LARGE SIEVE ESTIMATES ON ARCS OF A CIRCLE

LEONID GOLINSKII[1], DORON S. LUBINSKY[2], AND PAUL NEVAI[3]

Abstract. Let $0 \leq \alpha < \beta \leq 2\pi$ and let $\Delta \stackrel{\text{def}}{=} \left\{e^{i\theta} : \theta \in [\alpha, \beta]\right\}$. We show that for generalized (non–negative) polynomials P of degree r and $p > 0$, we have

$$\sum_{j=1}^{m} |P(a_j)|^p \left(\left|a_j - e^{i\alpha}\right| \left|a_j - e^{i\beta}\right| + \left(\frac{\beta - \alpha}{pr + 1}\right)^2 \right)^{1/2}$$
$$\leq c\tau (pr + 1) \int_{\alpha}^{\beta} \left|P\left(e^{i\theta}\right)\right|^p d\theta,$$

where $a_1, a_2, ..., a_m \in \Delta$, c is an absolute constant (and, thus, it is independent of α, β, p, m, r, P, $\{a_j\}$) and τ is an explicitly determined constant which measures the number of points $\{a_j\}$ in a small interval. This implies large sieve inequalities for generalized (non–negative) trigonometric polynomials of degree r on subintervals of $[0, 2\pi]$. The essential feature is the uniformity of the estimate in α and β.

1. Introduction and Results

The large sieve of number theory may be viewed as an inequality for algebraic polynomials $P(z) = \sum_{j=0}^{r} d_j z^j$ on the unit circle $\mathbb{T}$ of the form

$$\sum_{j=1}^{m} \left|P\left(e^{i\alpha_j}\right)\right|^2 \leq \left(\frac{r}{2\pi} + \frac{1}{\delta}\right) \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^2 d\theta, \tag{1.1}$$

where

$$0 \leq \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_m \leq 2\pi$$

and

$$\delta \stackrel{\text{def}}{=} \min\left\{\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, ..., \alpha_m - \alpha_{m-1}, 2\pi - (\alpha_m - \alpha_1)\right\} > 0.$$

This particular form may be deduced from Theorem 3 in [11, p. 559] by a substitution (see also [13, inequalities (2.29) and (2.30) on p. 221]). The large sieve has been extended in numerous directions. For instance, $|P|^2$ has been replaced by $|P|^p$ or, in more general form, by $\psi(|P|^p)$, where ψ is convex, non-negative, and non-decreasing function. Moreover, polynomials have been replaced by generalized polynomials (see [1], [6], [10], [11], and [15] for a variety of these extensions, and [9] for a survey of the related Marcinkiewicz-Zygmund inequalities).

Date: May 27, 2005.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-9706695 (P. N.). The research started while D. L. visited the Mathematical Research Institute at The Ohio State University in December, 1998, and while L. G. stayed at The Ohio State University during the 1998/1999 academic year.

The main focus of this paper is to establish inequalities like (1.1), but with integrals over arcs of the circle, rather than on the whole circle. These will have applications in estimates of trigonometric sums on short arcs of the circle, and also in problems of approximation over such arcs. We will deal not only with algebraic polynomials P, but also with *generalized (non–negative) algebraic polynomials*

$$P(z) = \kappa \prod_{j=1}^{\ell} |z - z_j|^{r_j} \,, \tag{1.2}$$

where $\kappa > 0$, $z_j \in \mathbb{C}$, and $r_j > 0$ for $j = 1, 2, \ldots, \ell$, and

$$r \stackrel{\text{def}}{=} r_1 + r_2 + ... + r_\ell$$

is called the *degree* of P. Note that neither r nor $\{r_j\}$ need to be integers. In addition, if $p > 0$, then $|P|^p$ is a generalized algebraic polynomial of degree pr. We will fix $p > 0$,

$$0 \le \alpha < \beta \le 2\pi, \tag{1.3}$$

and consider the arc

$$\Delta = \Delta(\alpha, \beta) = \left\{ e^{i\theta} : \theta \in [\alpha, \beta] \right\}. \tag{1.4}$$

The quadratic polynomial R defined by

$$R(z) \stackrel{\text{def}}{=} \left(z - e^{i\alpha}\right)\left(z - e^{i\beta}\right), \tag{1.5}$$

which has zeros at the endpoints of Δ, plays an essential role in our analysis, as does the function $\varepsilon(z) = \varepsilon(z, \alpha, \beta, p, r)$ defined by

$$\varepsilon(z) \stackrel{\text{def}}{=} \frac{1}{pr+1} \left[|R(z)| + \left(\frac{\beta - \alpha}{pr+1} \right)^2 \right]^{1/2}. \tag{1.6}$$

Note that although ε depends on the parameters α, β, p, and r, in what follows we will not display this dependence. One may view $\varepsilon(z)$ as an analogue of the Timan-type expression

$$\frac{1}{n} \left[\sqrt{1 - x^2} + \frac{1}{n} \right],$$

which plays a role in numerous estimates relating to algebraic polynomials on $[-1, 1]$.

We use $|A|$ to denote the number of elements of a set A. We denote the unit circle and the closed unit disk by $\mathbb{T}$ and by $\mathbb{D}$, respectively. The interior of $\mathbb{D}$, that is, $\mathbb{D} \setminus \mathbb{T}$ is denoted by $\mathbb{D}^i$, whereas the exterior of $\mathbb{D}$ by $\mathbb{D}^e \stackrel{\text{def}}{=} \mathbb{C} \setminus \mathbb{D}$.

Our main result is the following.

Theorem 1.1. *Let* $0 < p < \infty$, $0 < r < \infty$, *and assume* (1.3)–(1.6). *Let* $m \in \mathbb{N}$ *and*

$$a_j = e^{i\alpha_j} \in \Delta, \quad 1 \le j \le m. \tag{1.7}$$

Then for every generalized algebraic polynomial P *of degree* r, *we have*

$$\sum_{j=1}^{m} |P(a_j)|^p \, \varepsilon(a_j) \le c\tau \int_\alpha^\beta \left| P\left(e^{i\theta}\right) \right|^p d\theta, \tag{1.8}$$

where $\tau = \tau(\alpha, \beta, p, r, \{a_j\})$ is defined by

$$\tau \overset{\text{def}}{=} \max_{\gamma \in [\alpha,\beta]} \left| \left\{ j : \alpha_j \in \left[\gamma - \varepsilon\left(e^{i\gamma}\right), \gamma + \varepsilon\left(e^{i\gamma}\right) \right] \right\} \right| \tag{1.9}$$

and c *is an absolute constant. In particular,* c *is independent of* α, β, p, m, r, P, *and* $\{a_j\}$.

