

# Orthogonal Dirichlet Polynomials with Arctangent Density<sup>☆</sup>

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## Abstract

Let  $\{\lambda_j\}_{j=1}^\infty$  be a strictly increasing sequence of positive numbers with  $\lambda_1 = 1$ . We find a simple explicit formula for the orthogonal Dirichlet polynomials  $\{\phi_n\}$  formed from linear combinations of  $\{\lambda_j^{-it}\}_{j=1}^n$ , associated with the arctangent density. Thus

$$\int_{-\infty}^{\infty} \phi_n(t) \overline{\phi_m(t)} \frac{dt}{\pi(1+t^2)} = \delta_{mn}.$$

We obtain formulae for their Christoffel functions, and deduce their asymptotics, as well as universality limits, and spacing of zeros for their reproducing kernels. We also investigate the relationship between ordinary Dirichlet series, and orthogonal expansions involving the  $\{\phi_n\}$ , and establish Markov-Bernstein inequalities.

*Key words:* Dirichlet polynomials, orthogonal polynomials

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## 1. Introduction

Throughout, let

$$1 = \lambda_1 < \lambda_2 < \lambda_3 < \dots . \tag{1.1}$$

A *Dirichlet series* associated with this sequence of exponents has the form

$$\sum_{n=1}^{\infty} a_n \lambda_n^{-it} = \sum_{n=1}^{\infty} a_n e^{-i(\log \lambda_n)t}.$$

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In particular, when  $\lambda_j = j$ ,  $j \geq 1$ , we obtain the standard Dirichlet series, of which the Riemann zeta function is a special case.

It was Harald Bohr [2] who developed much of the theory of almost-periodic functions. One of its basic tools is that if  $0 < \alpha \leq \beta < \infty$ , then  $\alpha^{-it}$  and  $\beta^{-it}$  are orthonormal on  $(-\infty, \infty)$  in the mean, that is

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \alpha^{it} \overline{\beta^{-it}} dt = \delta_{\alpha\beta}. \quad (1.2)$$

Consequently, if  $\{a_n\}$  and  $\{b_n\}$  are square summable, and

$$f(t) = \sum_{n=1}^{\infty} a_n \lambda_n^{-it} \text{ and } g(t) = \sum_{n=1}^{\infty} b_n \lambda_n^{-it},$$

then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) \overline{g(t)} dt = \sum_{n=1}^{\infty} a_n \overline{b_n}.$$

Thus one can identify spaces of Dirichlet series with the sequence space  $\ell^2$ . The main results of the theory include existence (and uniqueness) of non-harmonic Fourier series for almost periodic functions, and their approximability by nonharmonic trigonometric polynomials. Notable contributors, in addition to Bohr, include Bochner, Stepanov, and Besicovitch [1], [2]. This has led to a very rich theory, in which *Dirichlet polynomials*

$$\mathcal{L}_m = \left\{ \sum_{n=1}^m a_n \lambda_n^{-it} : a_1, a_2, \dots, a_m \in \mathbb{C} \right\}, \quad m \geq 1, \quad (1.3)$$

have also been extensively studied [6], [7].

It is the purpose of this paper to investigate various properties of Dirichlet polynomials, using the arctangent density  $\frac{1}{\pi(1+t^2)}$ ,  $t \in (-\infty, \infty)$ . Our hope is that a more direct orthonormality relation than (1.2), might have some advantages. Our analysis uses the orthonormal polynomials  $\{\phi_n\}_{n=1}^{\infty}$  formed by applying the Gram-Schmidt process to  $\{\lambda_n^{-it}\}_{n=1}^{\infty}$  with respect to the arctangent density. Thus  $\phi_n \in \mathcal{L}_n$ , has positive leading coefficient, and

$$\int_{-\infty}^{\infty} \phi_n(t) \overline{\phi_m(t)} \frac{dt}{\pi(1+t^2)} = \delta_{mn}, \quad m, n \geq 1. \quad (1.4)$$

These Dirichlet orthogonal polynomials admit a very simple explicit expression:

**Theorem 1.1.** For  $n = 1$ ,  $\phi_1 = 1$ , and for  $n \geq 2$ ,

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}. \quad (1.5)$$

The author has searched the extensive literature of Dirichlet polynomials, and not found this, even in the special case  $\lambda_j = j$ . There of course,  $n^{1-it} - (n-1)^{1-it}$  arises in one of the standard ways of analytically continuing the Riemann zeta function, via summation by parts. We believe that even if (1.5) is known, at least the applications below are new.

Some elementary properties of  $\{\phi_n\}$  are given in the following proposition. In it, and in the sequel, we use the convention  $\lambda_0 = 0$ .

