APPLICATIONS OF NEW GERONIMUS TYPE IDENTITIES FOR REAL ORTHOGONAL POLYNOMIALS

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Abstract. Let μ be a positive measure on the real line, with associated orthogonal polynomials $\{p_n\}$. Let $\operatorname{Im} a \neq 0$. Then there is an explicit constant c_n such that for all polynomials P of degree at most $2n-2$,

$$c_n \int_{-\infty}^{\infty} \frac{P(t)}{|p_n(a)\,p_{n-1}(t) - p_{n-1}(a)\,p_n(t)|^2} dt = \int P \, d\mu.$$

In this paper, we provide a self-contained proof of the identity. Moreover, we apply the formula to deduce a weak convergence result, a discrepancy estimate, and also to establish a Gauss quadrature associated with μ with nodes at the zeros of $p_n(a)\,p_{n-1}(t) - p_{n-1}(a)\,p_n(t)$.

Orthogonal Polynomials on the real line, Geronimus formula, discerepancy, weak convergence, Gauss quadrature. 42C05

1. Introduction[1]

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, ...$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \ \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

In analysis and applications of orthogonal polynomials, the reproducing kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(x)\,p_j(y).$$

plays a key role. The Christoffel-Darboux fomula asserts that

$$K_n(x, y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\,p_{n-1}(y) - p_{n-1}(x)\,p_n(y)}{x - y}.$$

We shall also use the notation

$$L_n(x, y) = (x - y)\,K_n(x, y) = \frac{\gamma_{n-1}}{\gamma_n} (p_n(x)\,p_{n-1}(y) - p_{n-1}(x)\,p_n(y)) \tag{1.1}$$

and for non-real a,

$$E_{n,a}(z) = \sqrt{\frac{2\pi}{|L_n(a, \bar{a})|}} L_n(\bar{a}, z). \tag{1.2}$$

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In a recent paper [4], we used the theory of de Branges spaces [1] to show that for $\operatorname{Im} a > 0$, and all polynomials P of degree $\leq 2n-2$, we have

$$\int_{-\infty}^{\infty} \frac{P(t)}{|E_{n,a}(t)|^2} dt = \int P \, d\mu. \tag{1.3}$$

This may be regarded as an analogue of Geronimus' formula for the unit circle, where instead of $E_{n,a}$, we have a multiple of the orthonormal polynomial on the unit circle in the denominator [2, Thm. V.2.2, p. 198], [5, p. 95, 955]. The name Geronimus' formula is not universal, some talk of continuous analogues of quadrature, or Bernstein-Szegő approximations. There is an earlier real line analogue, rediscovered by Barry Simon [6, Theorem 2.1, p. 5], namely

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(t)}{\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 p_n^2(t) + p_{n-1}^2(t)} dt = \int P \, d\mu.$$

Simon calls this a real line orthogonal polynomial analogue of *Carmona's formula* and refers to earlier work of Krutikov and Remling [3]. The latter seems to be a special case of (1.3) with $(p_{n-1}/p_n)(\bar{a}) = \pm i\gamma_{n-1}/\gamma_n$. As far as the author is aware, (1.3) is new. At least, the author could not find it in a search of the orthogonal polynomial and orthogonal rational function literature.

In this paper, we present a self-contained proof of (1.3), and deduce results on weak convergence, discrepancy estimates, and a Gauss quadrature type formula with complex nodes. Recall that μ is said to be *determinate* if the moment problem

$$\int x^j d\nu(x) = \int x^j d\mu(x), \; j = 0, 1, 2, ...,$$

has the unique solution $\nu = \mu$ from the class of positive measures. We also say a function f has *polynomial growth at* ∞ if for some $L > 0$ and for large enough $|x|$,

$$|f(x)| \leq |x|^L.$$

We shall prove:

