On an asymptotic equality for reproducing kernels and sums of squares of orthonormal polynomials

A. Ignjatovic and D. S. Lubinsky*

Abstract In a recent paper, the first author considered orthonormal polynomials $\{p_n\}$ associated with a symmetric measure with unbounded support, and with recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + A_{n-1} p_{n-1}(x), \quad n \geq 0.$$

Under appropriate restrictions on $\{A_n\}$, the first author established the identity

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2(x)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x) + p_{2n+1}^2(x)}{A_{2n}^{-1} + A_{2n+1}^{-1}},$$

uniformly for x in compact subsets of the real line. In this paper, we establish and evaluate this limit for a class of even exponential weights, and also investigate analogues for weights on a finite interval, and for some non-even weights.

This paper is dedicated to the memory of Q. I. Rahman

1 Introduction

Let μ be a symmetric positive measure on the real line, with all finite power moments. Then we may define orthonormal polynomials $p_n(x) = \gamma_n x^n + \cdots$,

A. Ignjatovic
School of Computer Science and Engineering, University of New South Wales, Sydney, NSW 2052 Australia e-mail: ignjat@cse.unsw.edu.au

D. S. Lubinsky
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 USA e-mail: lubinsky@math.gatech.edu

* Research of second author supported by NSF grant DMS136208.

$\gamma_n > 0$, $n \geq 0$, satisfying

$$\int p_n p_m d\mu = \delta_{mn}.$$

Because of the symmetry, the three term recurrence relation takes a simple form:

$$xp_n(x) = A_n p_{n+1}(x) + A_{n-1} p_{n-1}(x), \quad n \geq 1,$$

where

$$A_n = \frac{\gamma_n}{\gamma_{n+1}}, \quad n \geq 1.$$

The asymptotic behavior of p_n as $n \to \infty$ has been intensively investigated for over a century, and has a myriad of applications. In a recent paper [4], the first author presented a novel approach, and placed the following hypotheses on the recurrence coefficients:

(C1) $\lim_{n\to\infty} A_n = \infty$;
(C2) $\lim_{n\to\infty} (A_{n+1} - A_n) = 0$;
(C3) There exist m_0, n_0 such that $A_{m+n} > A_n$ for all $n \geq n_0$ and $m \geq m_0$;
(C4)

$$\sum_{j=0}^{\infty} \frac{1}{A_j} = \infty;$$

(C5) There exists $k > 1$ such that

$$\sum_{j=0}^{\infty} \frac{1}{A_j^k} < \infty;$$

(C6)

$$\sum_{j=0}^{\infty} \frac{|A_{j+1} - A_j|}{A_j^2} < \infty;$$

(C7)

$$\sum_{j=0}^{\infty} \frac{|A_{j+2} - 2A_{j+1} + A_j|}{A_j} < \infty.$$

He proved [4]:

Theorem A. *Under the hypotheses* (C1)–(C7)*, the following limits exist, are finite and positive, and satisfy*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2(x)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x) + p_{2n+1}^2(x)}{A_{2n}^{-1} + A_{2n+1}^{-1}}, \tag{1.1}$$

uniformly for x in compact subsets of the real line.

It was shown in [4], that if $0 < p < 1$, $c > 0$, and

$$A_n = c\,(n+1)^p\,, \quad n \geq 0,$$

then (C1)–(C7) hold. This rate of growth of recurrence coefficients is typically associated with an exponential weight such as $\exp\left(-|x|^{-1/p}\right)$, $0 < p < 1$. Indeed the asymptotics for recurrence coefficients given in [2, p. 50, Theorem 1.3] show that (C1)–(C7) are valid for these specific exponential weights.

In this paper, we shall evaluate the limit in (1.1), showing that it equals $(2\pi\mu'(x))^{-1}$, for a large class of exponential weights. We do this by using asymptotics for orthonormal polynomials and Christoffel functions that were established in [5].