Since every trigonometric polynomial s of degree r can be represented in the form $s(\theta) = e^{-ir\theta} P\left(e^{i\theta}\right)$, where P is an algebraic polynomial of degree $2r$, we deduce that

$$\sum_{j=1}^{m} |s(\alpha_j)|^p \left[\left| \sin\left(\frac{\alpha_j - \alpha}{2}\right) \sin\left(\frac{\alpha_j - \beta}{2}\right) \right| + \left(\frac{\beta - \alpha}{pr+1}\right)^2 \right]^{1/2} \tag{1.10}$$
$$\leq c\tau(pr+1) \int_\alpha^\beta |s(\theta)|^p d\theta,$$

where c is an absolute constant. The same relation holds more generally for generalized (non-negative) trigonometric polynomials

$$s(\theta) = \kappa \prod_{j=1}^{\ell} \left| \sin\left(\frac{\theta - u_j}{2}\right) \right|^{r_j},$$

where $\kappa > 0$, $u_j \in \mathbb{C}$, and $r_j > 0$ for $j = 1, 2, \ldots, \ell$. To see this, just set $z_j = \exp(iu_j)$ and $P(e^{i\theta}) = s(\theta)$. Another immediate consequence is the inequality

$$\sum_{j=1}^{m} |P(a_j)|^p |R(a_j)|^{1/2} \leq c\tau(pr+1) \int_\alpha^\beta \left| P\left(e^{i\theta}\right) \right|^p d\theta.$$

When $\alpha = 0$ and $\beta = 2\pi$, this gives

$$\sum_{j=1}^{m} |P(a_j)|^p |a_j - 1| \leq c_1\tau(pr+1) \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta,$$

where c_1 is an absolute constant. The factor $|a_j - 1|$ should not be there. By splitting the whole circle into two semicircles, we will deduce from Theorem 1.1 the following.

Corollary 1.2. *Let* $0 < p < \infty$, $0 < r < \infty$, *and assume* (1.3)–(1.6). *Let* $m \in \mathbb{N}$ *and*

$$a_j = e^{i\alpha_j} \in \Delta, \quad 1 \leq j \leq m.$$

Then for every generalized algebraic polynomial P *of degree* r, *we have*

$$\sum_{j=1}^{m} |P(a_j)|^p \leq c\tau^* (pr+1) \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta, \tag{1.11}$$

where

$$\tau^* \overset{\text{def}}{=} \max_{\gamma \in [0,2\pi]} \left| \left\{ j : \alpha_j \in \left[\gamma - \frac{1}{pr+1}, \gamma + \frac{1}{pr+1} \right] \right\} \right| \tag{1.12}$$

and c *is an absolute constant.*

How do the factors $\tau(pr+1)$ and $\tau^*(pr+1)$ relate to the term

$$\frac{r}{2\pi} + \frac{1}{\delta}, \quad \delta = \min\left\{ \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, ..., \alpha_m - \alpha_{m-1}, 2\pi - (\alpha_m - \alpha_1) \right\} \tag{1}$$

which appears in the more familiar form of the large sieve inequalities? The obvious advantage of τ and τ^* is that they remain bounded if two of the a_j's approach one another, while $1/\delta$ approaches ∞. In general, we claim that

$$\tau^* (pr + 1) \leq pr + \frac{2\pi}{\delta} , \tag{1.13}$$

so that we do in fact have an improvement on the traditional form (at the expense of introducing a multiplicative constant c). Indeed, if $\tau^* = 1$, the inequality is immediate. If $\tau^* \geq 2$, then

$$\delta \leq \frac{2}{(\tau^* - 1)(pr + 1)} ,$$

which again implies (1.13).

We are certain that the presence of $R(z)$ in (1.6) is essential, as it reflects endpoint effects which occur since Δ is not a closed curve. This is analogous to the presence of the factor $\sqrt{1 - x^2}$ in estimates relating to $[-1, 1]$.

Corollary 1.2 is an improvement of a result published by Joung in [6, Theorem 2.2] because of the use of the factor τ^*. However, in [6] there are more explicit simple constants which are close to being optimal. In addition, not only $|P|^p$ is considered in [6], but also $\psi(|P|^p)$, where ψ is a convex, non-decreasing, and non-negative function. Note that [6, Theorem 2.2] is derived by using the method of [10] and inequality (6) in [3, p. 606].

Our method of proof can be used to give a numerical value for c, once one knows the numerical constant which appears in an inequality of Carleson. Carleson measures have been used before in the context of quadrature sum estimates by Zhong and his coauthors (cf. [14] and [15]). However our use of Carleson measures here is closer to that from [7] and [8] where they were used in proving Markov-Bernstein inequalities in weighted L^p spaces.

We present the proof of Theorem 1.1 in Section 2, whereas we defer some technical estimates to the subsequent sections. In Section 3, we present estimates involving the function ε and the conformal map Ψ of $\mathbb{C} \setminus \Delta$ onto $\mathbb{D}^e$. In Section 4, we estimate the norms of certain Carleson measures, and in Section 5, we prove Corollary 1.2.

2. The Proof of Theorem 1.1

Throughout, $c, c_0, c_1, \ldots,$ denote absolute constants (and thus do not involve dependence on any parameters). The same symbol does not necessarily denotes the same constant in different formulas. We will prove Theorem 1.1 in several steps.

(i) **Reduction to the case** $p = 2$.

We first note that it suffices to prove (1.8) for $p = 2$. For, if $p > 0$, and P is a generalized polynomial of degree r, then we write

$$|P|^p = \left(|P|^{p/2}\right)^2 ,$$

where $|P|^{p/2}$ is a generalized polynomial of degree $pr/2$. Note that the definition of ε is unchanged, since p and r occur in all our estimates only in the form of the product pr.

(ii) **Reduction to integer** r. Note that p and r occur in (1.8) via (1.9) only in the factor $1 + pr \equiv 1 + 2r$ in $\varepsilon(z)$ of (1.6). If r, which is the degree of P, is not an integer then we can replace P by

$$P^* \stackrel{\text{def}}{=} |z|^{r^*-r} P$$

where r^* is the smallest integer which is at least r. Since both P and P^* take the same values in Δ, if we prove

$$\sum_{j=1}^{m} |P^*(a_j)|^p \varepsilon(a_j) \leq c^* \tau \int_{\alpha}^{\beta} \left|P^*\left(e^{i\theta}\right)\right|^p d\theta,$$

then (1.8) follows for P with a constant $c = 9c^*$. The reason for this is that ε in (1.8) is at most 9 times ε in the above inequality, and τ in (1.8) is at least τ above.

The reason for this step is, that it allows us to choose a single valued branch of a certain analytic function below.

(iii) **Reduction to the case** $0 < \alpha < \pi$ **and** $\beta = 2\pi - \alpha$.

If necessary, after a rotation of the circle, we may assume that Δ has the form

$$\Delta = \left\{e^{i\theta} : \theta \in [\alpha', 2\pi - \alpha']\right\},$$

where $0 \leq \alpha' < \pi$. Then Δ is symmetric about the real line, and this simplifies the use of a conformal map below. Moreover, then

$$\beta - \alpha' = 2(\pi - \alpha').$$

Thus, dropping the prime, it suffices to prove (1.8) with $0 < \alpha < \pi$ and $\beta - \alpha$ replaced everywhere by $2(\pi - \alpha)$. Thus in what follows, we will assume that

$$\Delta = \left\{e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha]\right\}, \tag{2.1}$$

$$R(z) = \left(z - e^{i\alpha}\right)\left(z - e^{-i\alpha}\right) = z^2 - 2z\cos\alpha + 1, \tag{2.2}$$

and

$$\varepsilon(z) = \frac{1}{2r+1}\left[|R(z)| + \left(\frac{2(\pi - \alpha)}{2r+1}\right)^2\right]^{1/2}.$$

In fact, we are going to simplify ε to

$$\varepsilon(z) = \frac{1}{r+1}\left[|R(z)| + \left(\frac{2(\pi - \alpha)}{r+1}\right)^2\right]^{1/2}, \tag{2.3}$$

which incurs an extra constant factor of 4 in (1.8).