Theorem 1.1
Let μ be a positive measure on the real line with all finite power moments and let μ be determinate. Let $\{a_n\}$ be a sequence of complex numbers with non-zero imaginary part. Then for all functions $f : \mathbb{R} \to \mathbb{R}$ having polynomial growth at ∞, and that are Riemann-Stieltjes integrable with respect to μ, we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \frac{dx}{\left|E_{n,a_n}(x)^2\right|} = \int f \, d\mu. \tag{1.4}$$

Of course, if f is continuous on the real line, it will be locally Riemann-Stieltjes integrable with respect to μ. Simon [6] noted the weak convergence involving his Carmona type formula. When μ is indeterminate, the weak convergence will fail, since then $E_{n,a}$ has a finite limiting value in the plane. In this case, the limit (1.4) should probably hold only for a limited class of entire functions.

One consequence of the weak convergence is that $1/|E_{n,a}|^2 \to 0$ outside the support, in some sense, yielding information on the behavior of K_n :

Corollary 1.2
Assume the hypotheses of Theorem 1.1.
(a) Let J be a closed subset of $\mathbb{R}\backslash\operatorname{supp}[\mu]$. Then

$$\lim_{n\to\infty} |\operatorname{Im} a_n| \, K_n(a_n, \bar{a}_n) \int_J \frac{dt}{\left(t^2 + |a_n|^2\right) |K_n(t, a_n)|^2} = 0. \tag{1.5}$$

(b) Assume, in addition, that supp$[\mu]$ is compact and that J is a compact set disjoint from supp$[\mu]$. Then

$$\limsup_{n\to\infty} \left\{ \frac{|\operatorname{Im} a_n|}{1 + |a_n|^2} K_n(a_n, \bar{a}_n) \int_J \frac{dt}{|K_n(t, a_n)|^2} \right\}^{1/n} < 1. \tag{1.6}$$

We can also prove a discrepancy type estimate for the measure $\frac{dt}{|E_{n,a}(t)|^2} - d\mu(t)$. The main tool here is the Markov-Stieltjes inequalities, and the formulation involves the Christoffel function

$$\lambda_n(x) = \frac{1}{K_n(x, x)} = \inf_{\deg(P) \leq n-1} \frac{\int P^2 d\mu}{P^2(x)}.$$

Theorem 1.3
Assume the hypotheses of Theorem 1.1. Let $c, d \in$supp$[\mu]$ and $\varepsilon > 0$. Then for large enough n, we have

$$\sup_{x\in[c,d]} \left| \int_{-\infty}^{x} \left(\frac{dt}{|E_{n,a_n}(t)|^2} - d\mu(t) \right) \right| \leq 3 \sup_{x\in[c-\varepsilon, d+\varepsilon]} \lambda_n(x). \tag{1.7}$$

We note that the same estimate (1.7) holds when $\frac{dt}{|E_{n,a_n}(t)|^2}$ is replaced by any positive measure sharing the same first $2n-2$ power moments with μ. Another consequence of (1.3) is a Gauss type quadrature formula with complex nodes. Recall that if we fix real ξ, then $L_n(t, \xi)$ has n or $n-1$ real zeros $\{t_{jn}\}$, one of which is ξ. There is an associated Gauss quadrature rule [2, Thm. I.3.2, p. 21]:

$$\sum_j \lambda_n(t_{jn}) P(t_{jn}) = \int P \, d\mu, \tag{1.8}$$

valid for all polynomials P of degree $\leq 2n-2$. The classical Gauss quadrature, involving the zeros $\{x_{jn}\}$ of p_n, is the case where ξ is a zero of p_n. By using elementary properties of the Poisson kernel, one can show that if we let $\operatorname{Im} a$ approach 0 in (1.3), then we we obtain this last quadrature formula. In general, when a has non-zero imaginary part, one obtains an analogue of (1.8) with complex nodes. In the formulation, we need the Schwarz reflection of a function g,

$$g^*(z) = \overline{g(\bar{z})}. \tag{1.9}$$

Theorem 1.4
Let μ be a positive measure on the real line with at least $n+1$ points in its support and the first $2n$ finite power moments. Let $a \in \mathbb{C}\backslash\mathbb{R}$ and $\{z_j\}_{j=1}^{n}$ denote the zeros of $L_n(a, \cdot)$. Assume they are simple, and let

$$\lambda_j = \frac{2\pi i}{E_{n,a}(z_j) E_{n,a}^{*\prime}(z_j)}, 1 \leq j \leq n. \tag{1.10}$$

