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# ORTHOGONAL DIRICHLET POLYNOMIALS WITH CONSTANT WEIGHT

*Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.*

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Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers. We analyze the orthogonal Dirichlet polynomials $\{\psi_{n,T}\}$ formed from linear combinations of $\left\{\lambda_j^{-it}\right\}_{j=1}^{n}$, associated with constant (or Legendre) weight on $[-T,T]$. Thus

$$\frac{1}{2T}\int_{-T}^{T}\psi_{n,T}\left(t\right)\overline{\psi_{m,T}\left(t\right)}dt=\delta_{mn}.$$

Moreover, we analyze how these polynomials behave as $T$ varies.

## 1. Introduction

Throughout, let

(1) $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers.

Given $m\geq 1$, a *Dirichlet polynomial of degree* $\leq n$ [**16**], [**17**] associated with this sequence of exponents has the form

$$\sum_{n=1}^{m}a_n\lambda_n^{-it}=\sum_{n=1}^{m}a_n e^{-i(\log\lambda_n)t},$$

where $\{a_n\}\subset\mathbb{C}$. We denote the set of all such polynomials by $\mathcal{L}_n$.

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The traditional orthogonal Dirichlet polynomials are just the "monomials" $\left\{\lambda_n^{-it}\right\}$ themselves. Indeed, in the theory of almost-periodic functions [**1**], [**2**], heavy use is made of orthogonality in the mean:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \lambda_j^{-it}\overline{\lambda_k^{-it}}dt=\delta_{jk}.$$

In the hope that a more standard orthogonality relation might have some advantages, the author [**6**], investigated Dirichlet orthogonal polynomials associated with the arctangent density. Thus $\phi_n\in\mathcal{L}_n$ has positive leading coefficient, and

$$\int_{-\infty}^{\infty}\phi_n(t)\overline{\phi_m(t)}\frac{dt}{\pi(1+t^2)}=\delta_{mn},\quad m,n\geq 1.$$

These Dirichlet orthogonal polynomials admit a very simple explicit expression, at least when $0<\lambda_1<\lambda_2<\cdots$ : for $n\geq 2$,

$$\phi_n(t)=\frac{\lambda_n^{1-it}-\lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2-\lambda_{n-1}^2}}.$$

These orthonormal polynomials have been applied in several questions by Weber and Dimitrov as well as the author [**4**], [**7**], [**15**], [**17**], [**18**], [**19**]. In a subsequent paper [**8**], the author considered orthogonal Dirichlet polynomials for the Laguerre weight, though it turned out that much of the material there was already subsumed by Mű̈ntz orthogonal polynomials [**3**]. Mű̈ntz orthogonal polynomials have also been a topic investigated by Gradimir Milovanovic [**10**], [**11**], [**12**], to whom this paper is dedicated.

Very recently [**9**], we investigated Dirichlet orthogonal polynomials for rational weights

$$w(t)=\sum_{m=1}^{L}\frac{c_m}{\pi\left(1+(b_mt)^2\right)}$$

and appropriately chosen $\{c_j\}$. Here $L\geq 1$, and $1=b_1<b_2<\cdots<b_L$. We obtained a simple explicit determinantal expression for the orthonormal polynomials, but could only resolve positivity of the weight for the case $L=2$.

In this paper, we let $T>0$, and consider $\psi_{n,T}\in\mathcal{L}_n$, with positive leading coefficient $\gamma_{n,T}$, such that

$$(\psi_{n,T},\psi_{m,T})_T=\frac{1}{2T}\int_{-T}^{T}\psi_{n,T}(t)\overline{\psi_{m,T}(t)}dt=\delta_{mn}. \tag{2}$$

