Asymptotics associated with Exponential Weights

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Abstract

We announce some asymptotics for orthogonal and extremal polynomials associated with exponential weights $W = \exp(-Q)$.

1 Classes of Weights

Let I be a finite or infinite interval and let $Q : I \to [0, \infty)$ be convex. Let

$$W := \exp(-Q)$$

and assume that all power moments

$$\int_I x^n W^2(x) dx, \ n = 0, 1, 2, 3, ...$$

are finite. Then we may define orthonormal polynomials

$$p_n(x) = p_n(W^2, x) = \gamma_n(W^2) x^n + \dots, \ \gamma_n(W^2) > 0,$$

satisfying

$$\int_I p_n p_m W^2 = \delta_{mn},\ m,n = 0,1,2,\dots\ .$$

In the last twenty years, there has been a remarkable development of quantitative analysis around exponential weights W, and in particular around asymptotics for $p_n(x)$. See [1], [4], [6], [12–15], [18], [21–23] for references and reviews.

In this paper, we announce some of the results established in the monograph [10]. The main focus there is to treat in a unified fashion, finite or infinite intervals I, and Q of whatever convex rate of growth at the endpoints of I. In particular, our aim was to treat in a unified fashion $W = \exp(-Q)$ for a class of weights including the following two examples. Let

$$\exp_0(x) := x$$

and for $j \geq 1$, recursively define the jth iterated exponential

$$\exp_j(x) := \exp\left(\exp_{j-1}(x)\right).$$

Let k, ℓ be nonnegative integers.

(I) Let $I = \mathbb{R}$ and for $\alpha,\ \beta > 1$, let

$$Q(x) = Q_{\ell,k,\alpha,\beta}(x) := \begin{cases} \exp_\ell(x^\alpha) - \exp_\ell(0), & x \in [0,\infty) \\ \exp_k(|x|^\beta) - \exp_k(0), & x \in (-\infty,0) \end{cases}. \tag{1}$$

(II) Let $I = (-1,1)$ and for $\alpha,\ \beta > 0$, let

$$Q(x) = Q^{(\ell,k,\alpha,\beta)}(x) := \begin{cases} \exp_\ell((1-x^2)^{-\alpha}) - \exp_\ell(1), & x \in [0,1) \\ \exp_k((1-x^2)^{-\beta}) - \exp_k(1), & x \in (-1,0) \end{cases}. \tag{2}$$

In both cases, the subtraction of a constant ensures continuity of Q at 0.

Note that in the special case $k = \ell = 0$ and $\alpha = \beta$, the first example becomes

$$Q_\alpha(x) := Q_{0,0,\alpha,\alpha}(x) = |x|^\alpha,\ x \in \mathbb{R}.$$

This is the archetypal "Freud" exponent, much investigated by Freud and his coworkers in the 1960's and 1970's [18]. One of its important features is homogeneity that allows for easy normalisation and scaling: if $c > 0$,

$$Q_\alpha\left((cn)^{1/\alpha}x\right) = cnQ_\alpha(x),$$

so that for $W_\alpha := \exp(-Q_\alpha)$,

$$W_\alpha\left((cn)^{1/\alpha}x\right) = W_\alpha^{cn}(x).$$

This relation allows one to treat the weight W_α on $\left[-(cn)^{1/\alpha}, (cn)^{1/\alpha}\right]$ as W_α^{cn} on $[-1,1]$. Thus instead of a weight on a growing sequence of intervals, one may deal with a varying sequence of weights on the fixed interval $[-1,1]$. In the potential theory associated with polynomials of degree n and the weights W_α^{cn}, one invariably takes nth roots and then returns to a problem involving the fixed weight W_α^c on $[-1,1]$. Such a problem admits analysis far more easily than the original problem on an unbounded interval or a sequence of intervals. Indeed, many of the fundamental advances in orthogonal and extremal polynomials in the last twenty years have used such a device, though often in a more complicated form.