This paper is organized as follows: in Section 2, we briefly discuss the case of weights on $[-1,1]$. This simple case illustrates some of the ideas of proof. Our main results, for even exponential weights, are stated and proved in Section 3. In Section 4, we discuss some limited extensions to non-even weights. In the sequel $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, polynomials of degree $\leq n$, and possibly other parameters. We use $\sim$ in the following sense: given sequences of non-zero real numbers $\{x_n\}$ and $\{y_n\}$, we write $x_n \sim y_n$ if there exists a constant $C > 1$ such that

$$C^{-1} \leq x_n/y_n \leq C \quad \text{for } n \geq 1.$$

Similar notation is used for functions and sequences of functions.

2 Weights on $[-1,1]$

The result of this section is:

Theorem 2.1. *Let μ be a positive measure supported on $[-1,1]$ that satisfies Szegő's condition*

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}}\,dx > -\infty. \tag{2.1}$$

Assume, that for some $\eta \in \left(0, \frac{1}{2}\right)$, μ is absolutely continuous in $[-\eta,\eta]$, that μ' is positive and continuous in $[-\eta,\eta]$, and satisfies for some $C > 0$, $\rho > 1$, and $x, y \in [-\eta,\eta]$,

$$|\mu'(x) - \mu'(y)| \leq C\,|\log|x-y||^{-\rho}\,. \tag{2.2}$$

Let $\{x_n\}$ be a sequence of real numbers with limit 0 as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2(x_n)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x_n) + p_{2n+1}^2(x_n)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi}. \tag{2.3}$$

Of course this result is quite restricted as we need $x_n \to 0$.

We turn to

The Proof of Theorem 2.1. Under the assumptions of Theorem 2.1, there is the asymptotic as $m \to \infty$,

$$p_m(x)\,\mu'(x)^{1/2}\left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}}\cos\left(m\theta+\gamma(\theta)\right)+o(1), \tag{2.4}$$

uniformly for $x = \cos\theta$ in a compact subset of $(-\eta, \eta)$, where

$$\gamma(\theta) = \frac{1}{4\pi}PV\int_{-\pi}^{\pi}\left[\log f(t) - \log f(\theta)\right]\cot\frac{\theta - t}{2}dt.$$

Here PV denotes principal value, and

$$f'(\theta) = \mu'(\cos\theta)\left|\sin\theta\right|.$$

This follows from Theorem 2 in [1, p. 41]. We note that other criteria for asymptotics are given in, for example [3, p. 246, Table II(a)], or Theorem 5 in [6, p. 77].

Now let $\{x_n\}$ be a sequence with limit 0, and for $n \geq 1$, write $x_n = \cos\theta_n$, where $\theta_n \in (0, \pi)$. We see that $\theta_n \to \frac{\pi}{2}$ as $n \to \infty$. The asymptotic (2.4) gives

$$\begin{aligned}
&\left[p_{2n}^2(x_n) + p_{2n+1}^2(x_n)\right]\mu'(x_n)^{1/2}\left(1-x_n^2\right)^{1/2} \\
&= \frac{2}{\pi}\left[\cos^2\left(2n\theta_n + \gamma(\theta_n)\right) + \cos^2\left((2n+1)\theta_n + \gamma(\theta_n)\right)\right] + o(1) \\
&= \frac{2}{\pi}\left[\cos^2\left(2n\theta_n + \gamma(\theta_n)\right) + \cos^2\left(2n\theta_n + \gamma(\theta_n) + \frac{\pi}{2}\right)\right] + o(1) \\
&= \frac{2}{\pi}\left[\cos^2\left(2n\theta_n + \gamma(\theta_n)\right) + \sin^2\left(2n\theta_n + \gamma(\theta_n)\right)\right] + o(1) = \frac{2}{\pi} + o(1).
\end{aligned}$$

Next, the limit

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{\gamma_n}{\gamma_{n+1}} = \frac{1}{2} \tag{2.5}$$

is an immediate consequence of the fact that μ satisfies Szegő's condition [11, p. 309]. Thus the second part in (2.3) satisfies

$$\lim_{n\to\infty}\frac{p_{2n}^2(x_n) + p_{2n+1}^2(x_n)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi}. \tag{2.6}$$