**Proposition 1.2.** Let  $n \geq 2$ .

(a)

$$\sup_{t \in \mathbb{R}} |\phi_n(t)| = \sqrt{\frac{\lambda_n + \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}}. \quad (1.6)$$

(b) The zeros of  $\phi_n$  are simple and have the form

$$-i + \frac{2k\pi}{\log(\lambda_n/\lambda_{n-1})}, \quad k \in \mathbb{Z}. \quad (1.7)$$

(c)

$$\sup_{t \in \mathbb{R}} |\phi'_n(t)| = \frac{(\log \lambda_n) \lambda_n + (\log \lambda_{n-1}) \lambda_{n-1}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}. \quad (1.8)$$

(d)

$$\lambda_m^{1-it} = \sum_{j=1}^m \sqrt{\lambda_j^2 - \lambda_{j-1}^2} \phi_j(t). \quad (1.9)$$

(e)

$$\int_{-\infty}^{\infty} \frac{|\phi'_n(t)|^2}{\pi(1+t^2)} dt = (\log \lambda_n)^2 + \frac{\lambda_{n-1}^2}{\lambda_n^2 - \lambda_{n-1}^2} \left( \log \frac{\lambda_{n-1}}{\lambda_n} \right)^2. \quad (1.10)$$

(f)

$$\inf_{c_1, c_2, \dots, c_{n-1} \in \mathbb{C}} \int_{-\infty}^{\infty} \left| \lambda_n^{-it} - \sum_{j=1}^{n-1} c_j \lambda_j^{-it} \right|^2 \frac{dt}{\pi(1+t^2)} = 1 - \left( \frac{\lambda_{n-1}}{\lambda_n} \right)^2. \quad (1.11)$$

The machinery of orthogonal functions and the simplicity of the formula above allow us to analyze reproducing kernels, Christoffel functions, and Markov-Bernstein inequalities. The  $m$ th reproducing kernel is

$$K_m(x, t) = \sum_{n=1}^m \phi_n(x) \overline{\phi_n(t)},$$

and  $m$ th Christoffel function is

$$\Lambda_m(x) = \frac{1}{K_m(x, x)} = \frac{1}{\sum_{n=1}^m |\phi_n(x)|^2}. \quad (1.12)$$

It admits the extremal property

$$\Lambda_m(x) = \inf \left\{ \frac{\int_{-\infty}^{\infty} |P(t)|^2 \frac{dt}{\pi(1+t^2)}}{|P(x)|^2} : P \in \mathcal{L}_m \right\}. \quad (1.13)$$

Christoffel functions are an essential tool in analysis of orthogonal polynomials [5]. We need the sinc and hyperbolic sinc kernels

$$S(z) = \frac{\sin \pi z}{\pi z}; \quad \tilde{S}(z) = \frac{\sinh \pi z}{\pi z} \quad (1.14)$$

in describing limits of Christoffel functions.

**Theorem 1.3.** (a) For  $s, t \in \mathbb{C}$ , and  $m \geq 1$ ,

$$\begin{aligned} K_m(s, t) &= 1 + 4 \sum_{n=2}^m \frac{(\lambda_{n-1} \lambda_n)^{1 + \frac{i}{2}(s-t)}}{\lambda_n^2 - \lambda_{n-1}^2} \\ &\times \left[ \sin^2 \left( \frac{s+t}{4} \log \frac{\lambda_n}{\lambda_{n-1}} \right) - \sin^2 \left( \left[ \frac{t-s}{4} + \frac{i}{2} \right] \log \frac{\lambda_n}{\lambda_{n-1}} \right) \right]. \end{aligned} \quad (1.15)$$

(b) For real  $x$ , and  $m \geq 1$ ,

$$\begin{aligned} K_m(x, x) &= 1 + \sum_{n=2}^m \frac{1}{\lambda_n^2 - \lambda_{n-1}^2} \\ &\times \left[ (\lambda_n - \lambda_{n-1})^2 + 4\lambda_n \lambda_{n-1} \sin^2 \left( \frac{x}{2} \log \frac{\lambda_n}{\lambda_{n-1}} \right) \right]. \end{aligned} \quad (1.16)$$

(c) For all real  $x$ , and  $m \geq 1$ ,

$$K_m(x, x) \leq 1 + \sum_{n=2}^m \frac{\lambda_n - \lambda_{n-1}}{\lambda_n + \lambda_{n-1}} \left(1 + x^2 \frac{\lambda_n}{\lambda_{n-1}}\right). \quad (1.17)$$

(d) Moreover, if as  $m \rightarrow \infty$ ,

$$\lambda_m \rightarrow \infty \text{ and } \frac{\lambda_m}{\lambda_{m-1}} = 1 + o(1), \quad (1.18)$$

then as  $m \rightarrow \infty$ , uniformly for  $s, t$ , in compact subsets of the complex plane,

$$\begin{aligned} K_m(s, t) &= \frac{1}{2} (1 + is)(1 - it) \lambda_m^{i(s-t)/2} (\log \lambda_m) S\left(\frac{s-t}{2\pi} \log \lambda_m\right) \\ &\quad + o\left(\lambda_m^{|Im(s-t)|/2} (\log \lambda_m) \tilde{S}\left(\frac{|Im(s-t)|}{2\pi} \log \lambda_m\right)\right). \end{aligned} \quad (1.19)$$