We are especially interested in how $\psi_{n,T}$ behaves as $T$ varies, and in particular how it behaves as $T\to\infty$. Next, define the $n$th reproducing kernel

$$K_{n,T}(u,v)=\sum_{j=1}^{n}\psi_{j,T}(u)\overline{\psi_{j,T}(v)}. \tag{3}$$

In the sequel, we also use

$$\mathbb{S}\left(u\right)=\frac{\sin u}{u}.$$

From the simple relation

$$\left(\lambda_j^{-it},\lambda_k^{-it}\right)_T=\frac{1}{2T}\int_{-T}^{T}\left(\lambda_j/\lambda_k\right)^{-it}dt=\mathbb{S}\left(T\log\left(\lambda_j/\lambda_k\right)\right), \tag{4}$$

and standard determinantal representations for orthonormal functions with respect to a given inner product, we see that

$$\begin{aligned}\psi_{n,T}\left(x\right)&=\frac{\left(-1\right)^{n+1}}{\sqrt{A_{n-1,T}A_{n,T}}}\\ &\times\det\begin{bmatrix}\lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix}\\ 1 & \mathbb{S}(T\log\lambda_1/\lambda_2) & \mathbb{S}(T\log\lambda_1/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_1/\lambda_n)\\ \mathbb{S}(T\log\lambda_2/\lambda_1) & 1 & \mathbb{S}(T\log\lambda_2/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_2/\lambda_n)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbb{S}(T\log\lambda_{n-1}/\lambda_1) & \mathbb{S}(T\log\lambda_{n-1}/\lambda_2) & \mathbb{S}(T\log\lambda_{n-1}/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_{n-1}/\lambda_n)\end{bmatrix},\end{aligned} \tag{5}$$

so the leading coefficient of $\psi_{n,T}\left(x\right)$ is

$$\gamma_{n,T}=\sqrt{\frac{A_{n-1,T}}{A_{n,T}}}, \tag{6}$$

where

$$A_{n,T}=\det\left[\mathbb{S}\left(T\log\lambda_j/\lambda_k\right)\right]_{1\le j,k\le n}. \tag{7}$$

It follows easily from the determinantal expression and the fact that $\lim_{x\to\infty}\mathbb{S}\left(x\right)=0$, that

$$\lim_{T\to\infty}\psi_{n,T}\left(x\right)=\lambda_n^{-ix}$$

and that $\psi_{n,T}$ is an infinitely differentiable function of $T$.

One of the motivations for our study is the celebrated Montgomery-Vaughan inequality and its ramifications. In one form its asserts that [**14**, p. 74, Corollary 2], [**13**, p. 128, Thm. 1]

$$\int_0^T\left|\sum_{j=1}^{n}a_j\lambda_j^{-it}\right|^2dt=\left(T+2\pi\varepsilon\delta^{-1}\right)\sum_{j=1}^{n}\left|a_j\right|^2, \tag{8}$$

where

$$\delta=\min\left\{\left|\log\lambda_j-\log\lambda_k\right|:1\le j,k\le n\text{ and }j\ne k\right\},$$

while $|\varepsilon| \leq 1$. We hope that a theory of orthogonal Dirichlet polynomials might contribute to this circle of ideas and to estimates involving Dirichlet polynomials. We begin with a simple result related to the Montgomery-Vaughan inequality: write for $j \geq 1$, $T > 0$,

$$\lambda_j^{-it} = \sum_{k=1}^{j} c_{T,j,k} \psi_{k,T}(t) . \tag{9}$$

Also write

$$\psi_{n,T}(t) = \sum_{j=1}^{n} d_{T,n,j} \lambda_j^{-it} . \tag{10}$$

Let

$$C_{T,n} = \begin{bmatrix} c_{T,1,1} & c_{T,2,1} & c_{T,3,1} & \cdots & c_{T,n,1} \\ 0 & c_{T,2,2} & c_{T,3,2} & \cdots & c_{T,n,2} \\ 0 & 0 & c_{T,3,3} & \cdots & c_{T,n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{T,n,n} \end{bmatrix} . \tag{11}$$

**Theorem 1.** *(a) For any complex numbers $\{a_j\}_{j=1}^n$,*

$$\frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt = \| C_{T,n} \mathbf{a} \|^2 \tag{12}$$