Next, asymptotics for Christoffel functions and the continuity of μ' in $[-\eta, \eta]$ yield that[7], [10, Thm. 3.11.9, p. 220] that uniformly for $x \in [-\eta, \eta]$,

$$\lim_{n\to\infty}\frac{1}{n+1}\left(\sum_{k=0}^{n}p_k^2(x)\right)\mu'(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

In particular, then, as μ' is continuous at 0,

$$\lim_{n\to\infty} \frac{1}{n+1} \left(\sum_{k=0}^{n} p_k^2 (x_n) \right) = \frac{1}{\pi}. \tag{2.7}$$

Finally (2.5) gives

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} A_k^{-1} = 2.$$

Combining the last two limits, we have

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2 (x_n)}{\sum_{k=0}^{n} A_k^{-1}} = \frac{1}{2\pi},$$

so the result follows using (2.6). ■

3 Even Weights on $(-\infty, \infty)$

Following is the class of even weights we shall consider. It is a subclass of that in [5, Definition 1.1, p. 7]:

Definition 3.1. Let $\mu'(x) = e^{-2Q(x)}$, $x \in \mathbb{R}$, where Q is even,

(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$;
(b) Q'' exists and is positive in $\mathbb{R}\setminus\{0\}$;
(c)

$$\lim_{t\to\infty} Q(t) = \infty;$$

(d) The function $T(t) = \frac{tQ'(t)}{Q(t)}$, $t \in (0, \infty)$ is quasi-increasing in $(0, \infty)$, in the sense that for some constant C and $0 \leq x < y \Rightarrow$

$$T(x) \leq CT(y);$$

In addition we assume that T is bounded below in $\mathbb{R}\setminus\{0\}$ by a constant Λ larger than 1.
(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{Q'(x)}{Q(x)} \quad \text{, a.e. } x \in (0\,\infty).$$

Then we write $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2, even\right)$.

Examples of Q satisfying the conditions above on $(-\infty, \infty)$ include [5, pp. 8–9]

$$Q(x) = |x|^{\alpha},$$

where $\alpha > 1$, and

$$Q(x) = \exp_\ell(|x|^\alpha) - \exp_\ell(0),$$

where $\alpha > 1$, $\ell \geq 0$, and $\exp_k(x) = \exp(\exp(\cdots(\exp(x)))$ is the kth iterated exponential. We could actually allow a more general (but more technical) class of weights, namely the even weights of class $\mathcal{F}\left(lip\frac{1}{2}\right)$ from [5]. We shall prove

Theorem 3.2. *Let $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2, even\right)$. Then uniformly for x in compact subsets of the real line,*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n p_k^2(x)}{\sum_{k=0}^n A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x) + p_{2n+1}^2(x)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi e^{-2Q(x)}}. \tag{3.1}$$

In considering orthogonal polynomials associated with the measure $d\mu(t) = e^{-2Q(t)}dt$, a crucial role is played by the Mhaskar-Rakhmanov-Saff numbers $a_t, t > 0$. These are defined by the equations [5, p. 13], [8], [9]

$$t = \frac{1}{\pi}\int_0^1 \frac{a_t x Q'(a_t x)}{\sqrt{(1-x^2}} dx. \tag{3.2}$$

We note that a_t increases with t and $a_t \to \infty$ as $t \to \infty$. As an example of Mhaskar-Rakhmanov-Saff numbers, let $Q(x) = |x|^\alpha$, $x \in \mathbb{R}$, $\alpha > 0$. It is known that then [8], [9]

$$a_t = \left\{\frac{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2}{\Gamma(\alpha)}\right\}^{1/\alpha} t^{1/\alpha}, \quad t > 0.$$

Another important quantity associated with Q is the nth equilibrium density [5, p. 16]

$$\sigma_n(x) = \frac{1}{\pi^2}\sqrt{a_n^2 - x^2}\int_0^{a_n} \frac{sQ'(s) - xQ'(x)}{s^2 - x^2}\frac{ds}{\sqrt{a_n^2 - s^2}}, \quad x \in [-a_n, a_n]. \tag{3.3}$$

It has total mass n

$$\int_{-a_n}^{a_n} \sigma_n = n,$$

and satisfies the equilibrium equation

$$\int_{-a_n}^{a_n} \log\left|\frac{1}{x-s}\right| \sigma_n(s)ds + Q(x) = c_n, \quad x \in [-a_n, a_n].$$

Here c_n is a constant.