Then for all polynomials P of degree at most $2n-2$,

$$\sum_{j=1}^{n} \lambda_j P(z_j) = \int P \, d\mu.$$

We note that it is possible, for some finitely many exceptional choices of a, that $E_{n,a}$ has multiple zeros, see the remark after Lemma 2.2. In this case, the quadrature involves derivatives of P at the multiple zeros. We note too that as $\operatorname{Im} a \to 0$, this last formula reduces to (1.8).

2. Proof of (1.3)

The proof uses similar ideas to those in [4], but is easier to follow because it is self contained, and avoids use of the de Branges theory. Throughout, we assume the hypotheses of Theorem 1.1.

Theorem 2.1
Let $a \in \mathbb{C} \backslash \mathbb{R}$. For polynomials R of degree at most $2n-2$,

$$\int_{-\infty}^{\infty} \frac{R(t)}{|E_{n,a}(t)|^2} dt = \int R \, d\mu. \tag{2.1}$$

Recall that $L_n(z,v) = (z-v) K_n(z,v)$ and the notation (1.9) for the Schwarz reflection.

Lemma 2.2
(a) For all complex α, β, z, v,

$$L_n(z,v) L_n(\alpha,\beta) = L_n(\alpha,z) L_n(\beta,v) - L_n(\beta,z) L_n(\alpha,v). \tag{2.2}$$

(b) Let $\operatorname{Im} a > 0$. Then

$$K_n(z,v) = \frac{i}{2\pi} \frac{E_{n,a}(z) E_{n,a}^*(v) - E_{n,a}^*(z) E_{n,a}(v)}{z-v}. \tag{2.3}$$

(c) If $\operatorname{Im} a > 0$, all zeros of $E_{n,a}$ are in the lower half plane.

Proof
(a) Just substitute the definition (1.1) of L_n into the right-hand side of (2.2), then multiply out, cancel factors, and refactorize.
(b) The identity (2.2), with $\alpha = a; \beta = \bar{a}$; gives

$$L_n(z,v) L_n(a,\bar{a}) = L_n(a,z) L_n(\bar{a},v) - L_n(\bar{a},z) L_n(a,v). \tag{2.4}$$

Since $L_n(z,v)$ has real coefficients as a polynomial in z, v, and

$$L_n(a,\bar{a}) = 2i \operatorname{Im} a K_n(a,\bar{a}) = i |L_n(a,\bar{a})|,$$

we obtain

$$K_n(z,v) = \frac{i}{|L_n(a,\bar{a})|} \frac{L_n(\bar{a},z) \overline{L_n(\bar{a},\bar{v})} - \overline{L_n(\bar{a},\bar{z})} L_n(\bar{a},v)}{z-v},$$

and (2.3) follows on taking account of (1.2).
(c) It suffices to show that $K_n(\bar{a},\cdot)$ has all its zeros in the lower half plane, since $E_{n,a}$ is a multiple of $(\cdot - \bar{a}) K_n(\bar{a},\cdot)$. In turn, in view of the Christoffel-Darboux formula, and the fact that p_{n-1} and p_n have real zeros, it suffices to show that

$$\frac{p_{n-1}(z)}{p_n(z)} - \frac{p_{n-1}(\bar{a})}{p_n(\bar{a})}$$

cannot vanish for $\operatorname{Im} z \geq 0$. By the Lagrange interpolation formula at the zeros $\{x_{jn}\}$ of p_n, or by partial fraction decomposition,

$$\frac{p_{n-1}(z)}{p_n(z)} = \sum_{j=1}^{n} \frac{p_{n-1}(x_{jn})}{p_n'(x_{jn})} \frac{1}{z - x_{jn}}.$$