(e) As  $m \rightarrow \infty$ , uniformly for  $x$  in compact subsets of the real line,

$$K_m(x, x) = \frac{1}{2} (1 + x^2) (\log \lambda_m) (1 + o(1)). \quad (1.20)$$

We can deduce universality limits (cf. [4]) for the reproducing kernels, and asymptotics for their zeros:

**Theorem 1.4.** Assume (1.18).

(a) We have, uniformly for  $\alpha, \beta$  in compact subsets of  $\mathbb{C}$ , and  $x$  in compact subsets of the real line,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{\log \lambda_m} K_m\left(x + \frac{\alpha}{\log \lambda_m}, x + \frac{\beta}{\log \lambda_m}\right) \\ = [1 + x^2] e^{i(\alpha-\beta)/2} S\left(\frac{\alpha-\beta}{2\pi}\right). \end{aligned} \quad (1.21)$$

(b) Let  $x \in \mathbb{R}$ . Then for each fixed integer  $j = \pm 1, \pm 2, \pm 3, \dots$ , and large enough  $m$ ,  $K_m(x, t)$  has a simple zero  $t_{m,j}$ , which satisfies

$$\lim_{m \rightarrow \infty} (t_{m,j} - x) \log \lambda_m = 2j\pi. \quad (1.22)$$

Moreover, given  $r > 0$ , for large enough  $m$ , the only zeros of  $K_m(x, t)$  in  $\left\{z : |z - x| \leq \frac{r}{\log \lambda_m}\right\}$  are the zeros  $\{t_{m,j}\}$ .

Next, we turn to Markov-Bernstein inequalities, which estimate derivatives of Dirichlet polynomials. There is a substantial literature for such inequalities for Müntz polynomials [3], but the author has not found any such results for Dirichlet polynomials.

**Theorem 1.5 (Markov-Bernstein Inequality).** *For  $P \in \mathcal{L}_m$ ,*

$$\left( \int_{-\infty}^{\infty} \frac{|P'(t)|^2}{\pi(1+t^2)} dt \right)^{1/2} \leq \left( \int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt \right)^{1/2} \times \left( \log \lambda_m + \left( \sum_{j=2}^m \frac{\lambda_j - \lambda_{j-1}}{\lambda_j + \lambda_{j-1}} \right)^{1/2} \right). \quad (1.23)$$

*In particular,*

$$\left( \int_{-\infty}^{\infty} \frac{|P'(t)|^2}{\pi(1+t^2)} dt \right)^{1/2} \leq \left( \int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt \right)^{1/2} \left( \log \lambda_m + (\log \lambda_m)^{1/2} \right). \quad (1.24)$$

Proposition 1.2 (e) shows that this is essentially sharp with respect to the order of  $\log \lambda_m$ , and moreover, just a growth factor of  $\log \lambda_m$  is insufficient – we need an extra smaller term in the last right-hand side.

Finally, we turn to orthonormal expansions. Let

$$\mathcal{H} = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}. \quad (1.25)$$

This is a subspace of the weighted  $L^2$  space consisting of measurable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with

$$\|f\| = \left( \int_{-\infty}^{\infty} \frac{|f(t)|^2}{\pi(1+t^2)} dt \right)^{1/2} < \infty,$$

which we denote by  $\mathcal{G}$ . For  $f \in \mathcal{H}$  and  $m \geq 1$ , we denote the  $m$ th partial sum of its orthonormal expansion by

$$S_m[f] = \sum_{n=1}^m a_n \phi_n, \quad (1.26)$$

where for  $n \geq 1$ ,

$$a_n = a_n[f] = \int_{-\infty}^{\infty} f(t) \overline{\phi_n(t)} \frac{1}{\pi(1+t^2)} dt. \quad (1.27)$$

The relationship between formal orthonormal expansions and formal Dirichlet series is given in:

**Theorem 1.6.** (a) Let  $\{a_n\} \subset \mathbb{C}$  and  $f$  denote the formal orthonormal expansion

$$f = \sum_{n=1}^{\infty} a_n \phi_n. \quad (1.28)$$

For  $n \geq 1$ , let

$$b_n = \lambda_n \left( \frac{a_n}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} - \frac{a_{n+1}}{\sqrt{\lambda_{n+1}^2 - \lambda_n^2}} \right). \quad (1.29)$$

Then for  $m \geq 1$ ,

$$S_m[f] = \sum_{n=1}^{m-1} b_n \lambda_n^{-it} + \frac{a_m \lambda_m}{\sqrt{\lambda_m^2 - \lambda_{m-1}^2}} \lambda_m^{-it}. \quad (1.30)$$

(b) Conversely, let  $\{b_n\} \subset \mathbb{C}$ . Choose  $a_1 \in \mathbb{C}$ , and for  $m \geq 2$ , let

$$a_m = \sqrt{\lambda_m^2 - \lambda_{m-1}^2} \left( a_1 - \sum_{n=1}^{m-1} \frac{b_n}{\lambda_n} \right). \quad (1.31)$$

Define  $f$  by the formal orthonormal expansion (1.28). Then the partial sums  $S_m[f]$  satisfy (1.30) for  $m \geq 2$ .