Now we are ready to begin the main part of the proof.

(iv) **Use of subharmonicity of** $|P|^2$

If P is given by (1.2), then

$$|P|^2 = \exp\left(2\log|\kappa| + 2\sum_{j=1}^{\ell}\rho_j \log|z-z_j|\right)$$

is a subharmonic function. Thus,

$$|P(a)|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(a+\frac{\varepsilon(a)}{100}e^{i\theta}\right)\right|^2 d\theta, \qquad a\in\Delta,$$

so that

$$\sum_{j=1}^{m}|P(a_j)|^2\varepsilon(a_j) \le \int_{\mathbb{C}}|P|^2\,d\sigma, \tag{2.4}$$

where the measure σ is defined by

$$\int_{\mathbb{C}} f\,d\sigma \overset{\text{def}}{=} \sum_{j=1}^{m}\varepsilon(a_j)\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(a_j+\frac{\varepsilon(a_j)}{100}e^{i\theta}\right)d\theta, \tag{2.5}$$

that is, σ a linear combination with positive coefficients of Lebesgue measures on certain circles centered at a_j.

Remark. If ψ is non-negative, convex, and increasing function, then (2.4) holds with $|P|^2$ replaced by $\psi(|P|^p)$, since the latter is still subharmonic.

Our next goal is to pass from the right-hand side of (2.4) to an estimate over the entire unit circle. This passage would be permitted by fundamental result of L. Carleson, if P were analytic off the unit circle, and if it had an appropriate behavior at ∞. The next steps are mainly there to deal with the fact that P in general has neither of these properties.

(v) **The conformal map** Ψ **of** $\mathbb{C}\setminus\Delta$ **onto** $\{w : |w|>1\}$.

This map is given by

$$\Psi(z) \overset{\text{def}}{=} \frac{1}{2\cos\frac{\alpha}{2}}\left[z+1+\sqrt{R(z)}\right], \tag{2.6}$$

where the branch of $\sqrt{R}$ is chosen so that it is analytic off Δ and behaves like $z(1+o(1))$ for $z\to\infty$. Note that both $\sqrt{R}$ and Ψ have well defined (non-tangential and tangential) boundary values as z approaches Δ from either inside or outside the unit circle. We denote the boundary values from the inside by $\sqrt{R_+}$ and Ψ_+, and those from the outside by $\sqrt{R_-}$ and Ψ_-, respectively. Unless otherwise specified, we also set

$$\Psi(\zeta) \overset{\text{def}}{=} \Psi_-(\zeta), \quad \zeta\in\Delta.$$

For a detailed discussion and derivation of this conformal map Ψ, see [5]. In Lemma 3.2 we show that there is an absolute constant c_0 such that for $a\in\Delta$ we have

$$|\Psi(z)|^{2r+2} \le c_0, \tag{2.7}$$

as long as $|z-a| \le \varepsilon(a)/100$, and then we may rewrite (2.4) as

$$\sum_{j=1}^{m} |P(a_j)|^2 \varepsilon(a_j) \le c_0 \int_{\mathbb{C}} |Q(z)|^2 \, d\sigma, \qquad Q(z) \stackrel{\text{def}}{=} \frac{P(z)}{\Psi^{r+1}(z)} . \tag{2.8}$$

Since the version of Carleson's inequality that we are going to use involves analytic functions which are defined on $\mathbb{D}^i$, we will split σ into its parts with support inside and outside $\mathbb{T}$. For σ-measurable sets S, let

$$\sigma^{+}(S) \stackrel{\text{def}}{=} \sigma(S \cap \{z : |z| < 1\}), \qquad \sigma^{-}(S) \stackrel{\text{def}}{=} \sigma(S \cap \{z : |z| > 1\}). \tag{2.9}$$

Moreover, we need to be able to reflect σ^- through $\mathbb{T}$. Define $\sigma^{\#}$ by

$$\sigma^{\#}(S) \stackrel{\text{def}}{=} \sigma^{-}\left(S^{-1}\right), \tag{2.10}$$

where

$$S^{-1} \stackrel{\text{def}}{=} \left\{z : z^{-1} \in S\right\}.$$

Then, since for the unit circle $\mathbb{T}$ we have $\sigma(\mathbb{T}) = 0$, (2.8) becomes

$$\sum_{j=1}^{m} |P(a_j)|^2 \varepsilon(a_j) \le c_0 \left(\int_{\mathbb{C}} |Q(z)|^2 \, d\sigma^{+} + \int_{\mathbb{C}} \left|Q\left(z^{-1}\right)\right|^2 \, d\sigma^{\#} \right). \tag{2.11}$$

Next we focus on handling the first integral in (2.11).

(vi) **Evade the non-analyticity of P and estimate the integral involving σ^{+} in (2.11)**.

If P is the absolute value of a polynomial, then we can already apply Carleson's result. Since in general this is not the case, we proceed as follows. For each factor $z - z_j$ in P with $|z_j| < 1$, multiply it by the Blaschke factor

$$\frac{1 - \overline{z_j} z}{z - z_j},$$

obtaining a term with the same absolute value on $\mathbb{T}$, but not vanishing in $\mathbb{D}^i$, and, in fact, having larger absolute value in $\mathbb{D}^i$. The we can form a branch of

$$g_1(z) \stackrel{\text{def}}{=} \frac{\kappa}{\Psi^{r+1}(z)} \left[\prod_{|z_j|<1} \left((z - z_j) \frac{1 - \overline{z_j} z}{z - z_j} \right)^{r_j} \right] \left[\prod_{|z_j| \ge 1} (z - z_j)^{r_j} \right]$$

that is single valued and analytic in $\mathbb{D}^i$. For $a \in \mathbb{T}$ we have

$$\lim_{\substack{z \to a \\ z \in \mathbb{D}^i}} |g_1(z)| = |Q(a)|,$$

whereas

$$|g_1(z)| \ge |Q(z)|, \qquad z \in \mathbb{D}^i.$$

Now we are ready to apply Carleson's result. Recall that a positive Borel measure μ with support in $\mathbb{D}^i$ is called a *Carleson measure* if there exists constant $A > 0$ such that for every $0 < h < 1$ and every sector

$$S \stackrel{\text{def}}{=} \left\{ t e^{i\theta} \ : \ t \in [1-h, 1] \, ; \ |\theta - \theta_0| \le h \right\} ,$$

we have

$$\mu(S) \le Ah.$$

The smallest such A is called the Carleson norm of μ and denoted $N(\mu)$ (see, for instance, [4, sections I.5 and VI.3] for an introduction to Carleson measures). The striking feature of such a measure is the inequality

$$\int_{\mathbb{D}} |f|^2 \, d\mu \le c_1 N(\mu) \int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 d\theta\,, \qquad f \in H^2(\mathbb{D}), \tag{2.12}$$

where c_1 is an absolute constant (cf. [2, Theorem 1, p. 548]). Applying this to our function g_1 yields

$$\begin{aligned} \int_{\mathbb{D}} |Q|^2 \, d\sigma^+ &\le \int_{\mathbb{D}} |g_1|^2 \, d\sigma^+ \le c_1 N\left(\sigma^+\right) \int_0^{2\pi} \left| g_1\left(e^{i\theta}\right) \right|^2 d\theta \\ &= c_1 N\left(\sigma^+\right) \int_0^{2\pi} \left| Q\left(e^{i\theta}\right) \right|^2 d\theta. \end{aligned} \tag{2.13}$$