In many contexts, it is convenient to map σ_n onto a density function that is supported on $[-1, 1]$. Let

$$\sigma_n^*(t) = \frac{a_n}{n}\sigma_n(a_n t), \quad t \in [-1, 1]. \tag{3.4}$$

It satisfies

$$\int_{-1}^{1} \sigma_n^* = 1.$$

Proof of Theorem 3.2. In [5, p. 403, Theorem 15.3, (15.11)], it is shown that uniformly for $x = \cos\theta$ in a closed subinterval of $(-1,1)$ and $m = n-1, n$,

$$\begin{aligned} &a_n^{1/2} p_m (a_n x) W (a_n x) \left(1 - x^2\right)^{1/4} \\ &\quad = \sqrt{\frac{2}{\pi}} \cos\left(\left(m - n + \frac{1}{2}\right)\theta + n\pi \int_x^1 \sigma_n^* - \frac{\pi}{4}\right) + o(1). \end{aligned} \tag{3.5}$$

Note that the linear transformation L_n there reduces to $L_n(x) = x/a_n$ and $L_n^{[-1]}(t) = a_n t$. Setting

$$\Delta_n(x) = \frac{1}{2}\theta + n\pi \int_x^1 \sigma_n^* - \frac{\pi}{4},$$

we see that uniformly for x in a closed subset of $(-1,1)$,

$$\begin{aligned} &a_n W (a_n x)^2 \left(1 - x^2\right)^{1/2} \left\{p_n^2(a_n x) + p_{n-1}^2(a_n x)\right\} \\ &\quad = \frac{2}{\pi}\left\{(\cos \Delta_n(x))^2 + (\cos(\Delta_n(x) - \theta))^2\right\} + o(1). \end{aligned}$$

In particular, setting $a_n x = y$ where y lies in a compact set, so that $x = y/a_n = \cos\theta$ has $\theta = \arccos(y/a_n) = \arccos(o(1)) = \frac{\pi}{2} + o(1)$, we obtain

$$\begin{aligned} &a_n W(y)^2 \left(1 - \left(\frac{y}{a_n}\right)^2\right)^{1/2} \left\{p_n^2(y) + p_{n-1}^2(y)\right\} \\ &= \frac{2}{\pi}\left\{\left(\cos \Delta_n\left(\frac{y}{a_n}\right)\right)^2 + \left(\cos\left(\Delta_n\left(\frac{y}{a_n}\right) - \frac{\pi}{2}\right)\right)^2\right\} + o(1) \\ &= \frac{2}{\pi}\left\{\left(\cos \Delta_n\left(\frac{y}{a_n}\right)\right)^2 + \left(\sin \Delta_n\left(\frac{y}{a_n}\right)\right)^2\right\} + o(1) = \frac{2}{\pi} + o(1). \end{aligned}$$

Replacing y by x, and n by $2n+1$, we have that uniformly for x in a compact subset of $\mathbb{R}$,

$$a_{2n+1} W(x)^2 \left\{p_{2n+1}^2(x) + p_{2n}^2(x)\right\} = \frac{2}{\pi} + o(1). \tag{3.6}$$

Next, (1.124) of Theorem 1.23 in [5, p. 26] gives

$$\lim_{n\to\infty} \frac{A_n}{a_n} = \frac{1}{2}. \tag{3.7}$$

In addition, [5, p. 81, eqn. (3.50)]

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1.$$

Then the second part of (3.1) can be calculated as

$$\lim_{n\to\infty}\frac{p_{2n+1}^2(x)+p_{2n}^2(x)}{A_{2n+1}^{-1}+A_{2n}^{-1}}=\lim_{n\to\infty}\frac{a_{2n+1}}{4}\left\{p_{2n+1}^2(x)+p_{2n}^2(x)\right\}=\frac{1}{2\pi W^2(x)}, \tag{3.8}$$

uniformly for x in a compact subset of $\mathbb{R}$.