*where $\mathbf{a} = [a_1 \ a_2 \ \ldots a_n]^T$ and the norm is the usual Euclidean norm. In particular,*

$$\sup_{\{a_j\}} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt \Big/ \sum_{j=1}^{n} |a_j|^2 = \| C_{T,n} \|^2 , \tag{13}$$

*where the norm is the usual matrix norm induced by the Euclidean norm.*
*(b) The coefficients $\{c_{T,j,k}\}$ and $\{d_{T,n,k}\}$ are real.*
*(c) For $j,k \geq 1$,*

$$\sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell} c_{T,j,\ell} = \mathbb{S}\left(T \log \lambda_j / \lambda_k\right) . \tag{14}$$

Next, we consider $\psi'_{n,T}$:

**Theorem 2.**

*(a)*

$$\begin{aligned}\psi'_{n,T}(t) &= (-i\log\lambda_n)\,\psi_{n,T}(t)\\ &\quad+\frac{1}{2T}\Big(\psi_{n,T}(t)\,K_{n-1,T}(t,T)-\overline{\psi_{n,T}(t)}K_{n-1,T}(t,-T)\Big)\\ &= (-i\log\lambda_n)\,\psi_{n,T}(t)\\ &\quad+\frac{i}{T}\sum_{j=1}^{n-1}\psi_{j,T}(t)\,\mathrm{Im}\,\Big(\psi_{n,T}(T)\,\overline{\psi_{j,T}(T)}\Big).\end{aligned} \tag{15}$$

*(b)*

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left|\psi'_{n,T}\right|^2 &= (\log\lambda_n)^2+\frac{1}{T^2}\sum_{j=1}^{n-1}\left|\mathrm{Im}\,\Big(\psi_{n,T}(T)\,\overline{\psi_{j,T}(T)}\Big)\right|^2\\ &= (\log\lambda_n)^2+\frac{1}{T}\mathrm{Re}\,\Big(\psi_{n,T}\overline{\psi'_{n,T}}\Big)(T).\end{aligned} \tag{16}$$

Next, we compare the orthonormal polynomials $\psi_{n,T}$ and $\psi_{n,T}$ for different $S,T$:

**Theorem 3.** *Let* $S>T$.

*(a)*

$$\Delta_{n,T}=\frac{1}{2T}\int_{-T}^{T}\left|\psi_{n,S}(t)-\frac{\gamma_{n,S}}{\gamma_{n,T}}\psi_{n,T}(t)\right|^2dt\le\frac{S}{T}-\left(\frac{\gamma_{n,S}}{\gamma_{n,T}}\right)^2. \tag{17}$$

*(b)*

$$\frac{\gamma_{n,S}}{\gamma_{n,T}}\le\left(\frac{S}{T}\right)^{1/2}. \tag{18}$$

*(c)*

$$K_{n,T}(x,x)+\left(\frac{S}{T}-2\right)K_{n,S}(x,x)\ge 0. \tag{19}$$

Finally, we consider the rate of change of several quantities w.r.t. $T$:

**Theorem 4.**

*(a)*

$$\frac{\partial}{\partial T}K_{n,T}(x,x)=\frac{1}{T}K_{n,T}(x,x)-\frac{1}{2T}\Big(\left|K_n(x,T)\right|^2+\left|K_n(x,-T)\right|^2\Big). \tag{20}$$

*(b)*

$$\frac{\partial(\ln\gamma_{n,T})}{\partial T}=\frac{1}{2T}(1-\left|\psi_{n,T}(T)\right|^2). \tag{21}$$

*(c)*

$$\frac{\partial}{\partial T}\ln A_{n,T} = -\frac{1}{T}\left(n - K_{n,T}\left(T,T\right)\right). \tag{22}$$

*(d) Let* $c_{T,j,k}$ *be the connection coefficient as in* (9)*. Then*

$$\begin{aligned}\frac{\partial}{\partial T}c_{T,j,k} + \frac{1}{T}c_{T,j,k} &= \frac{1}{2T}\left[\lambda_j^{-iT}\overline{\psi_{k,T}\left(T\right)} + \lambda_j^{iT}\psi_{k,T}\left(T\right)\right] \\ &\quad + \frac{1}{2T}\int_{-T}^{T}\lambda_j^{-it}\frac{\partial}{\partial T}\overline{\psi_{k,T}\left(t\right)}dt.\end{aligned} \tag{23}$$

We prove Theorems 1 and 2 in Section 2, and Theorems 3 and 4 in Section 3.