(vii) **Estimate the integral involving $\sigma^{\#}$ in (2.11).**

Since P is of exact degree r and $\lim_{z\to\infty} \Psi(z)/z = (\cos\frac{\alpha}{2})^{-1} \neq 0$, we have $\lim_{z\to\infty} P(z)/\Psi^r(z) \neq 0$ as well. Hence $h(w) \stackrel{\text{def}}{=} Q(1/w)$ has zeros in $\mathbb{D}^i$ corresponding only to zeros of P outside the unit disk and a simple zero at $w = 0$, corresponding to the zero of Q at infinity. We may follow much the same procedure to h as we did to Q in Step (vi) to obtain a single-valued analytic function to which Carleson's inequality (2.12) can be applied. The consequence is that

$$\int_{\mathbb{D}} \left| Q\left(z^{-1}\right) \right|^2 d\sigma^{\#} \le c_1 N\left(\sigma^{\#}\right) \int_0^{2\pi} \left| Q\left(e^{-i\theta}\right) \right|^2 d\theta,$$

which, combined with (2.13) and (2.11), gives

$$\sum_{j=1}^{m} \left| P\left(a_j\right) \right|^2 \varepsilon\left(a_j\right) \le c_0 c_1 \left(N\left(\sigma^+\right) + N\left(\sigma^{\#}\right) \right) \int_0^{2\pi} \left| Q\left(e^{i\theta}\right) \right|^2 d\theta. \tag{2.14}$$

(viii) **Pass from the entire unit circle to Δ .**

Let $|d\zeta|$ denote arclength on $\mathbb{T}$. Suppose that we have an estimate of the form

$$\int_{\mathbb{T}\setminus\Delta} |g(\zeta)|^2 \, |d\zeta| \le \frac{c_2}{2} \left(\int_{\Delta} |g_+(\zeta)|^2 \, |d\zeta| + \int_{\Delta} |g_-(\zeta)|^2 \, |d\zeta| \right), \tag{2.15}$$

valid for all such functions g which are analytic in $\mathbb{C} \setminus \Delta$, satisfy $\lim_{z\to\infty} g(z) = 0$, and whose interior and exterior boundary values g_+ and g_- exist, where c_2 is an absolute constant. Such an inequality will be established in the next step with $c_2 = 1/2$. We would like to apply it to Q, but, as we have already experienced it, our problem is that Q is not analytic in $\mathbb{C} \setminus \Delta$. In order to remedy this, for each factor $z - z_j$ in P with $z_j \notin \Delta$, we define

$$b_j(z) \stackrel{\text{def}}{=} \begin{cases} (z - z_j) \left(\frac{1 - \overline{\Psi(z_j)}\Psi(z)}{\Psi(z) - \Psi(z_j)} \right), & z \neq z_j, \\ \left(1 - \left| \Psi(z_j) \right|^2 \right) / \Psi'(z_j), & z = z_j. \end{cases}$$

which is analytic in $\mathbb{C} \setminus \Delta$ and does not have any zeros there. Moreover, since $\lim_{z\to\Delta} |\Psi(z)| = 1$, we see that

$$|b_j(z)| = |z - z_j|\,, \; z \in \Delta$$

and

$$|b_j(z)| \geq |z - z_j|, \ z \in \mathbb{C} \setminus \Delta.$$

Recall that we extended Ψ to Δ as an exterior boundary value. Next we choose a branch of

$$g(z) \stackrel{\text{def}}{=} \frac{c}{\Psi^{r+1}(z)} \left(\prod_{z_j \notin \Delta} b_j^{r_j}(z) \right) \left(\prod_{z_j \in \Delta} (z - z_j)^{r_j} \right),$$

which is single valued and analytic in $\mathbb{C} \setminus \Delta$ such that $\lim_{z \to \infty} g(z) = 0$, $|(g)_\pm| = |Q| = |P|$ on Δ and $|g(z)| \geq |Q(z)|$ for $z \in \mathbb{C} \setminus \Delta$. It follows now from (2.15) that

$$\int_{\mathbb{T}\setminus\Delta} |Q(\zeta)|^2 |d\zeta| \leq \int_{\mathbb{T}\setminus\Delta} |g(\zeta)|^2 |d\zeta| \leq c_2 \int_\Delta |g(\zeta)|^2 |d\zeta| = c_2 \int_\Delta |P(\zeta)|^2 |d\zeta|$$

so that

$$\int_0^{2\pi} \left|Q\left(e^{i\theta}\right)\right|^2 d\theta \leq (1 + c_2) \left(\int_\alpha^{2\pi-\alpha} \left|P\left(e^{i\theta}\right)\right|^2 d\theta \right),$$

and then (2.14) becomes

$$\sum_{j=1}^m |P(a_j)|^2 \varepsilon(a_j) \leq c_0 c_1 (1 + c_2) \left(N\left(\sigma^+\right) + N\left(\sigma^\#\right)\right) \int_\alpha^{2\pi-\alpha} \left|P\left(e^{i\theta}\right)\right|^2 d\theta. \tag{2.16}$$

(ix) **Proof of (2.15).**

We note that inequalities such as (2.15) are an essential ingredient of the method used in [5, 6] for proving weighted Markov-Bernstein inequalities, although there the unit disk was replaced by a half-plane. We can nevertheless follow the same procedure. Of course, we may limit ourselves to functions g for which the right-hand side of (2.15) is finite. First, we may use the limiting version of Cauchy's integral formula to obtain that

$$g(z) = \frac{1}{2\pi i} \int_\Delta \frac{g_-(\zeta) - g_+(\zeta)}{\zeta - z} d\zeta, \qquad z \notin \Delta.$$

Let 1_Δ denote the characteristic function of Δ and for functions $f \in L^1(\mathbb{T})$, define the Cauchy type singular integral transform $H[f]$ by

$$H[f](z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} PV \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in \mathbb{T},$$

(cf. [12, formula (4.4) on p. 99]) which, by standard arguments, exists a.e. on $\mathbb{T}$. Here "PV" denotes Cauchy principal value. Then

$$g(z) = H[1_\Delta g_-](z) - H[1_\Delta g_+](z), \qquad z \in \mathbb{T} \setminus \Delta.$$

By comparing this transformaation to the standard conjugate function, we see that it is a bounded operator on $L^2(\mathbb{T})$. In fact, if $f \circ e_1 \sim \sum_{k=-\infty}^\infty c_k e_k$ where $e_k(t) \stackrel{\text{def}}{=} \exp(ikt)$, then it is easy to verify that $H[f] \circ e_1 \sim \frac{1}{2} \sum_{k=-\infty}^\infty \operatorname{sign}(k) c_k e_k$ so that

$$\int_{\mathbb{T}} |H[f](\zeta)|^2 |d\zeta| = \frac{1}{4} \int_{\mathbb{T}} |f(\zeta)|^2 |d\zeta|, \qquad f \in L^2(\mathbb{T})$$

(see [12, section 3.4.5 on pp. 111–112]). Therefore,

$$\int_{\mathbb{T}\setminus\Delta} |g(\zeta)|^2 |d\zeta| \leq \frac{1}{2} \left(\int_\Delta |g_+(\zeta)|^2 |d\zeta| + \int_\Delta |g_-(\zeta)|^2 |d\zeta| \right), \tag{2.17}$$

so that (2.15) holds with $c_2 = 1/2$.