It is more difficult to deal with the left-hand side in (3.1). We first note that

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}p_k^2(x)\,W^2(x)\,/\sigma_n(x)=1, \tag{3.9}$$

uniformly for x in a range that certainly includes compact subsets of the real line [5, Theorem 1.25, p. 26]. Our task is to compare this to

$$\sum_{k=0}^{n-1}A_k^{-1}=2\sum_{k=0}^{n-1}a_k^{-1}(1+o(1)).$$

We shall show that this last right-hand side behaves like

$$2\int_0^n\frac{1}{a_t}dt\,(1+o(1)),$$

and using an alternative representation for σ_n, due to Rakhmanov, that this in turn is close to $\sigma_n(x)$ when x is bounded. Let us now make this rigorous. First note that a_t is a differentiable increasing function of $t\in(0,\infty)$, with $a_t\to 0$ as $t\to 0+$ and $a_t\to\infty$ as $t\to\infty$. Define the inverse $b:(0,\infty)\to(0,\infty)$ of a by

$$b(a_t)=t,\quad t>0.$$

Rakhmanov's representation for σ_n for even weights asserts that [5, p. 46, eqn. (2.35)]

$$\sigma_n(x)=\frac{1}{\pi}\int_{|b(x)|}^n\frac{1}{\sqrt{a_s^2-x^2}}ds,\quad x\in(-a_n,a_n). \tag{3.10}$$

In particular,

$$\sigma_n(0)=\frac{1}{\pi}\int_0^n\frac{1}{a_s}ds. \tag{3.11}$$

(The convergence of the integral is established in Chapter 2 of [5].) Now let $m=m(n)=[\sqrt{n}]$, where $[x]$ denotes the greatest integer $\le x$. Assume that $x\in(0,r]$. For n large enough (the threshold depends on r), we have

$$
\begin{aligned}
|\sigma_n(x)-\sigma_n(0)| &= \frac{1}{\pi}\left|\int_{b(x)}^{n}\left\{\frac{1}{\sqrt{a_s^2-x^2}}-\frac{1}{a_s}\right\}ds-\int_0^{b(x)}\frac{1}{a_s}ds\right| \\
&\leq \frac{1}{\pi}\int_{b(x)}^{n}\frac{a_s-\sqrt{a_s^2-x^2}}{\sqrt{a_s^2-x^2}a_s}ds+\int_0^{b(r)}\frac{1}{a_s}ds \\
&= \frac{x^2}{\pi}\int_{b(x)}^{n}\frac{1}{\sqrt{a_s^2-x^2}a_s\left(a_s+\sqrt{a_s^2-x^2}\right)}ds+\int_0^{b(r)}\frac{1}{a_s}ds.
\end{aligned}
\tag{3.12}
$$

Here as $a_s \geq a_{b(x)} = x$ in the first integral, we see that

$$
\begin{aligned}
&\frac{x^2}{\pi}\int_{b(x)}^{n}\frac{1}{\sqrt{a_s^2-x^2}a_s\left(a_s+\sqrt{a_s^2-x^2}\right)}ds \\
&\quad\leq x^2\left\{\frac{1}{\pi x^2}\int_{b(x)}^{m}\frac{ds}{\sqrt{a_s^2-x^2}}+\frac{1}{a_m^2}\int_m^n\frac{1}{\sqrt{a_s^2-x^2}}ds\right\} \\
&\quad\leq \sigma_m(x)+\left(\frac{x}{a_m}\right)^2\sigma_n(x).
\end{aligned}
$$