Under additional conditions, we can give analytic meaning to these formal identities:

**Theorem 1.7.** (a) Let  $\{b_n\} \subset \mathbb{C}$  be a sequence for which

$$\sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} \quad (1.32)$$

converges. Define  $a_m$  for  $m \geq 1$  by

$$a_m = \sqrt{\lambda_m^2 - \lambda_{m-1}^2} \sum_{n=m}^{\infty} \frac{b_n}{\lambda_n}, \quad (1.33)$$

and  $f$  by (1.28). Then the conclusion (1.30) of Proposition 1.6(a) remains valid.

(b) *If in addition*

$$\sum_{m=1}^{\infty} (\lambda_m^2 - \lambda_{m-1}^2) \left| \sum_{n=m}^{\infty} \frac{b_n}{\lambda_n} \right|^2 < \infty, \quad (1.34)$$

then  $f$  defined by (1.28) has  $f \in \mathcal{H}$ , and this last sum equals  $\|f\|^2$ .

(c) *If in addition*

$$\lim_{m \rightarrow \infty} \frac{a_m \lambda_m}{\sqrt{\lambda_m^2 - \lambda_{m-1}^2}} = 0, \quad (1.35)$$

we have

$$\lim_{m \rightarrow \infty} \left\| S_m[f] - \sum_{n=1}^{m-1} b_n \lambda_n^{-it} \right\|_{L_{\infty}(\mathbb{R})} = 0, \quad (1.36)$$

and as functions in  $\mathcal{G}$ ,

$$f(t) = \sum_{n=1}^{\infty} b_n \lambda_n^{-it}. \quad (1.37)$$

In particular, if

$$P(t) = \sum_{n=1}^{\ell} b_n \lambda_n^{-it},$$

then Theorem 1.7(b) implies that

$$\int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt = \sum_{m=1}^{\ell} (\lambda_m^2 - \lambda_{m-1}^2) \left| \sum_{n=m}^{\ell} \frac{b_n}{\lambda_n} \right|^2. \quad (1.38)$$

## 2. Proofs of Theorems 1.1-1.4

**Proof of Theorem 1.1.** Let

$$\phi_n^{\#}(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}. \quad (2.1)$$

We use the following simple consequence of the residue theorem: for real  $\mu$ ,

$$\int_{-\infty}^{\infty} \frac{e^{i\mu t}}{\pi(1+t^2)} dt = e^{-|\mu|}. \quad (2.2)$$



Then for  $|\mu| \leq \log \lambda_{n-1}$ ,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \phi_n^\#(t) \frac{e^{i\mu t}}{\pi(1+t^2)} dt \\
&= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \int_{-\infty}^{\infty} \left( \lambda_n e^{i(\mu - \log \lambda_n)t} - \lambda_{n-1} e^{i(\mu - \log \lambda_{n-1})t} \right) \frac{dt}{\pi(1+t^2)} \\
&= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \left( \lambda_n e^{-|\mu - \log \lambda_n|} - \lambda_{n-1} e^{-|\mu - \log \lambda_{n-1}|} \right) \\
&= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} (e^\mu - e^\mu) = 0.
\end{aligned}$$

For  $\log \lambda_{n-1} < \mu < \log \lambda_n$ , instead

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi_n^\# \frac{e^{i\mu t}}{\pi(1+t^2)} dt &= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \left( \lambda_n e^{-|\mu - \log \lambda_n|} - \lambda_{n-1} e^{-|\mu - \log \lambda_{n-1}|} \right) \\
&= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \left( e^\mu - \frac{\lambda_{n-1}^2}{e^\mu} \right).
\end{aligned}$$

For  $\mu \geq \log \lambda_n$ , instead

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi_n^\# \frac{e^{i\mu t}}{\pi(1+t^2)} dt &= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \left( \lambda_n e^{-|\mu - \log \lambda_n|} - \lambda_{n-1} e^{-|\mu - \log \lambda_{n-1}|} \right) \\
&= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} (\lambda_n^2 e^{-\mu} - \lambda_{n-1}^2 e^{-\mu}).
\end{aligned}$$