(x) **Completion of the proof.**

We will show in Lemma 4.1 that

$$N\left(\sigma^{+}\right)+N\left(\sigma^{\#}\right)\leq c_3\tau.$$

Then, since by (2.17) we have $c_2 = 1/2$ in (2.15), inequality (2.16) becomes

$$\sum_{j=1}^{m}\left|P\left(a_j\right)\right|^2\varepsilon\left(a_j\right)\leq\frac{3}{2}\,c_0c_1c_3\tau\int_{\alpha}^{2\pi-\alpha}\left|P\left(e^{i\theta}\right)\right|^2d\theta.$$

Thus, we have (1.8) with a constant c that depends only on the absolute constants c_0, c_1, and c_4 that arise from the bound (2.7) on the conformal map Ψ, Carleson's inequality (2.12), and the upper bound on the Carleson norms of σ^{+} and $\sigma^{\#}$. $\square$

3. Auxiliary Estimates

Throughout we assume the notation given in (2.1)–(2.3). As in (i) and (ii) of Section 2, we will assume that $p = 2$ and $r \geq 1$. We begin with estimates on R and ε originally given by (1.5) and (1.6), respectively, but then simplified in (2.2) and (2.3).

Lemma 3.1. *Let $v, a \in \Delta$. Then*

$$|R(v)-R(a)|\leq 8|v-a|\cos\frac{\alpha}{2}\tag{3.1}$$

and

$$|R(a)|\leq 4\cos^2\frac{\alpha}{2}\leq(\pi-\alpha)^2.\tag{3.2}$$

In addition,

$$|\varepsilon(v)-\varepsilon(a)|\leq|v-a|\tag{3.3}$$

and

$$\varepsilon(a)\leq\frac{8\cos\frac{\alpha}{2}}{r+1}\leq\frac{4(\pi-\alpha)}{r+1}.\tag{3.4}$$

Proof.

Write $v = e^{i\theta}$ and $a = e^{is}$. Then

$$R(a)=-4a\sin\left(\frac{s-\alpha}{2}\right)\sin\left(\frac{s+\alpha}{2}\right)=-4a\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{s}{2}\right),\tag{3.5}$$

so that

$$R(v)-R(a)=-4(v-a)\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}\right)+4a\left(\cos^2\frac{\theta}{2}-\cos^2\frac{s}{2}\right).\tag{3.6}$$

Then, since $s, \theta \in [\alpha, 2\pi - \alpha]$,

$$\begin{aligned}|R(v) - R(a)| &\leq 4|v-a|\cos^2\frac{\alpha}{2} + 4\left|\sin\left(\frac{s-\theta}{2}\right)\sin\left(\frac{s+\theta}{2}\right)\right| \\ &\leq 4|v-a|\cos\frac{\alpha}{2} + 4\left|\sin\left(\frac{s-\theta}{2}\right)\right|\left(\left|\sin\frac{\theta}{2}\right|\left|\cos\frac{s}{2}\right| + \left|\cos\frac{\theta}{2}\right|\left|\sin\frac{s}{2}\right|\right) \\ &\leq 4|v-a|\cos\frac{\alpha}{2} + 8\left|\sin\left(\frac{s-\theta}{2}\right)\right|\cos\frac{\alpha}{2} = 8|v-a|\cos\frac{\alpha}{2}.\end{aligned}$$

Thus we have (3.1). For (3.2), we note the inequality

$$\frac{\pi-\alpha}{\pi} \leq \cos\frac{\alpha}{2} = \sin\left(\frac{\pi-\alpha}{2}\right) \leq \frac{\pi-\alpha}{2}, \qquad \alpha \in [0,\pi]. \tag{3.7}$$

This and (3.5) prove (3.2).

By (3.1) and (3.7), we have

$$\begin{aligned}|\varepsilon(v) - \varepsilon(a)| &= \frac{1}{r+1}\left|\frac{\left[|R(v)| + \left(\frac{2(\pi-\alpha)}{r+1}\right)^2\right] - \left[|R(a)| + \left(\frac{2(\pi-\alpha)}{r+1}\right)^2\right]}{\left[|R(v)| + \left(\frac{2(\pi-\alpha)}{r+1}\right)^2\right]^{1/2} + \left[|R(a)| + \left(\frac{2(\pi-\alpha)}{r+1}\right)^2\right]^{1/2}}\right| \\ &\leq \frac{|R(v)-R(a)|}{4(\pi-\alpha)} \leq \frac{2|v-a|\cos\frac{\alpha}{2}}{\pi-\alpha} \leq |v-a|.\end{aligned}$$

Furthermore, from (3.2) and (3.7),

$$\begin{aligned}\varepsilon(a) &\leq \frac{1}{r+1}\left(4\cos^2\frac{\alpha}{2} + 4(\pi-\alpha)^2\right)^{1/2} \leq \frac{2\cos\frac{\alpha}{2}}{r+1}(1+\pi^2)^{1/2} \\ &\leq \frac{8\cos\frac{\alpha}{2}}{r+1} \leq \frac{4(\pi-\alpha)}{r+1}\end{aligned}$$

which proves (3.4). The proof is complete. $\square$

Lemma 3.2. *Let Ψ be given by (2.6). Then there is an absolute constant c_0 such that for $a \in \Delta$ and $|z-a| \leq \varepsilon(a)/100$ we have*

$$|\Psi(z)|^{2r+2} \leq c_0. \tag{3.8}$$

Proof.

We will assume that $|z| \geq 1$. The case when $z \in \mathbb{D}^i$ is similar. Write $z = te^{i\theta}$ and set $v = e^{i\theta}$. It is clear that $|z-v| \leq |z-a|$ and $|v-a| \leq |z-a|$.

We distinguish two subcases.

(i) Suppose that $v \in \Delta$, that is, $\alpha \leq \theta \leq 2\pi - \alpha$.

We will show that for some absolute constant c_1 we have

$$|\Psi(z) - \Psi(v)| = |\Psi(z) - \Psi_-(v)| \leq \frac{c_1}{r+1}, \tag{3.9}$$

and then, since $|\Psi(v)| = 1$, we obtain

$$|\Psi(z)|^{2r+2} \leq \left(1 + \frac{c_1}{r+1}\right)^{2r+2} \leq \exp(2c_1) = c_0.$$

In order to prove (3.9), we proceed as follows. First, by (2.6),

$$|\Psi(z)-\Psi(v)| \le \frac{|z-v|}{2\cos\frac{\alpha}{2}} + \frac{\left|\sqrt{R(z)}-\sqrt{R(v)}\right|}{2\cos\frac{\alpha}{2}}. \tag{3.10}$$

Here

$$\frac{|z-v|}{2\cos\frac{\alpha}{2}} \le \frac{|z-a|}{2\cos\frac{\alpha}{2}} \le \frac{\varepsilon(a)}{200\cos\frac{\alpha}{2}} \le \frac{1}{25(r+1)} \tag{3.11}$$

by (3.4). Next we turn to the more difficult estimation of

$$T \stackrel{\text{def}}{=} \frac{\left|\sqrt{R(z)}-\sqrt{R(v)}\right|}{2\cos\frac{\alpha}{2}}.$$