Applying l'Hospital's rule to the Christoffel-Darboux formula gives

$$K_n(x,x) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n'(x) p_{n-1}(x) - p_{n-1}'(x) p_n(x) \right),$$

and in particular,

$$K_n(x_{jn}, x_{jn}) = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_{jn}) p_{n-1}(x_{jn}).$$

Thus

$$\frac{p_{n-1}(z)}{p_n(z)} = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=1}^{n} \frac{p_{n-1}^2(x_{jn})}{K_n(x_{jn}, x_{jn})} \frac{1}{z - x_{jn}} \tag{2.5}$$

so

$$\operatorname{Im}\left(\frac{p_{n-1}(z)}{p_n(z)} \right) = -(\operatorname{Im} z) \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=1}^{n} \frac{p_{n-1}^2(x_{jn})}{K_n(x_{jn}, x_{jn})} \frac{1}{|z - x_{jn}|^2}.$$

In particular, for $\operatorname{Im} z > 0$, $\operatorname{Im}\left(\frac{p_{n-1}(z)}{p_n(z)} \right) < 0$, while as $\operatorname{Im} \bar{a} < 0$, $\operatorname{Im}\left(\frac{p_{n-1}(\bar{a})}{p_n(\bar{a})} \right) > 0$, so $\frac{p_{n-1}(z)}{p_n(z)} - \frac{p_{n-1}(\bar{a})}{p_n(\bar{a})}$ cannot be zero. ■

Remark
It is possible for $E_{n,a}$ to have multiple zeros. Indeed, we see this occurs iff both

$$\begin{aligned} p_n(z) p_{n-1}(\bar{a}) - p_{n-1}(z) p_n(\bar{a}) &= 0; \\ p_n'(z) p_{n-1}(\bar{a}) - p_{n-1}'(z) p_n(\bar{a}) &= 0. \end{aligned}$$

These latter two relations are equivalent to

$$p_n'(z) p_{n-1}(z) - p_{n-1}'(z) p_n(z) = 0 \text{ and } \frac{p_{n-1}}{p_n}(z) = \frac{p_{n-1}}{p_n}(\bar{a}).$$

Let us choose a z that is one of the $n-1$ zeros of $p_n' p_{n-1} - p_{n-1}' p_n$ in the lower half-plane. (It is easily seen by differentiating (2.5) that there are none on the real line, and of course, they occur in conjugate pairs). Then let us choose a with $\operatorname{Im} a > 0$ such that

$$\frac{p_{n-1}}{p_n}(z) = \frac{p_{n-1}}{p_n}(\bar{a}).$$

There are n choices for a, counting multiplicity. For this choice of a, $E_{n,a}$ will have at least a double zero at z. Of course, there are only finitely many such exceptional a.

Proof of Theorem 2.1
We shall assume $\operatorname{Im} a > 0$. The case $\operatorname{Im} a < 0$ follows by taking conjugates. We first prove the reproducing kernel relation

$$P(z) = \int_{-\infty}^{\infty} \frac{P(t) K_n(t,z)}{|E_{n,a}(t)|^2} dt = \int_{-\infty}^{\infty} \frac{P(t) K_n(t,z)}{E_{n,a}(t) E_{n,a}^*(t)} dt. \tag{2.6}$$

Here z is any complex number, and P is any polynomial of degree at most $n-1$. Let us assume first that $\operatorname{Im} z > 0$. From the formula (2.3) for K_n, we see that

$$\int_{-\infty}^{\infty} \frac{P(t) K_n(t,z)}{E_{n,a}(t) E_{n,a}^*(t)} dt$$

$$(2.7) = \frac{i}{2\pi}\left(E_{n,a}^*(z)\int_{-\infty}^{\infty}\frac{P(t)}{E_{n,a}^*(t)(t-z)}dt - E_{n,a}(z)\int_{-\infty}^{\infty}\frac{P(t)}{E_{n,a}(t)(t-z)}dt\right).$$