In summary,

$$\int_{-\infty}^{\infty} \phi_n^\# \frac{e^{i\mu t}}{\pi(1+t^2)} dt = \begin{cases} 0, & |\mu| \leq \log \lambda_{n-1} \\ \frac{1}{e^\mu \sqrt{\lambda_n^2 - \lambda_{n-1}^2}} (e^{2\mu} - \lambda_{n-1}^2), & \log \lambda_{n-1} \leq \mu < \log \lambda_n \\ \frac{1}{e^\mu} \sqrt{\lambda_n^2 - \lambda_{n-1}^2}, & \mu \geq \log \lambda_n \end{cases}. \quad (2.3)$$

This immediately yields the desired orthogonality relations for  $\phi_n = \phi_n^\#$ . Finally, (2.3) shows that

$$\int_{-\infty}^{\infty} \left| \phi_n^\#(t) \right|^2 \frac{dt}{\pi(1+t^2)} = \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \int_{-\infty}^{\infty} \phi_n^\#(t) \frac{\lambda_n e^{i(\log \lambda_n)t}}{\pi(1+t^2)} dt = 1.$$

□

**Proof of Proposition 1.2.**

(a)

$$\begin{aligned} |\phi_n(t)| &= \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \left| \lambda_n - \lambda_{n-1} (\lambda_{n-1}/\lambda_n)^{-it} \right| \\ &\leq \frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} (\lambda_n + \lambda_{n-1}) = \sqrt{\frac{\lambda_n + \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}}, \end{aligned}$$

with equality if  $t = \pi / (\log \lambda_{n-1}/\lambda_n)$ .

(b) This is immediate.

(c)

$$\phi_n'(t) = \frac{-i}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} (\lambda_n^{1-it} \log \lambda_n - \lambda_{n-1}^{1-it} \log \lambda_{n-1}), \quad (2.4)$$

so the result follows as in (a).

(d) Using our convention  $\lambda_0 = 0$ ,

$$\sum_{j=1}^m \sqrt{\lambda_j^2 - \lambda_{j-1}^2} \phi_j(t) = \sum_{j=1}^m (\lambda_j^{1-it} - \lambda_{j-1}^{1-it}) = \lambda_m^{1-it}.$$

(e) From (2.4) and (d),

$$\begin{aligned} &\phi_n'(t) \\ &= \frac{-i}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \left( \log \lambda_n \sum_{j=1}^n \sqrt{\lambda_j^2 - \lambda_{j-1}^2} \phi_j(t) - \log \lambda_{n-1} \sum_{j=1}^{n-1} \sqrt{\lambda_j^2 - \lambda_{j-1}^2} \phi_j(t) \right) \\ &= \frac{-i}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \sum_{j=1}^{n-1} \sqrt{\lambda_j^2 - \lambda_{j-1}^2} \phi_j(t) (\log \lambda_n - \log \lambda_{n-1}) - i \log \lambda_n \phi_n(t). \end{aligned}$$

So by orthonormality,

$$\int_{-\infty}^{\infty} \frac{|\phi_n'(t)|^2}{\pi(1+t^2)} dt = \sum_{j=1}^{n-1} \frac{\lambda_j^2 - \lambda_{j-1}^2}{\lambda_n^2 - \lambda_{n-1}^2} (\log \lambda_n - \log \lambda_{n-1})^2 + (\log \lambda_n)^2.$$

This telescopes to the right-hand side of (1.10).

(f) Let

$$\begin{aligned}\psi_n(t) &= \lambda_n^{-it} - \left(\frac{\lambda_{n-1}}{\lambda_n}\right) \lambda_{n-1}^{-it} \\ &= \lambda_n^{-1} \sqrt{\lambda_n^2 - \lambda_{n-1}^2} \phi_n(t)\end{aligned}$$

denote the  $n$ th “monic” Dirichlet orthogonal polynomial. The orthonormality relations show that for any “monic” Dirichlet polynomial  $P(t) = \psi_n(t) + \sum_{j=1}^{n-1} a_j \phi_j(t)$ , we have

$$\int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt = \frac{\lambda_n^2 - \lambda_{n-1}^2}{\lambda_n^2} + \sum_{j=1}^{n-1} |a_j|^2.$$

Thus, the inf over such monic polynomials  $P$  is  $\frac{\lambda_n^2 - \lambda_{n-1}^2}{\lambda_n^2}$ , with equality iff  $P(t) = \psi_n(t)$ .  $\square$

