorname{Re}\left[e^{-in\theta} \varphi_{2n}^*(e^{i\theta}) \right]$$

which follows from (2.1), and using Leibniz's formula, we obtain, uniformly for $\theta \in \tilde{I}_1$,

$$\begin{aligned} \mathcal{D}^r \left[p_n(\cos\theta) \right] &= 2\beta_{2n} \sum_{j=0}^{r} \binom{r}{j} \operatorname{Re}\left[(-in)^{r-j} e^{-in\theta} \mathcal{D}^j \left[\varphi_{2n}^*(e^{i\theta}) \right] \right] \\ &= 2\beta_{2n} \operatorname{Re}\left[(-in)^r e^{-in\theta} \varphi_{2n}^*(e^{i\theta}) \right] + o(n^r). \end{aligned} \tag{2.10}$$

In particular then,

$$\sup_{\theta \in \tilde{I}_1} \left| \mathcal{D}^r \left[p_n(\cos\theta) \right] \right| \leqq C n^r. \tag{2.11}$$

Next, Faa di Bruno's formula for derivatives of a composition of functions [5, p. 19], gives

$$\mathcal{D}^r \left[p_n(\cos\theta) \right] \tag{2.12}$$

$$= \sum \frac{r!}{j_1! j_2! \dots j_m!} p_n^{(\ell)}(\cos\theta) \left(\frac{-\sin\theta}{1!} \right)^{j_1} \left(\frac{-\cos\theta}{2!} \right)^{j_2} \dots \left(\frac{\mathcal{D}^m \cos\theta}{m!} \right)^{j_m},$$

where the sum is over all $m \geqq 1$ and m-tuples $(j_1, j_2, \dots, j_m)$ of positive integers with $j_1 + 2j_2 + \dots + mj_m = r$, while $\ell = j_1 + j_2 + \dots + j_m$. From this, we see that $p_n^{(r)}$ arises only when $m = 1$, $j_1 = r$. Thus

$$\mathcal{D}^r \left[p_n(\cos\theta) \right] = p_n^{(r)}(\cos\theta)(-\sin\theta)^r + \Sigma,$$

where Σ is a linear combination of $p_n^{(k)}(\cos\theta)$, $0 \leqq k \leqq r-1$, multiplied by powers of sin and cos. We then see that

$$n^{-r} \left| \mathcal{D}^r \left[p_n(\cos\theta) \right] - p_n^{(r)}(\cos\theta)(-\sin\theta)^r \right| \leqq C_2 \max_{0 \leqq k \leqq r-1} n^{-r} \left| p_n^{(k)}(\cos\theta) \right|. \tag{2.13}$$

Applying (2.11), this last inequality, and using induction on r, we see that

$$\sup_{x \in I_1} \left| p_n^{(r)}(x) \right| \leqq Cn^r,$$

and hence

$$\sup_{x \in I_1} \left| \mathcal{D}^r \left[p_n(\cos\theta) \right] - p_n^{(r)}(\cos\theta)(-\sin\theta)^r \right| \leqq Cn^{r-1}.$$

Finally (2.10) gives

$$p_n^{(r)}(\cos\theta)(-\sin\theta)^r = 2\beta_{2n} \operatorname{Re} \left[(-in)^r e^{-in\theta} \varphi_{2n}^*(e^{i\theta}) \right] + o(n^r)$$

and hence from (2.9),

$$n^{-r} p_n^{(r)}(x)(1-x^2)^{r/2} = \operatorname{Re} \left[i^r e^{-in\theta} f(e^{-i\theta}) \right] + o(1).$$

Since (2.9) implies $f(e^{-i\theta}) = \overline{f(e^{i\theta})}$ and since for any complex number u,

$$\operatorname{Re}[i^r u] = \begin{cases} (-1)^{r/2} \operatorname{Re} \overline{u}, & \text{if } r \text{ is even,} \\ (-1)^{(r-1)/2} \operatorname{Im} \overline{u}, & \text{if } r \text{ is odd} \end{cases}$$

$$= (-1)^{[r/2]} \mathcal{I}_r(\overline{u}),$$

the result follows. □

3. Proof of Theorems 4 and 6

From (2.10),

$$\left[\int_0^\pi \left|\mathcal{D}^r\left[p_n(\cos\theta)\right] - 2\beta_{2n}\operatorname{Re}\left[(-in)^r e^{-in\theta}\varphi_{2n}^*(e^{i\theta})\right]\right|^2 d\sigma(\theta)\right]^{1/2} \tag{3.1}$$

$$\leqq 2|\beta_{2n}|\sum_{j=1}^r \binom{r}{j} n^{r-j}\left[\int_0^\pi \left|\mathcal{D}^j\left[\varphi_{2n}^*(e^{i\theta})\right]\right|^2 d\sigma(\theta)\right]^{1/2}$$