## 2. Proof of Theorems 1 and 2

**Proof of Theorem 1**

(a)

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left|\sum_{k=1}^{n}a_k\lambda_k^{-it}\right|^2 dt &= \frac{1}{2T}\int_{-T}^{T}\left|\sum_{k=1}^{n}a_k\left(\sum_{j=1}^{k}c_{T,k,j}\psi_{j,T}\left(t\right)\right)\right|^2 dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left|\sum_{j=1}^{n}\psi_{j,T}\left(t\right)\left\{\sum_{k=j}^{n}a_kc_{T,k,j}\right\}\right|^2 dt \\ &= \sum_{j=1}^{n}\left|\sum_{k=j}^{n}a_kc_{T,k,j}\right|^2 \\ &= \sum_{j=1}^{n}\left|\left(C_{T,n}\mathbf{a}\right)_j\right|^2 = \left\|C_{T,n}\mathbf{a}\right\|^2.\end{aligned} \tag{24}$$

(b) First, if $j < k$,

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T}\lambda_j^{-it}\overline{\overline{\psi_{k,T}\left(-t\right)}}dt \\ &= \frac{1}{2T}\int_{-T}^{T}\lambda_j^{is}\psi_{k,T}\left(s\right)ds \\ &= \overline{\frac{1}{2T}\int_{-T}^{T}\lambda_j^{-is}\overline{\psi_{k,T}\left(s\right)}ds} = 0.\end{aligned}$$

Also,

$$\overline{\psi_{k,T}\left(-t\right)} = \sum_{j=1}^{k}\overline{d_{T,k,j}}\lambda_j^{-it}$$

so is also an orthonormal polynomial (recall the leading coefficient is positive). By uniqueness,

(25)
$$\overline{\psi_{k,T}(-t)} = \psi_{k,T}(t),$$

and hence the $\{d_{T,k,j}\}$ are real. Next, by orthogonality,

$$\begin{aligned}\overline{c_{T,j,k}} &= \overline{\frac{1}{2T}\int_{-T}^{T} \lambda_j^{-it}\overline{\psi_{k,T}(t)}dt} \\ &= \frac{1}{2T}\int_{-T}^{T} \lambda_j^{it}\psi_{k,T}(t)\,dt \\ &= \frac{1}{2T}\int_{-T}^{T} \lambda_j^{-it}\psi_{k,T}(-t)\,dt \\ &= \frac{1}{2T}\int_{-T}^{T} \lambda_j^{-it}\overline{\psi_{k,T}(t)}dt = c_{T,j,k}.\end{aligned}$$

Thus the $\{c_{T,k,j}\}$ are real.
(c) From (4), (24), and (a),

$$\begin{aligned}&\sum_{1\le j,k\le n} a_j\overline{a_k}\mathbb{S}(T\log\lambda_j/\lambda_k) \\ &= \sum_{\ell=1}^{n}\left|\sum_{k=\ell}^{n} a_k c_{T,k,\ell}\right|^2 \\ &= \sum_{1\le k,j\le n} a_j\overline{a_k}\sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell}c_{T,j,\ell}.\end{aligned}$$

Choosing some $a_j = 1$ and all remaining $a's = 0$ gives

$$\sum_{\ell=1}^{j} c_{T,j,\ell}^2 = 1.$$

Next choose distinct $j,k$ and $a_j = a_k = 1$ with all remaining $a's = 0$. Then we obtain

$$2\mathbb{S}(T\log\lambda_j/\lambda_k) + 1 = 2\sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell}c_{T,j,\ell} + \sum_{\ell=1}^{j} c_{T,j,\ell}^2.$$