**Proof of Theorem 1.3.** (a) Let  $n \geq 2$  and  $\tau = \log \frac{\lambda_n}{\lambda_{n-1}}$ . Elementary trigonometric identities give

$$\begin{aligned}& (\lambda_n^2 - \lambda_{n-1}^2) \phi_n(t) \overline{\phi_n(s)} \\ &= (\lambda_n^{1-it} - \lambda_{n-1}^{1-it}) (\lambda_n^{1+is} - \lambda_{n-1}^{1+is}) \\ &= (\lambda_{n-1} \lambda_n)^{1+i\frac{s-t}{2}} \left[ \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\frac{1}{2}-i\frac{t}{2}} - \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\frac{1}{2}-i\frac{t}{2}} \right] \\ &\quad \times \left[ \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\frac{1}{2}+i\frac{s}{2}} - \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\frac{1}{2}+i\frac{s}{2}} \right] \\ &= 4 (\lambda_{n-1} \lambda_n)^{1+i\frac{s-t}{2}} \sin\left((t+i)\frac{\tau}{2}\right) \sin\left((s-i)\frac{\tau}{2}\right) \\ &= 2 (\lambda_{n-1} \lambda_n)^{1+i\frac{s-t}{2}} \left[ \cos\left((t-s+2i)\frac{\tau}{2}\right) - \cos\left((s+t)\frac{\tau}{2}\right) \right] \\ &= 4 (\lambda_{n-1} \lambda_n)^{1+i\frac{s-t}{2}} \left[ \sin^2\left((s+t)\frac{\log \frac{\lambda_n}{\lambda_{n-1}}}{4}\right) \right. \\ &\quad \left. - \sin^2\left((t-s+2i)\frac{\log \frac{\lambda_n}{\lambda_{n-1}}}{4}\right) \right].\end{aligned}$$

Now add for  $n = 2, 3, \dots, m$ , and recall  $\phi_1(t) = 1$ .

(b) When  $s = t = x$ , the above identity simplifies to

$$\begin{aligned} (\lambda_n^2 - \lambda_{n-1}^2) |\phi_n(x)|^2 &= 4(\lambda_{n-1}\lambda_n) \left[ \sin^2\left(\frac{x}{2} \log \frac{\lambda_n}{\lambda_{n-1}}\right) - \sin^2\left(\frac{i}{2} \log \frac{\lambda_n}{\lambda_{n-1}}\right) \right] \\ &= (\lambda_n - \lambda_{n-1})^2 + 4\lambda_{n-1}\lambda_n \sin^2\left(\frac{x}{2} \log \frac{\lambda_n}{\lambda_{n-1}}\right). \end{aligned}$$

Now add over  $n = 2, 3, \dots, m$ , and recall  $\phi_1(t) = 1$ .

(c) Using first  $|\sin u| \leq |u|$ , for all real  $u$ , and then  $\log(1+u) \leq u$  for  $u \geq 0$ ,

$$\begin{aligned} K_m(x, x) &\leq 1 + \sum_{n=2}^m \frac{1}{\lambda_n^2 - \lambda_{n-1}^2} \left[ (\lambda_n - \lambda_{n-1})^2 + 4\lambda_n\lambda_{n-1} \left[ \frac{x}{2} \log(\lambda_n/\lambda_{n-1}) \right]^2 \right] \\ &\leq 1 + \sum_{n=2}^m \frac{1}{\lambda_n^2 - \lambda_{n-1}^2} \left[ (\lambda_n - \lambda_{n-1})^2 + 4\lambda_n\lambda_{n-1} \left[ \frac{x}{2} \left( \frac{\lambda_n}{\lambda_{n-1}} - 1 \right) \right]^2 \right] \\ &= 1 + \sum_{n=2}^m \frac{(\lambda_n - \lambda_{n-1})^2}{\lambda_n^2 - \lambda_{n-1}^2} \left[ 1 + \lambda_n\lambda_{n-1} \left[ \frac{x}{\lambda_{n-1}} \right]^2 \right]. \end{aligned}$$

(d) Using  $\sin t = t(1 + o(1))$  as  $t \rightarrow 0$ , we see that as  $m \rightarrow \infty$ , with the  $o(1)$  term below having limit 0 as  $n \rightarrow \infty$ ,

$$\begin{aligned} &K_m(s, t) - 1 \\ &= 4 \sum_{n=2}^m \frac{\lambda_n^{2+i(s-t)} (1 + o(1))}{\lambda_n^2 - \lambda_{n-1}^2} \left( \log \frac{\lambda_n}{\lambda_{n-1}} \right)^2 \\ &\quad \times \left[ \left( \frac{s+t}{4} \right)^2 (1 + o(1)) - \left( \frac{t-s+2i}{4} \right)^2 (1 + o(1)) \right] \\ &= 4 \sum_{n=2}^m \frac{\lambda_n^{2+i(s-t)} (1 + o(1))}{\lambda_n^2 - \lambda_{n-1}^2} \left( \frac{\lambda_n}{\lambda_{n-1}} - 1 \right)^2 \\ &\quad \times \left[ \left( \frac{s+t}{4} \right)^2 (1 + o(1)) - \left( \frac{t-s+2i}{4} \right)^2 (1 + o(1)) \right] \\ &= \sum_{n=2}^m \frac{\lambda_n^{i(s-t)} (\lambda_n - \lambda_{n-1})^2}{\lambda_n^2 - \lambda_{n-1}^2} [1 + i(s-t) + st + o(1)] \\ &= \sum_{n=2}^m \frac{\lambda_n^{i(s-t)} (\lambda_n - \lambda_{n-1})}{\lambda_n + \lambda_{n-1}} [(1 + is)(1 - it) + o(1)], \end{aligned}$$