Recall that $E_{n,a}$ has all its zeros in the lower-half plane, so $E_{n,a}^*$ has all its zeros in the upper-half plane. Then the integrand $\frac{P(t)}{E_{n,a}^*(t)(t-z)}$ in the first integral is analytic as a function of t in the closed lower-half plane, and is $O\left(|t|^{-2}\right)$ as $|t| \to \infty$. By the residue theorem, or Cauchy's integral theorem, the first integral is 0. Next, the integrand $\frac{P(t)}{E_{n,a}(t)(t-z)}$ in the second integral is analytic as a function of t in the closed upper-half plane, except for a simple pole at z (unless $P(z)=0$) and is $O\left(|t|^{-2}\right)$ as $|t| \to \infty$. The residue theorem shows that

$$\int_{-\infty}^{\infty}\frac{P(t)}{E_{n,a}(t)(t-z)}dt = 2\pi i\frac{P(z)}{E_{n,a}(z)}.$$

Substituting this into (2.7) gives (2.6) for $\operatorname{Im} z > 0$. As both sides of (2.6) are polynomials in z, analytic continuation gives it for all z.

Now we can prove (2.1). We can write $R = PS$ where both P and S are polynomials of degree $\leq n-1$. We multiply the identity in (2.6) by S and then integrate with respect to μ. We obtain

$$\begin{aligned}
\int R\, d\mu &= \int (PS)(z)\, d\mu(z) \\
&= \int S(z)\left[\int_{-\infty}^{\infty} P(t)\frac{K_n(t,z)}{|E_{n,a}(t)|^2}dt\right] d\mu(z) \\
&= \int_{-\infty}^{\infty} P(t)\frac{1}{|E_{n,a}(t)|^2}\left[\int S(z) K_n(t,z)\, d\mu(z)\right] dt \\
&= \int_{-\infty}^{\infty} P(t)\frac{1}{|E_{n,a}(t)|^2} S(t)\, dt \\
&= \int_{-\infty}^{\infty}\frac{R}{|E_{n,a}|^2}.
\end{aligned}$$

Here, we have used the reproducing kernel formula for the measure μ. Moreover, the interchange of integrals is justified by absolute convergence of all integrals involved. ■

3. Weak Convergence, Discrepancy, Gauss Quadrature

Proof of Theorem 1.1

Let f be Riemann-Stieltjes integrable with respect to μ and of polynomial growth at ∞, and let $\varepsilon > 0$. Since μ is determinate, there exist upper and lower polynomials P_u and P_ℓ such that

$$P_\ell \leq f \leq P_u \text{ in } (-\infty,\infty)$$

and

$$\int (P_u - P_\ell)\, d\mu < \varepsilon.$$

See, for example, [2, Theorem 3.3, p. 73]. Then for n so large that $2n-2$ exceeds the degree of P_u and P_ℓ, (1.3) gives

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{f}{|E_{n,a_n}|^2} - \int f\ d\mu \\
= & \int_{-\infty}^{\infty} \frac{f - P_\ell}{|E_{n,a_n}|^2} - \int (f - P_\ell\ )\, d\mu \\
\leq & \int_{-\infty}^{\infty} \frac{P_u - P_\ell}{|E_{n,a_n}|^2} - 0 \\
= & \int (P_u - P_\ell)\, d\mu < \varepsilon.
\end{aligned}$$

Similarly, for large enough n,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{f}{|E_{n,a_n}|^2} - \int f\ d\mu \\
= & \int_{-\infty}^{\infty} \frac{f - P_u}{|E_{n,a_n}|^2} + \int (P_u - f)\, d\mu \\
\geq & \int_{-\infty}^{\infty} \frac{P_\ell - P_u}{|E_{n,a_n}|^2} + 0 \\
= & \int (P_\ell - P_u)\, d\mu > -\varepsilon.
\end{aligned}$$