Thus we obtain (14) in full generality. □

**Proof of Theorem 2**
(a) Write

$$\psi_{n,T}'(t) = (-i\log\lambda_n)\,\psi_{n,T}(t) + \sum_{j=1}^{n-1}\beta_j\psi_{j,T}(t).$$

Here, integrating by parts, for $j \leq n-1$,

$$\begin{aligned}\beta_j &= \frac{1}{2T}\int_{-T}^{T} \psi'_{n,T}(t)\,\overline{\psi_{j,T}(t)}dt \\ &= \frac{1}{2T}\left\{\psi_{n,T}(T)\,\overline{\psi_{j,T}(T)} - \psi_{n,T}(-T)\,\overline{\psi_{j,T}(-T)}\right\} \\ &\quad - \frac{1}{2T}\int_{-T}^{T} \psi_{n,T}(t)\,\overline{\psi'_{j,T}(t)}dt \\ &= \frac{1}{2T}\left\{\psi_{n,T}(T)\,\overline{\psi_{j,T}(T)} - \overline{\psi_{n,T}(T)}\psi_{j,T}(T)\right\} \\ &= \frac{i}{T}\text{Im }\left(\psi_{n,T}(T)\,\overline{\psi_{j,T}(T)}\right).\end{aligned}$$

So

$$\begin{aligned}&\psi'_{n,T}(t) = (-i\log\lambda_n)\,\psi_{n,T}(t) \\ &\quad + \frac{1}{2T}\left\{\psi_{n,T}(T)\sum_{j=1}^{n-1}\overline{\psi_{j,T}(T)}\psi_{j,T}(t) - \psi_{n,T}(-T)\sum_{j=1}^{n-1}\overline{\psi_{j,T}(-T)}\psi_{j,T}(t)\right\} \\ &= (-i\log\lambda_n)\,\psi_{n,T}(t) + \frac{1}{2T}\left\{\psi_{n,T}(T)\,K_{n-1}(t,T) - \overline{\psi_{n,T}(T)}K_{n-1}(t,-T)\right\}.\end{aligned}$$

Also,

$$\psi'_{n,T}(t) = (-i\log\lambda_n)\,\psi_{n,T}(t) + \frac{i}{T}\sum_{j=1}^{n-1}\text{Im }\left(\psi_{n,T}(T)\,\overline{\psi_{j,T}(T)}\right)\psi_{j,T}(t).$$

(b) The first identity in (16) follows from the second identity in (15). Next, integrating by parts gives

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T}\left|\psi'_{n,T}\right|^2 \\ &\quad = \frac{1}{2T}\int_{-T}^{T}\psi'_{n,T}\overline{\psi'_{n,T}} \\ &\quad = \frac{1}{2T}\left\{\left(\psi_{n,T}\overline{\psi'_{n,T}}\right)(T) - \left(\psi_{n,T}\overline{\psi'_{n,T}}\right)(-T)\right\} - \frac{1}{2T}\int_{-T}^{T}\psi_{n,T}\overline{\psi''_{n,T}}.\end{aligned} \tag{26}$$

Here using (a) twice,

$$\psi''_{n,T} = -\left(\log\lambda_n\right)^2\psi_{n,T} + P,$$

where $P \in \mathcal{L}_{n-1}$, so

$$\frac{1}{2T}\int_{-T}^{T}\psi_{n,T}\overline{\psi''_{n,T}} = -\left(\log\lambda_n\right)^2. \tag{27}$$

Next,

$$\psi'_{n,T}(t) = -i \sum_{j=1}^{n} f_j (\log \lambda_j) \lambda_j^{-it},$$

where all $f_j$ are real, so

$$\begin{aligned}\overline{\psi'_{n,T}(-T)} &= \overline{-i \sum_{j=1}^{n} f_j (\log \lambda_j) \lambda_j^{iT}} \\ &= i \sum_{j=1}^{n} f_j (\log \lambda_j) \lambda_j^{-iT} = -\psi'_{n,T}(T).\end{aligned}$$