uniformly for  $s, t$  in compact subsets of the plane. Again using (1.18), we continue this as

$$\begin{aligned}
&= \sum_{n=2}^m \int_{\lambda_{n-1}}^{\lambda_n} \frac{u^{i(s-t)}}{2u} (1 + o(1)) [(1 + is)(1 - it) + o(1)] du \\
&= \frac{1}{2} (1 + is)(1 - it) \sum_{n=2}^m \int_{\lambda_{n-1}}^{\lambda_n} u^{i(s-t)-1} du + \sum_{n=2}^m o \left( \int_{\lambda_{n-1}}^{\lambda_n} |u^{\operatorname{Im}(t-s)-1}| du \right) \\
&= \frac{1}{2} (1 + is)(1 - it) \int_1^{\lambda_m} u^{i(s-t)-1} du + o \left( \int_1^{\lambda_m} u^{|\operatorname{Im}(t-s)-1|} du \right).
\end{aligned}$$

Here we are also using that  $\lambda_m \rightarrow \infty$  as  $m \rightarrow \infty$ , so that the  $o$  term grows at least as fast as  $\log \lambda_m$ . Simple calculations show that for complex  $\alpha$ , real nonnegative  $\beta$ , and for  $T \geq 1$ ,

$$\begin{aligned}
\int_1^T u^{i\alpha-1} du &= T^{i\alpha/2} (\log T) S \left( \frac{\alpha}{2\pi} \log T \right); \\
\int_1^T u^{\beta-1} du &= T^{\beta/2} (\log T) \tilde{S} \left( \frac{\beta}{2\pi} \log T \right).
\end{aligned}$$

Hence

$$\begin{aligned}
K_m(s, t) &= \frac{1}{2} (1 + is)(1 - it) \lambda_m^{i(s-t)/2} (\log \lambda_m) S \left( \frac{s-t}{2\pi} \log \lambda_m \right) \\
&\quad + o \left( \lambda_m^{|\operatorname{Im}(s-t)|/2} (\log \lambda_m) \tilde{S} \left( \frac{|\operatorname{Im}(s-t)|}{2\pi} \log \lambda_m \right) \right)
\end{aligned}$$

(e) Setting  $s = t = x$ , we also obtain (1.20). □

#### Proof of Theorem 1.4.

(a) We choose  $s = x + \frac{\alpha}{\log \lambda_m}$  and  $t = x + \frac{\beta}{\log \lambda_m}$  in (1.19). We see that

$$(1 + is)(1 - it) = 1 + x^2 + o(1),$$

and

$$\lambda_m^{i(s-t)/2} S \left( \frac{s-t}{2\pi} \log \lambda_m \right) = e^{i(\alpha-\beta)/2} S \left( \frac{\alpha-\beta}{2\pi} \right).$$

Then (1.21) follows from (1.19).

(b) This follows directly from (a), from Hurwitz' theorem, and the fact that the only zeros of  $S(z)$  are the non-zero integers. □

### 3. Proof of Theorem 1.5

**Proof of Theorem 1.5.** Now for  $j \geq 1$ , (1.5) and (2.4) show that

$$\phi'_j(t) + i(\log \lambda_j) \phi_j(t) = \frac{1}{\sqrt{\lambda_j^2 - \lambda_{j-1}^2}} (-i)(\log \lambda_j - \log \lambda_{j-1}) \lambda_{j-1}^{1-it}, \quad (3.1)$$

so

$$|\phi'_j(t) + i(\log \lambda_j) \phi_j(t)| = \frac{\lambda_{j-1}}{\lambda_j} \left(1 - \left(\frac{\lambda_{j-1}}{\lambda_j}\right)^2\right)^{-1/2} \log \frac{\lambda_j}{\lambda_{j-1}}. \quad (3.2)$$

Next if

$$P(t) = \sum_{j=1}^n a_j \phi_j(t),$$

we recall that  $\phi_1(t) = 1$  and write

$$\begin{aligned} P'(t) &= \sum_{j=2}^m a_j [\phi'_j(t) + i(\log \lambda_j) \phi_j(t)] - \sum_{j=2}^m a_j i(\log \lambda_j) \phi_j(t) \\ &=: T_1(t) + T_2(t). \end{aligned}$$