Write $z=e^{i\xi}$ where $\xi=\theta-i\log t$. We see from (3.6) that

$$R(v)-R(z) = -4(v-z)\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}\right)+4z\left(\cos^2\frac{\theta}{2}-\cos^2\frac{\xi}{2}\right)$$

and, hence,

$$|R(v)-R(z)| \le 4|v-z|\cos^2\frac{\alpha}{2}+4t\left|\sin\left(\frac{\theta-\xi}{2}\right)\sin\left(\frac{\theta+\xi}{2}\right)\right|. \tag{3.12}$$

Here

$$\left|\sin\left(\frac{\theta-\xi}{2}\right)\right| = \frac{1}{2}\left|e^{-\frac{i}{2}(\theta+\xi)}\right|\left|e^{i\theta}-e^{i\xi}\right| = \frac{1}{2\sqrt{t}}|v-z|.$$

In addition,

$$\begin{aligned}\left|\sin\left(\frac{\theta+\xi}{2}\right)\right| &= \left|\sin\left(\frac{\xi-\theta}{2}+\theta\right)\right| \le \left|\sin\left(\frac{\xi-\theta}{2}\right)\right|+\left|\cos\left(\frac{\xi-\theta}{2}\right)\right||\sin\theta| \\ &= \frac{1}{2\sqrt{t}}|v-z|+2\left|\cosh\left(\frac{\log t}{2}\right)\right|\left|\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right| \le \frac{1}{2\sqrt{t}}|v-z|+2\sqrt{t}\cos\frac{\alpha}{2}\end{aligned}$$

(here we use $v\in\Delta$). Then (3.12) gives

$$|R(v)-R(z)| \le 4|v-z|\cos^2\frac{\alpha}{2}+|v-z|^2+4t|v-z|\cos\frac{\alpha}{2}.$$

By (3.11), $|z-v|<\cos\frac{\alpha}{2}\le 1$ so that $t<2$. Thus (3.11) yields

$$|R(v)-R(z)| \le 13|v-z|\cos\frac{\alpha}{2}.$$

We have $|a-v|\le|a-z|\le\varepsilon(a)/100$. Hence, by (3.3) in Lemma 3.1,

$$\varepsilon(v) \ge \frac{99}{100}\varepsilon(a)$$

and

$$|z-v| \le |z-a| \le \frac{\varepsilon(v)}{99}.$$

Therefore,

$$|R(z)-R(v)| \le \frac{\varepsilon(v)}{7}\cos\frac{\alpha}{2}. \tag{3.13}$$

Assume that the first term in the right-hand side of (2.3) prevails

$$|R(v)| \ge 4\left(\frac{\pi-\alpha}{r+1}\right)^2 \ge \frac{16\cos^2\frac{\alpha}{2}}{(r+1)^2}, \qquad |R(v)|^{\frac{1}{2}} \ge \frac{4\cos\frac{\alpha}{2}}{(r+1)}, \tag{3.14}$$

and, hence,

$$\varepsilon(v) \leq \frac{\sqrt{2}}{r+1}\, |R(v)|^{1/2}. \tag{3.15}$$

This, (3.14) and (3.13) give

$$|R(z) - R(v)| \leq \frac{\sqrt{2}\cos\frac{\alpha}{2}}{7(r+1)}\, |R(v)|^{1/2} \leq \frac{|R(v)|}{14\sqrt{2}}.$$

In this case the circle with diameter $[R(z), R(v)]$ lies inside the disk $\{w : |w - R(v)| < |R(v)|/20\}$. For each semicircle γ from $R(z)$ to $R(v)$ we have

$$|w| \geq \frac{1}{2}|R(v)|, \qquad w \in \gamma,$$

the function $w \to \sqrt{w}$ is analytic and single valued in some open set containing γ, and moreover, the limit of $\sqrt{w}$ as w approaches the relevant endpoint is the value assigned to $\sqrt{R(z)}$ or $\sqrt{R(v)}$ above. Then

$$\left|\sqrt{R(z)} - \sqrt{R(v)}\right| = \left|\int_\gamma \frac{dw}{2\sqrt{w}}\right| \leq \frac{\mathit{length}(\gamma)}{2\sqrt{|R(v)|/2}}$$
$$= \pi\frac{|R(z) - R(v)|}{\sqrt{2|R(v)|}} \leq \frac{\pi}{7\sqrt{2}}\frac{\varepsilon(v)\cos\frac{\alpha}{2}}{\sqrt{|R(v)|}} \leq \frac{\pi}{7}\frac{\cos\frac{\alpha}{2}}{r+1},$$

by (3.13) and (3.15). Hence

$$T \leq \frac{\pi}{14(r+1)},$$

and together with (3.10) and (3.11), this gives (3.9).

If (3.14) fails, then

$$\varepsilon(v) < 2\sqrt{2}\,\frac{\pi - \alpha}{(r+1)^2},$$

and (3.7), (3.13) give

$$|R(v)| \leq c_2\frac{\cos^2\frac{\alpha}{2}}{(r+1)^2}, \qquad |R(z)| \leq c_2\frac{\cos^2\frac{\alpha}{2}}{(r+1)^2},$$

whence it follows that

$$T \leq \frac{\sqrt{|R(z)|} + \sqrt{|R(v)|}}{2\cos\frac{\alpha}{2}} \leq \frac{\sqrt{c_2}}{r+1},$$

and again (3.9) holds.

(ii) Suppose that $v \notin \Delta$. Then $\theta \in [0, \alpha)$ or $\theta \in (2\pi - \alpha, 2\pi]$. Without loss of generality we will examine only the case when $\theta \in [0, \alpha)$ and $a = e^{is}$ with $s \in [\alpha, \pi]$. Since now $|\Psi(v)| \neq 1$, relation (3.9)does not imply (3.8). Instead, let us consider the difference

$$\left|\Psi(z) - \Psi\left(e^{i\alpha}\right)\right| \leq \frac{\left|z - e^{i\alpha}\right|}{2\cos\frac{\alpha}{2}} + \frac{\sqrt{|R(z)|}}{2\cos\frac{\alpha}{2}}. \tag{2}$$

We have

$$\left|a - e^{i\alpha}\right| < |a - v| \leq |a - z| \leq \frac{\varepsilon(a)}{100}$$

and by (3.3) in Lemma 3.1 with $v = e^{i\alpha}$

$$\varepsilon(a) \leq \frac{100}{99}\varepsilon\left(e^{i\alpha}\right) = \frac{200}{99}\frac{\pi-\alpha}{(r+1)^2}\,.$$