While this idea of scaling is still crucial for the examples in (1) and (2), some of the simplicity is lost because of non–homogeneity, or a very complicated scaling. One of the main features of our work has been to cope with this complexity, simultaneously dealing with Q of all convex rates of growth at ± 1 or $\pm\infty$.

In terms of applications of exponential weights, undoubtedly the Freud weights on $\mathbb{R}$, for which Q is even and of polynomial growth, have the lion's share. However, there is a growing body of applications of other rates of growth of Q. For example, symmetric exponential weights $W = \exp(-Q)$ that decay rapidly at ± 1, such as

$$Q(x) := Q^{(0,0,\alpha,\alpha)}(x) = \left(1 - x^2\right)^{-\alpha} - 1, \ x \in (-1,1)$$

are being investigated for applications in the numerical solution of ill–posed equations [5]. A somewhat older example is the Stieltjes–Wigert weight, or log–normal distribution in statistics, which has the form

$$W(x) := \exp\left(-k\,(\log x)^2\right), \ x \in (0,\infty), \ k > 0$$

[7]. We expect the rapidly decaying weights on the whole real line will also be applied in several contexts.

To define our classes of weights that include the exponents in (1) and (2), we need the notion of a quasi–decreasing/ quasi–increasing function. A function $g : (0,b) \to (0,\infty)$ is said to be *quasi–increasing* if there exists $C > 0$ such that

$$g(x) \leq C g(y), \ 0 < x \leq y < b.$$

Similarly we may define the notion of a *quasi–decreasing* function. The notation

$$f(x) \sim g(x)$$

means that there are positive constants C_1, C_2 such that for the relevant range of x,

$$C_1 \leq f(x)/g(x) \leq C_2.$$

Similar notation is used for seqences and sequences of functions. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and polynomials P of degree at most n.

Throughout we work on an interval $I = (c, d)$ where

$$-\infty \leq c < 0 < d \leq \infty.$$

We emphasize that c, d may be finite or infinite. Our smallest, but most explicit, class of weights is:

Definition 1

Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$ satisfies the following properties:

(a) *Q' is continuous and $Q(0) = 0$;*

(b) *Q'' exists and is positive in $I \backslash \{0\}$;*

(c)

$$\lim_{t \to c+} Q(t) = \lim_{t \to d-} Q(t) = \infty;$$

(d) *The function*

$$T(t) := \frac{tQ'(t)}{Q(t)},\ t \neq 0 \tag{3}$$

is quasi–increasing in $(0, d)$, and quasi–decreasing in $(c, 0)$, with

$$T(t) \geq \Lambda > 1,\ t \in I \backslash \{0\}. \tag{4}$$

(e) *There exists $C_1 > 0$ such that*

$$\frac{Q''(x)}{\mid Q'(x) \mid} \leq C_1 \frac{\mid Q'(x) \mid}{Q(x)}\ ,\ a.e.\ x \in I \backslash \{0\}. \tag{5}$$

Then we write $W \in \mathcal{F}\left(C^2\right)$.

We note that both examples (1) and (2) lie in $\mathcal{F}\left(C^2\right)$. For many results we may relax the restrictions on Q in Definition 1, dealing instead with:

Definition 2

Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$ satisfies the following properties:

(a) *Q' is continuous and $Q(0)=0$;*

(b) *Q' is non–decreasing in I;*

(c)

$$\lim_{t\to c+} Q(t) = \lim_{t\to d-} Q(t) = \infty;$$

(d) *The hypothesis (d) of Definition 1 holds and there exists $\varepsilon_0 \in (0,1)$ such that for $y \in I\backslash\{0\}$,*

$$T(y) \sim T\left(y\left[1-\frac{\varepsilon_0}{T(y)}\right]\right). \tag{6}$$

Then we write $W \in \mathcal{F}$.