Here, using Cauchy-Schwarz, (3.2) and the inequality  $\log(1+u) \leq u$ ,  $u \geq 0$ ,

$$\begin{aligned} |T_1(t)| &\leq \left(\sum_{j=2}^m |a_j|^2\right)^{1/2} \left(\sum_{j=2}^m \left(\frac{\lambda_{j-1}}{\lambda_j}\right)^2 \left(1 - \left(\frac{\lambda_{j-1}}{\lambda_j}\right)^2\right)^{-1} \left(\frac{\lambda_j}{\lambda_{j-1}} - 1\right)^2\right)^{1/2} \\ &\leq \left(\int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt\right)^{1/2} \left(\sum_{j=2}^m \frac{\lambda_j - \lambda_{j-1}}{\lambda_j + \lambda_{j-1}}\right)^{1/2}, \end{aligned}$$

so the triangle inequality and orthonormality, and our bound on  $T_1$  give

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \frac{|P'(t)|^2}{\pi(1+t^2)} dt\right)^{1/2} &\leq \left(\int_{-\infty}^{\infty} \frac{|T_1(t)|^2}{\pi(1+t^2)} dt\right)^{1/2} + \left(\int_{-\infty}^{\infty} \frac{|T_2(t)|^2}{\pi(1+t^2)} dt\right)^{1/2} \\ &\leq \left(\int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt\right)^{1/2} \left(\sum_{j=2}^m \frac{\lambda_j - \lambda_{j-1}}{\lambda_j + \lambda_{j-1}}\right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& + \left( \sum_{j=2}^m |a_j|^2 (\log \lambda_j)^2 \right)^{1/2} \\
& \leq \left( \int_{-\infty}^{\infty} \frac{|P(t)|^2}{\pi(1+t^2)} dt \right)^{1/2} \\
& \quad \times \left( \log \lambda_m + \left( \sum_{j=2}^m \frac{\lambda_j - \lambda_{j-1}}{\lambda_j + \lambda_{j-1}} \right)^{1/2} \right).
\end{aligned}$$

Here

$$\sum_{j=2}^m \frac{\lambda_j - \lambda_{j-1}}{\lambda_j + \lambda_{j-1}} \leq \sum_{j=2}^m \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \leq \left( \sum_{j=2}^m \int_{\lambda_{j-1}}^{\lambda_j} \frac{dt}{t} \right) = \log \lambda_m.$$

We note that using our explicit expression for  $\phi'_j$ , it is possible to obtain an explicit orthonormal expansion for  $P'$  in terms of the  $\{\phi_j\}$ . However, estimation of that does not seem to lead to a better estimate than that in (1.23/1.24).  $\square$

#### 4. Proof of Theorems 1.6 and 1.7

##### Proof of Theorem 1.6.

(a) From Theorem 1.1,

$$\begin{aligned}
S_m[f] &= \sum_{n=1}^m \frac{a_n}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} [\lambda_n^{1-it} - \lambda_{n-1}^{1-it}] \\
&= \sum_{n=1}^{m-1} \lambda_n^{1-it} \left\{ \frac{a_n}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} - \frac{a_{n+1}}{\sqrt{\lambda_{n+1}^2 - \lambda_n^2}} \right\} + \frac{a_m}{\sqrt{\lambda_m^2 - \lambda_{m-1}^2}} \lambda_m^{1-it},
\end{aligned}$$

by a summation by parts. Our definition (1.29) of  $\{b_n\}$  gives the result.

(b) It is easily seen from (1.31) that  $b_n$  satisfies (1.29) for  $n \geq 1$ , so the result follows from (a).  $\square$

##### Proof of Theorem 1.7.

(a) If  $\{a_m\}$  are defined by (1.33) for  $m \geq 1$ , then it is easily seen that (1.29) is satisfied for  $n \geq 1$ , and Theorem 1.6(a) yields the result.

(b) Our hypothesis (1.34) asserts that

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

so indeed  $f = \sum_{n=1}^{\infty} a_n \phi_n \in \mathcal{H}$ , and

$$\|f\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{m=1}^{\infty} (\lambda_m^2 - \lambda_{m-1}^2) \left| \sum_{n=m}^{\infty} \frac{b_n}{\lambda_n} \right|^2.$$

(c) From (1.30),

$$\left\| S_m[f](t) - \sum_{n=1}^{m-1} b_n \lambda_n^{-it} \right\|_{L^\infty(\mathbb{R})} = \frac{|a_m| \lambda_m}{\sqrt{\lambda_m^2 - \lambda_{m-1}^2}}.$$

Then (1.36) follows from (1.35). Moreover, then

$$\begin{aligned} \left\| f(t) - \sum_{n=1}^{m-1} b_n \lambda_n^{-it} \right\| &\leq \|f - S_m[f]\| + \left\| S_m[f](t) - \sum_{n=1}^{m-1} b_n \lambda_n^{-it} \right\| \\ &\leq \left( \sum_{n=m}^{\infty} |a_n|^2 \right)^{1/2} + \left\| S_m[f](t) - \sum_{n=1}^{m-1} b_n \lambda_n^{-it} \right\|_{L^\infty(\mathbb{R})} \\ &\rightarrow 0, \quad m \rightarrow \infty. \end{aligned}$$

Then (1.37) follows. □

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