Substituting this and (25) into (26), gives

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T} \left|\psi'_{n,T}\right|^2 \\ &= \frac{1}{2T}\left\{\left(\psi_{n,T}\overline{\psi'_{n,T}}\right)(T) + \left(\overline{\psi_{n,T}}\psi'_{n,T}\right)(T)\right\} + (\log \lambda_n)^2 \\ &= \frac{1}{T}\mathrm{Re}\,\left(\psi_{n,T}\overline{\psi'_{n,T}}\right)(T) + (\log \lambda_n)^2.\end{aligned}$$

□

## 3. Proof of Theorems 3 and 4n

**Proof of Theorem 3**
(a)

$$\begin{aligned}\Delta_{n,T} &:= \frac{1}{2T}\int_{-T}^{T}\left|\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}}\psi_{n,T}(t)\right|^2 dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left(\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}}\psi_{n,T}(t)\right)\left(\overline{\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}}\psi_{n,T}(t)}\right) dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left(\psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}}\psi_{n,T}(t)\right)\left(\overline{\psi_{n,S}(t)}\right) dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left|\psi_{n,S}(t)\right|^2 dt - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}}\right)^2 \le \frac{S}{T} - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}}\right)^2.\end{aligned}$$

(b) Then also,

$$\frac{\gamma_{n,S}}{\gamma_{n,T}} \le \left(\frac{S}{T}\right)^{1/2}.$$

(c)

$$\frac{1}{2T}\int_{-T}^{T}\left|K_{n,S}\left(x,t\right)-K_{n,T}\left(x,t\right)\right|^{2}dt$$
$$=\frac{1}{2T}\int_{-T}^{T}\left|K_{n,S}\left(x,t\right)\right|^{2}dt-2K_{n,S}\left(x,x\right)+K_{n,T}\left(x,x\right). \tag{28}$$

Also,

$$\frac{1}{2T}\int_{-T}^{T}\left|K_{n,S}\left(x,t\right)\right|^{2}dt$$
$$=\frac{S}{T}\left[\frac{1}{2S}\int_{-S}^{S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt-\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt\right]$$
$$=\frac{S}{T}\left[K_{n,S}\left(x,x\right)-\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt\right]$$

So substituting in (28) above,

$$0\leq\frac{S}{T}\left[K_{n,S}\left(x,x\right)-\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt\right]$$
$$-2K_{n,S}\left(x,x\right)+K_{n,T}\left(x,x\right)$$
$$=K_{n,T}\left(x,x\right)+\left(\frac{S}{T}-2\right)K_{n,S}\left(x,x\right)$$
$$-\frac{1}{2T}\int_{T\leq|t|\leq S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt. \tag{29}$$

In particular, (19) follows. □

**Proof of Theorem 4**

(a) Let $S>T$. Now from (29) above,

$$K_{n,T}\left(x,x\right)-K_{n,S}\left(x,x\right)\geq\left(1-\frac{S}{T}\right)K_{n,S}\left(x,x\right)$$
$$+\frac{1}{2T}\int_{T\leq|t|\leq S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt$$

so

$$\frac{K_{n,T}\left(x,x\right)-K_{n,S}\left(x,x\right)}{T-S}\leq\frac{1}{T}K_{n,S}\left(x,x\right)$$
$$+\frac{1}{T-S}\frac{1}{2T}\int_{T\leq|t|\leq S}\left|K_{n,S}\left(x,t\right)\right|^{2}dt. \tag{30}$$