$$\leqq Cn^{r-1}\left[\int_0^\pi \left|\mathcal{D}\varphi_{2n}^*(e^{i\theta})\right|^2 d\sigma(\theta)\right]^{1/2},$$

by repeated application of our Markov inequality (1.10). Next,

$$\left[\int_0^\pi \left|\mathcal{D}\varphi_{2n}^*(e^{i\theta})\right|^2 d\sigma(\theta)\right]^{1/2} \leqq \left[\int_0^\pi \left|\mathcal{D}\big(\varphi_{2n}^* - \varphi_{[\sqrt{n}]}^*\big)(e^{i\theta})\right|^2 d\sigma(\theta)\right]^{1/2}$$

$$+\left[\int_0^\pi \left|\mathcal{D}\varphi_{[\sqrt{n}]}^*(e^{i\theta})\right|^2 d\sigma(\theta)\right]^{1/2} \leqq Cn\left[\int_0^\pi \left|\big(\varphi_{2n}^* - \varphi_{[\sqrt{n}]}^*\big)(e^{i\theta})^2\right| d\sigma(\theta)\right]^{1/2}$$

$$+ C\sqrt{n}\left[\int_0^\pi \left|\varphi_{[\sqrt{n}]}^*(e^{i\theta})\right|^2 d\sigma(\theta)\right]^{1/2} = o(n),$$

because of the classical Szegő asymptotics [11, p. 144] and (2.9)

$$\lim_{m\to\infty}\int_{-\pi}^\pi \left|\varphi_m^*(e^{i\theta}) - D^{-1}(e^{i\theta})\right|^2 d\sigma(\theta) = 0. \tag{3.2}$$

Recall here that $D(z)$ is the Szegő function, given by (1.3). Thus, from (3.1),

$$\left[\int_0^\pi \left|n^{-r}\mathcal{D}^r\left[p_n(\cos\theta)\right] - 2\beta_{2n}\operatorname{Re}\left[(-i)^r e^{-in\theta}\varphi_{2n}^*(e^{i\theta})\right]\right|^2 d\sigma(\theta)\right]^{1/2} = o(1).$$

Using the just stated Szegő asymptotics (3.2), and the fact that $\beta_{2n} \to 1/\sqrt{2\pi}$, we can restate this as

$$\left[\int_0^\pi \left|n^{-r}\mathcal{D}^r\left[p_n(\cos\theta)\right] - \sqrt{\frac{2}{\pi}}\operatorname{Re}\left[(-i)^r e^{-in\theta}D^{-1}(e^{i\theta})\right]\right|^2 d\sigma(\theta)\right]^{1/2} = o(1). \tag{3.3}$$

The first part (1.11) of the theorem now follows. Next, let I be a compact subinterval of $(-1,1)$. For each r, and all $n \geqq 1$,

$$\int_0^\pi \left|n^{-r}\mathcal{D}^r\left[p_n(\cos\theta)\right]\right|^2 d\sigma(\theta) \leqq C. \tag{3.4}$$

This implies that for each r,

$$\sup_{n\geqq 1}\int_{\tilde I} \left|n^{-r}p_n^{(r)}(\cos\theta)\right|^2 d\sigma(\theta) \leqq C. \tag{3.5}$$

This follows by an easy induction on r, using the identity (2.12), and the fact that $\sin\theta = \sqrt{1-x^2}$ is bounded below in I. Next, as at (2.13),

$$\left[\int_{\tilde I}\left|n^{-r}\left|\mathcal{D}^r\left[p_n(\cos\theta)\right] - p_n^{(r)}(\cos\theta)(-\sin\theta)^r\right|\right|^2 d\sigma(\theta)\right]^{1/2}$$

$$\leqq C\sum_{\ell=0}^{r-1}\left[\int_{\tilde I}\left|n^{-r}p_n^{(\ell)}(\cos\theta)\right|^2 d\sigma(\theta)\right]^{1/2} = O(n^{-1}),$$

by (3.5). This and (3.3) give

$$\int_{\tilde I}\left|n^{-r}p_n^{(r)}(\cos\theta)(-\sin\theta)^r - \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left[(-i)^r e^{-in\theta}D^{-1}(e^{i\theta})\right]\right|^2 d\sigma(\theta) = o(1).$$

Transferring this to the interval I and taking account of whether r is even or odd, gives (1.12). $\square$

Proof of Corollary 5. This is the case $r=1$ of (1.11), after a substitution. $\square$

Proof of Theorem 6. This is the same as that of Theorem 4, with obvious modifications. $\square$

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