■

Proof of Corollary 1.2
(a) For all real x, let

$$f(x) = \frac{dist\,(x, \operatorname{supp}[\mu])}{1 + dist\,(x, \operatorname{supp}[\mu])}.$$

Here dist is the usual Euclidean distance between a point and a set. Then f is continuous, $0 \leq f \leq 1$ for all real x, $f = 0$ in supp$[\mu]$, and for $x \in J$,

$$f(x) \geq \frac{dist\,(J, \operatorname{supp}[\mu])}{1 + dist\,(J, \operatorname{supp}[\mu])} = c_0 > 0,$$

say. By the weak convergence,

$$\begin{aligned}
0 \quad \leq \quad & c_0 \limsup_{n\to\infty} \int_J \frac{1}{|E_{n,a}|^2} \leq \limsup_{n\to\infty} \int_J \frac{f}{|E_{n,a}|^2} \\
\leq \quad & \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{f}{|E_{n,a}|^2} \\
= \quad & \int f\ d\mu = 0.
\end{aligned}$$

Finally,

$$\begin{aligned} |E_{n,a_n}(t)|^2 &= \frac{\pi |t-a_n|^2 |K_n(t,\overline{a_n})|^2}{|\operatorname{Im} a_n| K_n(a_n,\overline{a_n})} \\ &\leq 4\pi \frac{\left(t^2+|a_n|^2\right) |K_n(t,\overline{a_n})|^2}{|\operatorname{Im} a_n| K_n(a_n,\overline{a_n})}. \end{aligned}$$

(b) We can cover compact J by finitely many open intervals, each at a positive distance to supp$[\mu]$. It then suffices to prove the conclusion for the closure of just one of these intervals. So we assume J consists of a single bounded interval. Next, as supp$[\mu]$ is compact, we may choose a set K consisting of two intervals, that contains supp$[\mu]$, but is disjoint from J. We can then choose polynomials P_n of degree $\leq n-1$ such that

$$|P_n| \leq 1 \text{ in } K \tag{3.1}$$

but

$$\liminf_{n\to\infty} \left(\inf_J |P_n|\right)^{1/n} > r > 1. \tag{3.2}$$

In the special case where $J=[-\alpha,\alpha]$ and $K=[-1,-\beta]\cup[\beta,1]$, and $0<\alpha<\beta<1$, we can just choose

$$P_n(x) = T_{\left[\frac{n-1}{2}\right]}\left(-1+2\left(\frac{x^2-\beta^2}{1-\beta^2}\right)\right).$$

Here T_m is the usual Chebyshev polynomial for $[-1,1]$ and $\left[\frac{n-1}{2}\right]$ denotes the integer part of $\frac{n-1}{2}$. The general case can be reduced to this special case, by enlarging the intervals of K so that they become symmetric about J, and then using a linear transformation.

Armed with the $\{P_n\}$ satisfying (3.1) and (3.2), we apply (1.3): for large enough n,

$$\begin{aligned} r^{2n}\int_J \frac{1}{|E_{n,a_n}|^2} &\leq \int_J \frac{P_n^2}{|E_{n,a_n}|^2} \\ &\leq \int P_n^2\, d\mu \leq \int d\mu. \end{aligned}$$

Finally, for t in the compact set J, we have for some $C>0$ depending only on the compact set,

$$|E_{n,a_n}(t)|^2 \leq C\frac{\left(1+|a_n|^2\right)|K_n(t,\overline{a_n})|^2}{|\operatorname{Im} a_n| K_n(a_n,\overline{a_n})}$$

where C is independent of n and t. Together with the previous estimate, this gives (1.6). ■

Proof of Theorem 1.3

The Markov-Stieltjes inequalities [2, p. 33] assert that

$$\sum_{j:x_{jn}<x_{kn}} \lambda_n(x_{jn}) \leq \int_{-\infty}^{x_{kn}} d\mu \leq \sum_{j:x_{jn}\leq x_{kn}} \lambda_n(x_{jn}).$$