Combining this and (3.12) gives

$$
\left|1-\frac{\sigma_n(0)}{\sigma_n(x)}\right| \leq \frac{\sigma_m(x)}{\sigma_n(x)}+\left(\frac{x}{a_m}\right)^2+\frac{C}{\sigma_n(x)}.
$$

Here as $x \in [0,r]$ and $m \to \infty$, we have $\left(\frac{x}{a_m}\right)^2 \to 0$ uniformly for $x \in [0,r]$. In addition, it follows from Theorem 5.2(b) in [5, p. 110] and then Lemma 3.5(c) in [5, p. 72] that uniformly for $x \in [-r,r]$,

$$
\frac{\sigma_m(x)}{\sigma_n(x)} \leq C\frac{(m/a_m)}{(n/a_n)} \leq C\left(\frac{m}{n}\right)^{1-1/\Lambda} \leq C\left(\frac{1}{\sqrt{n}}\right)^{1-1/\Lambda}. \tag{3.13}
$$

Here $\Lambda > 1$ is a lower bound for T in $\mathbb{R}$. Thus, using also evenness of σ_n, we have

$$
\lim_{n\to\infty}\sup_{x\in[-r,r]}\left|1-\frac{\sigma_n(0)}{\sigma_n(x)}\right| = 0.
$$

This, (3.9), and (3.11) give, uniformly for $x \in [-r,r]$, as $n \to \infty$,

$$
\sum_{k=0}^{n-1}p_k^2(x)W^2(x) = \frac{1}{\pi}\int_0^n\frac{1}{a_s}ds\,(1+o(1)). \tag{3.14}
$$

Finally, [5, p. 81, eqn. (3.50)] gives, for $k \geq 1$,

$$\left|\int_{k-1}^{k}\frac{1}{a_s}ds-\frac{1}{a_k}\right|\leq\int_{k-1}^{k}\frac{1}{a_s}\left|1-\frac{a_s}{a_k}\right|ds$$
$$\leq C\int_{k-1}^{k}\frac{1}{a_s}\left|1-\frac{s}{k}\right|ds\leq\frac{C}{k}\int_{k-1}^{k}\frac{1}{a_s}ds,$$

so from (3.7), and using monotonicity of a_t in t,

$$\begin{aligned}\left|\sum_{k=0}^{n-1}A_k^{-1}-2\int_0^n\frac{1}{a_s}ds\right| &= \left|2\sum_{k=2}^{n-1}\left(a_k^{-1}-\int_{k-1}^{k}\frac{1}{a_s}ds\right)+o\left(\sum_{k=2}^{n-1}a_k^{-1}\right)+O\left(1\right)\right|\\ &\leq C\sum_{k=2}^{n-1}\frac{1}{k}\int_{k-1}^{k}\frac{1}{a_s}ds+o\left(\int_1^n\frac{1}{a_s}ds\right)+O\left(1\right)\\ &\leq C\int_1^m\frac{1}{a_s}ds+\frac{C}{m}\int_m^n\frac{1}{a_s}ds+o\left(\int_1^n\frac{1}{a_s}ds\right)\\ &\leq C\sigma_m\left(0\right)+\frac{C}{m}\sigma_n\left(0\right)+o\left(\sigma_n\left(0\right)\right)\\ &\leq C\sigma_n\left(0\right)\left\{\left(\frac{1}{\sqrt{n}}\right)^{1-1/\Lambda}+\frac{1}{\sqrt{n}}+o\left(1\right)\right\}\\ &= o\left(\frac{1}{\pi}\int_0^n\frac{1}{a_s}ds\right),\end{aligned}$$

by (3.13). Thus

$$\left(\sum_{k=0}^{n-1}A_k^{-1}\right)/\left(2\int_0^n\frac{1}{a_s}ds\right)=1+o\left(1\right).$$

This and (3.14) give uniformly for x in $[-r,r]$,

$$\frac{\sum_{k=0}^{n-1}p_k^2\left(x\right)W^2\left(x\right)}{\left(\sum_{k=0}^{n-1}A_k^{-1}\right)}=\frac{1}{2\pi}\left(1+o\left(1\right)\right).$$