The condition (6) is automatically satisfied for "most" y for T, but we need it to hold on a set of full measure. It is not immediately obvious that it is satisfied for the examples in (1), (2), but it certainly is, since the conditions in Definition 1 imply those in Definition 2. Even more, the weights in $\mathcal{F}\left(C^2\right)$ also satisfy the hypotheses of the following two definitions, which involve Dini–Lipschitz or Lipschitz–$\frac{1}{2}$ type conditions.

Definition 3
Let $W \in \mathcal{F}$.

(a) *Assume that there exist C, $\varepsilon_1 > 0$ such that $\forall\, x \in I\backslash\,\{0\}$,*

$$\int_{x-\frac{\varepsilon_1|x|}{T(x)}}^{x+\frac{\varepsilon_1|x|}{T(x)}} \frac{Q'(s)-Q'(x)}{s-x}ds \le C\left|Q'(x)\right|. \tag{7}$$

Then we write $W \in \mathcal{F}(Dini)$.

(b) *Assume that $\forall\,\varepsilon > 0$, there exists $\delta > 0$ such that $\forall\, x \in I\backslash\,\{0\}$,*

$$\int_{x-\frac{\delta|x|}{T(x)}}^{x+\frac{\delta|x|}{T(x)}} \frac{Q'(s)-Q'(x)}{s-x}ds \le \varepsilon\left|Q'(x)\right|. \tag{8}$$

Then we write $W \in \mathcal{F}(dini)$.

It is clear that $\mathcal{F}(Dini)$ contains $\mathcal{F}(dini)$. Moreover, the defining conditions for both classes is clearly a local Dini–Lipschitz condition, with specified bounds. For bounds on orthogonal polynomials, we shall need to assume more, namely a local Lip $\frac{1}{2}$ condition:

Definition 4
Let $W \in \mathcal{F}$.

(a) *Assume that there exist* C, $\varepsilon_1 > 0$ *such that* $\forall\, x \in I\backslash\{0\}$,

$$\int_{x-\frac{\varepsilon_1|x|}{T(x)}}^{x} \frac{|Q'(s)-Q'(x)|}{|s-x|^{3/2}} ds \leq C\,|Q'(x)|\sqrt{\frac{T(x)}{|x|}}. \tag{9}$$

Then we write $W \in \mathcal{F}(Lip\frac{1}{2})$.

(b) *Assume that* $\forall\, \varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\forall\, x \in I\backslash\{0\}$,

$$\int_{x-\frac{\delta|x|}{T(x)}}^{x+\frac{\delta|x|}{T(x)}} \frac{|Q'(s)-Q'(x)|}{|s-x|^{3/2}} ds \leq \varepsilon\,|Q'(x)|\sqrt{\frac{T(x)}{|x|}}. \tag{10}$$

Then we write $W \in \mathcal{F}(lip\frac{1}{2})$.

We note that

$$\mathcal{F} \supseteq \mathcal{F}(Dini) \supseteq \mathcal{F}\,(dini) \supseteq \mathcal{F}\left(Lip\frac{1}{2}\right) \supseteq \mathcal{F}\left(lip\frac{1}{2}\right) \supseteq \mathcal{F}\left(C^2\right).$$

Most of the inclusions are fairly obvious, except for the third and the fifth.

In Section 2, we shall state some results for equilibrium measures and Christoffel functions, and in Section 3, we state results on asymptotics of orthonormal and extremal polynomials.

2 Equilibrium Measures and Christoffel Functions

In analysis of exponential weights, a crucial role is played by the Mhaskar–Rahmanov–Saff numbers a_t, $t \in \mathbb{R}$. For $t > 0$ and $W \in \mathcal{F}$, $c < a_{-t} < 0 < a_t < d$ are uniquely defined by the equations

$$t = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{xQ'(x)}{\sqrt{(x-a_{-t})(a_t-x)}} dx;$$

$$0 = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{Q'(x)}{\sqrt{(x-a_{-t})(a_t-x)}} dx.$$

It is a fairly basic result that a_t is an increasing function of $t \in \mathbb{R}$, with

$$\lim_{t\to-\infty} a_t = c;\ \lim_{t\to\infty} a_t = d.$$

We use the notation

$$\Delta_t = [a_{-t}, a_t].$$

One of the properties of $a_{\pm t}$ is the Mhaskar–Saff identity: for all polynomials P of degree at most n,

$$\| PW \|_{L_\infty(I)} = \| PW \|_{L_\infty[a_{-n}, a_n]} .$$

Moreover, $a_{\pm n}$ are essentially the smallest numbers for which this is true [12], [15–17], [21].