Next,

$$\begin{aligned}
0 &\leq \frac{1}{2S}\int_{-S}^{S}\left|K_{n,S}(x,t)-K_{n,T}(x,t)\right|^2 dt \\
&= K_{n,S}(x,x)-2K_{n,T}(x,x)+\frac{1}{2S}\int_{-S}^{S}\left|K_{n,T}(x,t)\right|^2 dt \\
&= K_{n,S}(x,x)-2K_{n,T}(x,x) \\
&\quad+\frac{T}{S}\left[K_{n,T}(x,x)+\frac{1}{2T}\int_{T\leq|t|\leq S}\left|K_{n,T}(x,t)\right|^2 dt\right] \\
&= K_{n,S}(x,x)+\left(\frac{T}{S}-2\right)K_{n,T}(x,x)+\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,T}(x,t)\right|^2 dt,
\end{aligned}$$

so

$$\begin{aligned}
K_{n,S}(x,x)-K_{n,T}(x,x) \geq &\left(1-\frac{T}{S}\right)K_{n,T}(x,x) \\
&-\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,T}(x,t)\right|^2 dt.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{K_{n,S}(x,x)-K_{n,T}(x,x)}{S-T} \geq &\frac{1}{S}K_{n,T}(x,x) \\
&-\frac{1}{S-T}\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,T}(x,t)\right|^2 dt.
\end{aligned}$$

Together with (30), this establishes

$$\begin{aligned}
&\frac{1}{S}K_{n,T}(x,x)-\frac{1}{S-T}\frac{1}{2S}\int_{T\leq|t|\leq S}\left|K_{n,T}(x,t)\right|^2 dt \\
&\quad\leq \frac{K_{n,S}(x,x)-K_{n,T}(x,x)}{S-T} \\
&\quad\leq \frac{1}{T}K_{n,S}(x,x)+\frac{1}{T-S}\frac{1}{2T}\int_{T\leq|t|\leq S}\left|K_{n,S}(x,t)\right|^2 dt. \qquad (31)
\end{aligned}$$

Inasmuch as $K_{n,T}(x,x)-K_{n,s}(x,x)\to 0$ as $|S-T|\to 0$, (indeed, the representation (5) shows that $\psi_{n,T}$ and hence $K_{n,T}$ are infinitely differentiable in $T$), this last inequality yields that

$$\frac{\partial}{\partial T}K_{n,T}(x,x)=\frac{1}{T}K_{n,T}(x,x)-\frac{1}{2T}\left(\left|K_n(x,T)\right|^2+\left|K_n(x,-T)\right|^2\right).$$

(b)

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\left|\psi_{n,S}\left(t\right)-\psi_{n,T}\left(t\right)\right|^{2}dt\\
&\quad=\frac{1}{2T}\int_{-T}^{T}\left|\psi_{n,S}\left(t\right)\right|^{2}dt-2\frac{\gamma_{n,S}}{\gamma_{n,T}}+1\\
&\quad=\frac{S}{T}\left(1-\frac{1}{2S}\int_{T\leq|t|\leq S}\left|\psi_{n,S}\left(t\right)\right|^{2}dt\right)-2\frac{\gamma_{n,S}}{\gamma_{n,T}}+1,
\end{aligned}$$

So

$$2\frac{\gamma_{n,T}-\gamma_{n,S}}{\gamma_{n,T}}\geq 1-\frac{S}{T}+\frac{1}{2T}\int_{T\leq|t|\leq S}\left|\psi_{n,S}\left(t\right)\right|^{2}dt.$$

Then recalling that $\psi_{n,S}\left(-t\right)=\overline{\psi_{n,S}\left(t\right)}$,

$$\frac{\gamma_{n,T}-\gamma_{n,S}}{T-S}\leq\frac{1}{2T}\gamma_{n,T}+\frac{1}{T-S}\gamma_{n,T}\frac{1}{2T}\int_{T}^{S}\left|\psi_{n,S}\left(t\right)\right|^{2}dt. \tag{32}$$