Hence

$$\left|z-e^{i\alpha}\right| \leq |z-a| + \left|a-e^{i\alpha}\right| \leq \frac{\varepsilon(a)}{50} \leq c_3\frac{\cos\frac{\alpha}{2}}{(r+1)^2}\,, \tag{3}$$

$$\begin{aligned}\sqrt{|R(z)|} &= \sqrt{\left|z-e^{i\alpha}\right|\left|z-e^{-i\alpha}\right|} \leq \sqrt{\left|z-e^{i\alpha}\right|\left(\left|z-e^{i\alpha}\right|+2\sin\alpha\right)} \\ &\leq \left|z-e^{i\alpha}\right| + 2\sqrt{\left|z-e^{i\alpha}\right|\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \leq c_4\frac{\cos\frac{\alpha}{2}}{r+1}\,,\end{aligned}$$

whence it follows that

$$\left|\Psi(z) - \Psi\left(e^{i\alpha}\right)\right| \leq \frac{c_5}{r+1} \tag{4}$$

and again (3.8) holds. □

4. Norms of the Carleson Measures

We will estimate the norms of the Carleson measures σ^+ and $\sigma^\#$ defined by (2.5), (2.9) and (2.10). Recall that the Carleson norm $N(\mu)$ of a positive measure μ with support in the unit disk is the least A such that

$$\mu(S) \leq Ah, \tag{4.1}$$

for every $0 < h < 1$ and for every sector

$$S \stackrel{\text{def}}{=} \left\{te^{i\theta} : t \in [1-h, 1]\,; |\theta - \theta_0| \leq h\right\}\,. \tag{4.2}$$

Lemma 4.1. We have

$$N\left(\sigma^+\right) \leq c_1\tau \tag{4.3}$$

and

$$N\left(\sigma^\#\right) \leq c_2\tau. \tag{4.4}$$

Proof.

In order to prove (4.3), we proceed similarly as in [7] or [8]. Let S be the sector (4.2) and let Γ be a circle with center a and radius $\varepsilon(a)/100 > 0$. A necessary condition for Γ to intersect S is that

$$\left|a-e^{i\theta_0}\right| \leq \left|a-te^{i\theta}\right| + \left|te^{i\theta}-e^{i\theta_0}\right| \leq \frac{\varepsilon(a)}{100} + h,$$

where $te^{i\theta} \in \Gamma \cap S$. Using (3.3) in Lemma 3.1, we see that

$$\left|a-e^{i\theta_0}\right| \leq \frac{\varepsilon\left(e^{i\theta_0}\right)}{100} + \frac{\left|a-e^{i\theta_0}\right|}{100} + h,$$

so that

$$\left|a-e^{i\theta_0}\right| \leq \lambda \stackrel{\text{def}}{=} \frac{\varepsilon\left(e^{i\theta_0}\right)}{99} + 2h. \tag{4.5}$$

Next, $\Gamma \cap S$ consists of at most two arcs (draw a picture!) and as each such arc is convex, it has length at most $4h$. Therefore the total angular measure of $\Gamma \cap S$, which obviously does not exceed 2π, is at most $800h/\varepsilon(a)$. Thus, if χ_S denotes the characteristic function of S,

$$\int_{-\pi}^{\pi} \chi_S\left(a+\varepsilon\left(a\right)e^{i\theta}\right)d\theta \leq \min\left\{2\pi,\ \frac{800h}{\varepsilon(a)}\right\}.$$

Then, from (2.5) and (2.9) we see that

$$\begin{aligned}\sigma^{+}\left(S\right) \leq \sigma\left(S\right) &\leq \sum_{j\,:\,\left|a_j-e^{i\theta_0}\right|\leq\lambda} \varepsilon\left(a_j\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_S\left(a_j+\frac{\varepsilon\left(a_j\right)}{100}e^{i\theta}\right)d\theta \\ &\leq c_1 \sum_{j\,:\,\left|a_j-e^{i\theta_0}\right|\leq\lambda} \min\left\{\varepsilon\left(a_j\right),h\right\}. \qquad (4.6)\end{aligned}$$

We now consider two subcases.

(i) $h \leq \varepsilon\left(e^{i\theta_0}\right)/100$.

In this case by (4.5) and (3.4)

$$\lambda < \frac{\varepsilon\left(e^{i\theta_0}\right)}{25} < 1.$$

With a suitable choice of $\alpha_j = \arg\left(a_j\right)$ we have for a_j in the sum in (4.6)

$$\frac{2}{\pi}\ \left|\alpha_j-\theta_0\right| \leq 2\left|\sin\left(\frac{\alpha_j-\theta_0}{2}\right)\right| = \left|a_j-e^{i\theta_0}\right| \leq \lambda < \frac{\varepsilon\left(e^{i\theta_0}\right)}{25}\ ,$$

so that

$$\left|\alpha_j-\theta_0\right| < \frac{\pi}{50}\,\varepsilon\left(e^{i\theta_0}\right)\ . \qquad (4.7)$$

Recalling the definition of τ in (1.9), we see that there are at most τ terms in the sum in (4.6), and, hence,

$$\sigma^{+}\left(S\right) \leq c_1 h\tau.$$

(ii) $h > \varepsilon\left(e^{i\theta_0}\right)/100$.

In this case $\lambda < 4h$. Let us now choose a partition

$$\alpha = \beta_0 < \beta_1 < ... < \beta_\ell = 2\pi-\alpha$$

as follows. Set $\beta_0 \stackrel{\text{def}}{=} \alpha$ and given β_{k-1}, choose β_k such that

$$\sin\frac{\beta_k-\beta_{k-1}}{2} = \frac{\varepsilon\left(e^{i\beta_{k-1}}\right)}{8\pi}\ , \qquad k\in\mathbb{N}\ .$$

Since $\varepsilon\left(z\right) \geq \left(\pi-\alpha\right)\left(r+1\right)^{-2}$, we obtain a finite ℓ with $\beta_{l-1} < 2\pi-\alpha \leq \beta_l$. If $2\pi-\alpha < \beta_l$, redefine $\beta_\ell \stackrel{\text{def}}{=} 2\pi-\alpha$. Note that $\beta_1 < 2\pi-\alpha$, so that the partition is nontrivial. Thus

$$\left|e^{i\beta_{k+1}}-e^{i\beta_k}\right| = \frac{\varepsilon\left(e^{i\beta_k}\right)}{4\pi}\ ,\ 0\leq k\leq \ell-2, \qquad \left|e^{i\beta_l}-e^{i\beta_{l-1}}\right| \leq \frac{\varepsilon\left(e^{i\beta_{l-1}}\right)}{4\pi} \qquad (4.8)$$

and

$$\beta_{k+1}-\beta_k \leq \frac{\varepsilon\left(e^{i\beta_k}\right)}{8}, \quad 0\leq k\leq \ell-1. \qquad (4.9)$$

Let I_k denote the closed arc of Δ with endpoints $e^{i\beta_k}$ and $e^{i\beta_{k+1}}$, $0 \le k \le l-1$. Then for $a \in I_k$ we have

$$\left|a - e^{i\beta_k}\right| \le \left|e^{i\beta_{k+1}} - e^{i\beta_k}\right| \le \frac{\varepsilon\left(e^{i\beta_k}\right)}{4\pi},$$

so that by (3.3) in Lemma 3.1,

$$\varepsilon\left(a\right) \le \varepsilon\left(e^{i\beta_k}\right) + \frac{\varepsilon\left(e^{i\beta_k}\right)}{4\pi} \le 2\varepsilon\left(e^{i\beta_k}\right).$$

Recalling the definition of τ (1.9), we see that $I_k \subset \left[\beta_k - \varepsilon\left(e^{i\beta_k}\right), \beta_k + \varepsilon\left(e^{i\beta_k}\right)\right]$, and, hence

$$\sum_{a_j \in I_k} \min\left\{\varepsilon\left(a_j\right), h\right\} \le 2\min\left\{\varepsilon\left(e^{i\beta_k}\right), h\right\}\tau. \tag{4.10}$$