Associated with Q, and $t > 0$, there is an equilibrium measure, whose density σ_t is given by

$$\sigma_t(x) = \frac{1}{\pi^2}\sqrt{(x-a_{-t})(a_t-x)}\int_{a_{-t}}^{a_t}\frac{Q'(s)-Q'(x)}{s-x}\frac{ds}{\sqrt{(s-a_{-t})(a_t-s)}},$$

$x \in [a_{-t}, a_t]$. The convexity of Q guarantees that σ_t is non–negative. The total mass of σ_t is t :

$$\int_{a_{-t}}^{a_t} \sigma_t = t.$$

Moreover, there is the crucial property that

$$V^{\sigma_t}(x) + Q(x) = c_t,\ x \in \Delta_t,$$

where the equilibrium potential V^{σ_t} is defined by

$$V^{\sigma_t}(z) = \int \log\left|\frac{1}{z-s}\right| \sigma_t(s)ds.$$

For further orientation on σ_t, the reader may consult [12], [15], [21]. One of our main estimates for σ_t is:

Theorem 5

Let $W \in \mathcal{F}$.

(a) *The following are equivalent:*

  (I) $W \in \mathcal{F}\left(Lip\frac{1}{2}\right)$.

  (II) *Uniformly for $t > 0$ and $x \in (a_{-t}, a_t)$,*

$$\sigma_t(x) \sim \frac{t\sqrt{(x-a_{-t})(a_t-x)}}{(x-a_{-2t})(a_{2t}-x)} .$$

(b) *The following are equivalent:*

  (I) $W \in \mathcal{F}\,(Dini)$.

(II) *There exists $C>0$ such that uniformly for $t>0$ and $x\in(a_{-t},a_t)$,*

$$\sigma_t(x)\le\frac{Ct}{\sqrt{(x-a_{-2t})(a_{2t}-x)}}.$$

(III) *There exists $C>0$ such that uniformly for $t>0$ and $x\in(a_{-t},a_t)$,*

$$\sigma_t(x)\le\frac{Ct}{\sqrt{(x-a_{-t})(a_t-x)}}.$$

Next, we turn to estimates of Christoffel functions. The classical Christoffel function associated with a weight W^2 is [18]

$$\begin{aligned}\lambda_n(W^2,x) &= \inf_{P\in\mathcal{P}_{n-1}}\int(PW)^2(t)dt/P^2(x)\\ &= 1\Big/\sum_{j=0}^{n-1}p_j^2(x).\end{aligned}$$

To state our estimates, we need more notation. Let

$$\delta_t:=\frac{1}{2}(a_t+|a_{-t}|);\ \beta_t:=\frac{1}{2}\left(a_t+a_{-t}\right),\ t>0;$$

$$\eta_{\pm t}:=\left[tT(a_{\pm t})\sqrt{\frac{|a_{\pm t}|}{\delta_t}}\right]^{-2/3},\ t>0;$$

and

$$\varphi_t(x):=\begin{cases}\frac{|x-a_{-2t}||x-a_{2t}|}{t\sqrt{[|x-a_{-t}|+|a_{-t}|\eta_{-t}][|x-a_t|+a_t\eta_t]}}, & x\in[a_{-t},a_t];\\ \varphi_t(a_t), & x\in(a_t,d);\\ \varphi_t(a_{-t}), & x\in(c,a_{-t}).\end{cases}\tag{11}$$

Note that the right–hand side in (11) is effectively the reciprocal of our estimate for σ_t in Theorem 5(a), adjusted by adding $\eta_{\pm t}$ so that the denominator does not become too small near $a_{\pm t}$. The function $\varphi_t(x)$ plays much the same role in our setting as does the function $\max\{n^{-2},\frac{\sqrt{1-x^2}}{n}\}$ in approximation on $(-1,1)$. In [10], we obtain estimates for generalized Christoffel functions, which in particular imply

Theorem 6

Let $0<\alpha<1$ and let $W\in\mathcal{F}(Dini)$.