Recall here that $\{x_{jn}\}$ are the zeros of p_n and λ_n is the nth Christoffel function for μ. Since the measure $\frac{dt}{|E_{n,a_n}(t)|^2}$ has the same first $2n-2$ power moments as $d\mu$, it has the same first $n-1$ orthogonal polynomials as $d\mu$, and the same nth Christoffel function. Thus it has the same Gauss quadrature involving $\{x_{jn}\}$ as $d\mu$, and so has the same Markov-Stieltjes inequalities (even though the $\{x_{jn}\}$ come from p_n, and there is no orthogonal polynomial of degree n for $\frac{dt}{|E_{n,a_n}(t)|^2}$). A cursory scan of the proof of Theorem 5.4 in [2, p. 32] verifies this. So

$$\sum_{j:x_{jn}<x_{kn}} \lambda_n(x_{jn}) \leq \int_{-\infty}^{x_{kn}} \frac{dt}{|E_{n,a_n}(t)|^2} \leq \sum_{j:x_{jn}\leq x_{kn}} \lambda_n(x_{jn}).$$

Combining the last two inequalities, we see that

$$\left| \int_{-\infty}^{x_{kn}} \left(\frac{dt}{|E_{n,a_n}(t)|^2} - d\mu(t) \right) \right| \leq \lambda_n(x_{kn}).$$

Then also, if $x \in (x_{kn}, x_{k-1,n})$, we deduce that

$$\begin{aligned} &\left| \int_{-\infty}^{x} \left(\frac{dt}{|E_{n,a_n}(t)|^2} - d\mu(t) \right) \right| \\ \leq\ & \lambda_n(x_{kn}) + \max\left\{ \int_{x_{kn}}^{x_{k-1,n}} \frac{dt}{|E_{n,a_n}(t)|^2}, \int_{x_{kn}}^{x_{k-1,n}} d\mu(t) \right\} \\ \leq\ & \lambda_n(x_{kn}) + \lambda_n(x_{k-1,n}) + \lambda_n(x_{kn}), \end{aligned}$$

again, by using the Markov-Stieltjes inequalities above. Now let c, d lie in supp$[\mu]$. As μ is determinate, both c and d attract zeros of p_n [2, Theorem 2.4, p. 67], so we can find for large enough n, and all $x \in [c,d]$, an index k such that $x \in (x_{kn}, x_{k-1,n})$ and both $x_{kn}, x_{k-1,n}$ lie in $[c-\varepsilon, d+\varepsilon]$. Then we obtain for large enough n,

$$\sup \left| \int_{-\infty}^{x} \left(\frac{dt}{|E_{n,a_n}(t)|^2} - d\mu(t) \right) \right| \leq 3 \sup_{[c-\varepsilon,d+\varepsilon]} \lambda_n.$$

■

Proof of Theorem 1.4
Let us assume that $\operatorname{Im} a > 0$ and P is a polynomial of degree $\leq 2n-2$. Then

$$\int_{-\infty}^{\infty} \frac{P}{|E_{n,a}|^2} = \int_{-\infty}^{\infty} \frac{P(t)}{E_{n,a}(t)\, E_{n,a}^*(t)} dt.$$

Here $E_{n,a}(z)$ has all its zeros in the lower-half plane. By contrast, $E_{n,a}^*(z)$ is a multiple of $L_n(a,z)$, which has all its zeros $\{z_j\}_{j=1}^n$ in the upper-half plane. By hypothesis, they are simple. Moreover, as $|t| \to \infty$,

$$\frac{P(t)}{E_{n,a}(t)\, E_{n,a}^*(t)} = O\left(t^{-2}\right).$$

We can then use the residue theorem to deduce that

$$\int_{-\infty}^{\infty} \frac{P(t)}{E_{n,a}(t)\, E_{n,a}^*(t)} dt = 2\pi i \sum_{j=1}^{n} \frac{P(z_j)}{E_{n,a}(z_j)\, E_{n,a}^{*\prime}(z_j)}.$$

■

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