Now let γ denote the arc $\left\{e^{i\theta} : \left|e^{i\theta} - e^{i\theta_0}\right| \le 4h\right\} \cap \Delta$, and let us choose the greatest non–negative integer L and smallest positive integer $M \le l$ such that

$$\gamma \subset \bigcup_{k=L}^{M-1} I_k.$$

Since now $\lambda < 4h$, it follows from (4.6) and (4.10) that

$$\sigma^+\left(S\right) \le \sigma\left(S\right) \le c_1 \sum_{k=L}^{M-1} \sum_{a_j \in I_k} \min\left\{\varepsilon\left(a_j\right), h\right\} \le 2c_1\tau \sum_{k=L}^{M-1} \min\left\{\varepsilon\left(e^{i\beta_k}\right), h\right\},$$

and, therefore,

$$\sigma^+\left(S\right) \le 2c_1\tau\left(\sum_{k=L+1}^{M-2} \varepsilon\left(e^{i\beta_k}\right) + 2h\right)$$

$$= c_2\tau\left(\sum_{k=L+1}^{M-2} \left|e^{i\beta_{k+1}} - e^{i\beta_k}\right| + 2h\right) \le c_3\tau\left(\text{length}\left(\gamma\right) + h\right) \le c_4\tau h.$$

Here we have used the equalities in (4.8) and the fact that the last sum is the length of a polygonal path with vertices on the unit circle and the latter is contained in the arc γ, that is, has length less than that of γ.

Thus we have proved that

$$N\left(\sigma^+\right) \le \sup_{h,S} \frac{\sigma\left(S\right)}{h} \le c_5\tau.$$

Next we prove (4.4). Recall that if S is the sector (4.2), then

$$\sigma^{\#}\left(S\right) = \sigma^-(S^{-1}) \le \sigma\left(S^{-1}\right),$$

where

$$S^{-1} = \left\{te^{i\theta} : t \in \left[1, (1-h)^{-1}\right]; \left|\theta + \theta_0\right| \le h\right\}.$$

For small h, say, for $h \in [0, 1/2]$ we have

$$(1-h)^{-1} \le 1 + 2h,$$

and exact the same argument as in the first part of the proof gives

$$\sigma^{\#}\left(S\right) \le \sigma\left(S^{-1}\right) \le c\tau h.$$

When $h \geq 1/2$, it is easier to use

$$\frac{\sigma^{\#}(S)}{h} \leq 2\sigma^{\#}(\mathbb{C}) \leq 2\sigma(\mathbb{C}) = 2\sum_{j=1}^{m} \varepsilon(a_j).$$

The argument applied in the proof of subcase (ii) shows that

$$\sigma(\mathbb{C}) = \sum_{k=0}^{l-1} \sum_{a_j \in I_k} \varepsilon(a_j) \leq 2\tau \sum_{k=0}^{l-1} \varepsilon\left(e^{i\beta_k}\right)$$

$$= 2\tau \left\{ \varepsilon\left(e^{i\beta_{l-1}}\right) + 4\pi \sum_{k=0}^{l-2} \left| e^{i\beta_{k+1}} - e^{i\beta_k} \right| \right\} \leq c\tau.$$

Therefore, the proof is complete. □

5. The Proof of Corollary 1.2

We obtain Corollary 1.2 from Theorem 1.1 by splitting the unit circle into two semicircles. Let

$$\Omega \stackrel{\text{def}}{=} \left\{ e^{i\theta} : \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \right\},$$

and let

$$\alpha = \frac{\pi}{4}, \quad \Delta = \left\{ e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha] \right\}.$$

Obviously $\Omega \subset \Delta$ and if R is the polynomial (2.2), then

$$0 < c_1 \leq \min_{z\in\Omega} |R(z)| < \max_{z\in\Delta} |R(z)| \leq c_2.$$

It is easy to see that $\varepsilon(z)$ defined by (1.6) satisfies $\varepsilon(z)(pr+1) \leq c_3$ for $z \in \Delta$ and

$$0 \leq c_4 \leq \varepsilon(z)(pr+1), \quad z \in \Omega.$$

Hence,

$$\tau \leq c_5 \tau^*.$$

Note that the latter inequality depends only on the upper bound for $\varepsilon(z)(pr+1)$, so it is true for arbitrary arc Δ. Applying Theorem 1.1 to Δ, it follows that

$$\sum_{a_j \in \Omega} |P(a_j)|^p \leq c\tau^*(pr+1) \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta.$$

Similarly, we obtain an estimate for the semicircle complementary to Ω, and, thereby, (1.11) follows. □

References

[1] C. K. Chui and Lefan Zhong, *Polynomial interpolation and Marcinkiewicz-Zygmund inequalities*, J. Math. Anal. Appl. 233 (1999), 387–405.

[2] L. Carleson, *Interpolation by bounded analytic functions and the Corona Problem*, Annals of Mathematics 76 (1962), 547–559.

[3] T. Erdélyi, A. P. Magnus, and P. Nevai *Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials*, SIAM J. Math. Anal. 25 (1994), 602–614.

[4] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[5] L. Golinskii, *Akhiezer's orthogonal polynomials and Bernstein-Szegő method for a circular arc*, J. Approx. Theory 95 (1998), 229–263.

[6] H. Joung, *Large sieve for generalized trigonometric polynomials*, Bulletin Korean Math. Soc. 36 (1999), 161–169.
[7] A. L. Levin and D. S. Lubinsky, *L_p Markov-Bernstein inequalities for Freud weights*, J. Approx. Theory 77 (1994), 229–248.
[8] A. L. Levin and D. S. Lubinsky, *Orthogonal polynomials associated with exponential weights*, to appear.
[9] D. S. Lubinsky, *Marcinkiewicz inequalities: methods and results*, (in) Recent Progress in Inequalities (ed. G. V. Milovanovic), Kluwer, Dordrecht, 1998, pp. 213–240.
[10] D. S. Lubinsky, A. Máté, and P. Nevai, *Quadrature sums involving p th powers of polynomials*, SIAM J. Math. Anal. 18 (1987), 531–544.
[11] H. L. Montgomery, *The analytic principle of the large sieve*, Bull. Amer. Math. Soc. 84 (1978), 547–567.
[12] S. Prössdorf, *Some Classes of Singular Equations*, North–Holland, 1978.
[13] A. Selberg, *Collected Papers*, vol. II, Springer-Verlag, 1991.
[14] L. Zhong and X. Shen, *Weighted Marcinkiewicz inequalities*, Advances in Mathematics (China) 23 (1994), 66–75.
[15] L. Zhong and L-Y. Zhu, *The Marcinkiewicz-Zygmund inequality on a simple arc*, J. Approx. Theory 83 (1995), 65–83.

[1]Mathematics Division, Institute for Low Temperature Physics, and Engineering, 47 Lenin Avenue, Kharkov 310164, Ukraine, E-mail: golinskii@ilt.kharkov.ua, [2]Mathematics Department, Witwatersrand University, Wits 2050, South Africa, E-mail: 036dsl@cosmos.wits.ac.za,, [3]Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210-1174, USA, E-mail: nevai@math.ohio-state.edu