(a) *Then uniformly for $n \geq 1$ and $x \in \Delta_{\alpha n}$,*

$$\lambda_n(W^2, x) \sim \varphi_n(x) W^2(x). \tag{12}$$

(b)

$$\begin{aligned} \max_{x \in I} \lambda_n(W^2, x)^{-1} W^2(x) &\sim \max_{x \in \Delta_{\alpha n}} \lambda_n(W^2, x)^{-1} W^2(x) \\ &\sim \frac{n}{\sqrt{\delta_n}} \max \left\{ \frac{T(a_{-n})}{|a_{-n}|}, \frac{T(a_n)}{a_n} \right\}^{1/2}. \end{aligned} \tag{13}$$

(c) *Assume in addition that $W \in \mathcal{F}\left(Lip\frac{1}{2}\right)$. Then uniformly for $n \geq n_0$ and $x \in \Delta_n$, we have (12).*

The methods of proof of both Theorem 5 and 6 follow those in [8], [9], but there are extra technical details because of the greater generality. Of course in turn the methods from [8], [9] depend on a host of ideas of many researchers, and we cannot adequately provide credit in this short paper.

3 Orthogonal and Extremal Polynomials

The main result of [10] is the following bound on the orthonormal polynomials $p_n(x)$:

Theorem 7
Let $W \in \mathcal{F}\left(lip\frac{1}{2}\right)$. Then uniformly for $n \geq 1$,

$$\sup_{x \in I} \left| p_n(W^2, x) \right| W(x) \left| (x - a_{-n})(a_n - x) \right|^{1/4} \sim 1. \tag{14}$$

Moreover,

$$\sup_{x \in I} \left| p_n(W^2, x) \right| W(x) \leq C n^{1/6} \delta_n^{-1/3} \max \left\{ \frac{T(a_n)}{a_n}, \frac{T(a_{-n})}{|a_{-n}|} \right\}^{1/6}. \tag{15}$$

We note that if, for example, we assume that $W \in \mathcal{F}\left(C^2\right)$ and that $\leq$ is replaced by $\sim$ in (5), except possibly for x near 0, then we have $\sim$ in (15). In particular, this is true for the examples in (1) and (2).

We also establish asymptotics for extremal errors and extremal polynomials. Recall that the L_p extremal error associated with monic polynomials of degree n is

$$E_{n,p}(W) := \inf_{P \in \mathcal{P}_{n-1}} \| \left[x^n - P(x) \right] W(x) \|_{L_p(I)}.$$

In the special case $p = 2$, it is well known that $E_{n,2}(W)$ is the reciprocal of the leading coefficient of $p_n(W^2, \cdot)$. Our asymptotic is:

Theorem 8
Let $W \in \mathcal{F}\,(dini)$ and $1 \le p \le \infty$. Let

$$\kappa_p := \begin{cases} \left[\sqrt{\pi}\Gamma(\frac{p+1}{2})/\Gamma(\frac{p}{2}+1)\right]^{1/p}, & p < \infty; \\ 1, & p = \infty. \end{cases}$$

Then as $n \to \infty$, we have

$$E_{n,p}(W) = 2\kappa_p \left(\frac{\delta_n}{2}\right)^{n+1/p} \exp\left(-\frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q(s)}{\sqrt{(s-a_{-n})(a_n-s)}}ds\right)\big(1+o(1)\big).$$