In the other direction,

$$\begin{aligned}
&\frac{1}{2S}\int_{-S}^{S}\left|\psi_{n,S}\left(t\right)-\psi_{n,T}\left(t\right)\right|^{2}dt\\
&\quad=1-2\frac{\gamma_{n,T}}{\gamma_{n,S}}+\frac{1}{2S}\int_{-S}^{S}\left|\psi_{n,T}\left(t\right)\right|^{2}dt\\
&\quad=1-2\frac{\gamma_{n,T}}{\gamma_{n,S}}+\frac{T}{S}+\frac{1}{2S}\int_{T\leq|t|\leq S}\left|\psi_{n,T}\left(t\right)\right|^{2}dt,
\end{aligned}$$

so

$$2\frac{\gamma_{n,S}-\gamma_{n,T}}{\gamma_{n,S}}\geq 1-\frac{T}{S}-\frac{1}{2S}\int_{T\leq|t|\leq S}\left|\psi_{n,T}\left(t\right)\right|^{2}dt,$$

and hence

$$\frac{\gamma_{n,S}-\gamma_{n,T}}{S-T}\geq\frac{\gamma_{n,S}}{2S}-\frac{\gamma_{n,S}}{2S}\frac{1}{S-T}\int_{T}^{S}\left|\psi_{n,T}\left(t\right)\right|^{2}dt.$$

Combined with (32), this gives

$$\begin{aligned}
\frac{\gamma_{n,S}}{2S}-\frac{\gamma_{n,S}}{2S}\frac{1}{S-T}\int_{T}^{S}\left|\psi_{n,T}\left(t\right)\right|^{2}dt&\leq\frac{\gamma_{n,S}-\gamma_{n,T}}{S-T}\\
&\leq\frac{\gamma_{n,T}}{2T}-\frac{1}{S-T}\frac{\gamma_{n,T}}{2T}\int_{T}^{S}\left|\psi_{n,S}\left(t\right)\right|^{2}dt.
\end{aligned}$$

This gives

$$\frac{\partial\gamma_{n,T}}{\partial T}=\frac{\gamma_{n,T}}{2T}-\frac{\gamma_{n,T}}{2T}\left|\psi_{n,T}\left(T\right)\right|^{2},$$

and hence the result.
(c) Recall that

$$\gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}},$$

so

$$\gamma_{2,T} \cdots \gamma_{n,T} = \sqrt{\frac{A_{1,T}}{A_{n,T}}} = \sqrt{\frac{1}{A_{n,T}}}.$$

Thus

$$\begin{aligned} \frac{\partial}{\partial T} \ln \sqrt{\frac{1}{A_{n,T}}} &= \sum_{j=2}^{n} \frac{\partial (\ln \gamma_{j,T})}{\partial T} = \sum_{j=2}^{n} \frac{1}{2T} (1 - |\psi_{j,T}(T)|^2) \\ \Rightarrow \frac{\partial}{\partial T} \ln A_{n,T} &= -\frac{1}{T} \sum_{j=2}^{n} (1 - |\psi_{j,T}(T)|^2) \\ &= -\frac{1}{T} (n - K_{n,T}(T,T)), \end{aligned}$$

recall that $\psi_{1,T}(x) = \lambda_1^{-ix}$ so $|\psi_{1,T}(x)| = 1$.
(d)

$$\begin{aligned} & c_{S,j,k} - c_{T,j,k} \\ &= \frac{1}{2S} \int_{-S}^{S} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt - \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \overline{\psi_{k,T}(t)} dt \\ &= \frac{1}{2S} \int_{T \le |t| \le S} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt + \frac{1}{2} \left( \frac{1}{S} - \frac{1}{T} \right) \int_{-T}^{T} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt \\ &\quad + \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \left[ \overline{\psi_{k,S}(t)} - \overline{\psi_{k,T}(t)} \right] dt \end{aligned}$$

so

$$\begin{aligned} & \frac{c_{S,j,k} - c_{T,j,k}}{S - T} \\ &= \frac{1}{S-T} \frac{1}{2S} \int_{T \le |t| \le S} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt - \frac{1}{2} \frac{1}{ST} \int_{-T}^{T} \lambda_j^{-it} \overline{\psi_{k,S}(t)} dt \\ &\quad + \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \left[ \frac{\overline{\psi_{k,S}(t)} - \overline{\psi_{k,T}(t)}}{S - T} \right] dt \end{aligned}$$

Now let $S \to T$. □

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