An L_p extremal polynomial of degree n associated with W is a monic polynomial $T_{n,p}(W,x)$ of degree n such that

$$\| T_{n,p}W \|_{L_p(I)} = \inf_{\substack{P\in P_n \\ P \text{ monic}}} \| PW \|_{L_p(I)} .$$

We also introduce the normalized extremal polynomial

$$p_{n,p}(W,x) := T_{n,p}(W,x)/E_{n,p}(W).$$

In our notation, the orthonormal polynomial $p_n\,(x) = p_n\left(W^2, x\right)$ satisfies

$$p_n(x) = p_{n,2}\,(W,x)\,.$$

The asymptotics for $p_{n,p}(W,x)$ involve the function

$$\Gamma(g;\theta) := \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1} \frac{\log g(s) - \log g(x)}{s-x}\,\frac{ds}{\sqrt{1-s^2}},\ x = \cos\theta;$$

the linear transformation L_n that maps $[a_{-n}, a_n]$ onto $[-1,1]$:

$$\begin{aligned} L_n(u) : &= (u-\beta_n)/\delta_n \Leftrightarrow \\ L_n^{[-1]}(u) &= \delta_n u + \beta_n,\ u \in [-1,1]; \end{aligned}$$

and the composition W_n^* of W and $L_n^{[-1]}$,

$$W_n^*(x) := W \circ L_n^{[-1]}(x),\ x \in [-1,1].$$

Theorem 9
Let $W \in \mathcal{F}(dini)$. Let $1 < p < \infty$. We have

$$\lim_{n\to\infty} \int_{-1}^{1} \Bigg| \delta_n^{1/p} p_{n,p}(W, L_n^{[-1]}(x)) W_n^*(x) - \frac{1}{\kappa_p (1-x^2)^{1/(2p)}} \times \\ \cos\left[\left(n + \frac{1}{p}\right) \arccos x + \Gamma(W_n^*; \arccos x) - \frac{\pi}{2p}\right] \Bigg|^p dx = 0.$$

This mean asymptotic implies asymptotics for $p_{n,p}\left(W, L_n^{[-1]}(z)\right)$ in $\mathbb{C}\backslash[-1,1]$. The latter were obtained in [10] for all $1 < p \leq \infty$ (the case $p = \infty$ involves a separate argument). By specialising to $p = 2$, one obtains asymptotics in the plane and mean asymptotics for $p_n(x) = p_n(W^2, x)$. These in turn imply asymptotics for the recurrence coefficients in the three term recurence relation:

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + A_{n-1} p_{n-1}(x).$$

Theorem 10
Let $W \in \mathcal{F}(dini)$. Then as $n \to \infty$,

$$\begin{aligned} \frac{A_n}{\delta_n} - \frac{1}{2} &= o(1); \\ \frac{B_n}{\delta_n} - \frac{\beta_n}{\delta_n} &= o(1). \end{aligned}$$

Finally, we record the pointwise asymptotics for $p_n(x)$:

Theorem 11
Let $W \in \mathcal{F}\left(lip\frac{1}{2}\right)$. There exists $\eta > 0$ such that as $n \to \infty$, we have uniformly for $|x| \leq 1 - n^{-\eta}$, $x = \cos\theta$,

$$\begin{aligned} &\delta_n^{1/2} p_n\left(L_n^{[-1]}(x)\right) W_n^*(x)(1-x^2)^{1/4} \\ &\quad = \sqrt{\frac{2}{\pi}} \cos\left(\left(n + \frac{1}{2}\right)\theta + \Gamma(W_n^*; \theta) - \frac{\pi}{4}\right) + O(n^{-\eta}). \end{aligned}$$

These pointwise asymptotics should be compared to the far more precise, but more special, results of E.A. Rahmanov [20] and the group of P. Deift [2–4], especially the most recent results of Kriecherbauer and McLaughlin [6], while the mean asymptotics should be compared to the remarkably precise results of Deift et al. [2–4, 6] and the remarkably general ones of Totik [22]. Our methods of proof largely follow those in [13–14], [22].

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