Combining this with (3.8), gives the result. ■

4 The Non-Even, Not Necessarily Unbounded Case

In this section, we briefly explore the extent to which we can extend Theorem 3.2 to non-even exponential weights, possibly not on an infinite interval. To this end, we first define the full class $\mathcal{F}\left(C^2\right)$ from [5, p. 7]:

Definition 4.1. Let $\mathcal{I}=(c,d)$ be a bounded or unbounded interval containing 0. Let $\mu'\left(x\right)=e^{-2Q(x)}$, $x\in\mathcal{I}$, where

(a) Q' is continuous in $\mathcal{I}$ and $Q(0) = 0$;
(b) Q'' exists and is positive in $\mathcal{I}\backslash\{0\}$;
(c)

$$\lim_{t\to c+} Q(t) = \infty = \lim_{t\to d-} Q(t);$$

(d) The function $T(t) = \frac{tQ'(t)}{Q(t)}$, $t \in \mathcal{I}\backslash\{0\}$is quasi-increasing in $(0,d)$, in the sense that for some constant C and $0 \le x < y < d \Rightarrow$

$$T(x) \le CT(y);$$

T is also assumed quasi-decreasing in $(c,0)$. In addition we assume that T is bounded below in $\mathcal{I}\backslash\{0\}$ by a constant larger than 1.
(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{Q'(x)}{Q(x)}, \qquad \text{a.e. } x \in \mathcal{I}\backslash\{0\}.$$

Then we write $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2\right)$.

Examples of Q satisfying the conditions above on $(-\infty,\infty)$ include [5, pp. 8–9]

$$Q(x) = \begin{cases} x^\alpha, & x \in [0,\infty) \\ |x|^\beta, & x \in (-\infty, 0), \end{cases}$$

where $\alpha,\beta > 1$. A more general example is

$$Q(x) = \begin{cases} \exp_\ell(x^\alpha) - \exp_\ell(0), & x \in [0,\infty) \\ \exp_k\left(|x|^\beta\right) - \exp_k(0) & x \in (-\infty,0), \end{cases}$$

where $\alpha,\beta > t$, $k,\ell \ge 0$, and $\exp_k(x) = \exp(\exp(\cdots(\exp(x)))$ is the kth iterated exponential. An example on $(-1,1)$ is

$$Q(x) = \begin{cases} \exp_\ell\left(\left(1-x^2\right)^{-\alpha}\right) - \exp_\ell(1), & x \in [0,1) \\ \exp_k\left(\left(1-x^2\right)^{-\beta}\right) - \exp_k(1), & x \in (-1,0), \end{cases}$$

where $\alpha,\beta > 0$ and $k,\ell \ge 0$.

Instead of just one Mhaskar-Rakhmanov-Saff number, there are now two: a_{-n}, a_n are defined by the equations [5, p. 13]

$$n = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx;$$

$$0 = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx.$$

The nth equilibrium density now takes the form [5, p. 16]

$$\sigma_n(x) = \frac{1}{\pi^2}\sqrt{(x-a_{-n})(a_n-x)}\int_{a_{-n}}^{a_n}\frac{Q'(s)-Q'(x)}{s-x}$$
$$\times\frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}},\quad x\in[a_{-n},a_n].$$

The contraction of σ_n to $[-1,1]$ is more complicated than for the even case: let

$$\beta_n := \frac{1}{2}(a_n+a_{-n});\quad \delta_n = \frac{1}{2}(a_n+|a_{-n}|).$$

We can then define the linear map of $[a_{-n},a_n]$ onto $[-1,1]$ by

$$L_n(x) = (x-\beta_n)/\delta_n,\quad x\in[a_{-n},a_n] \Leftrightarrow x = L_n^{[-1]}(t) = \beta_n+\delta_n t,\quad t\in[-1,1].$$

The transformed (and renormalized) density is

$$\sigma_n^*(t) = \frac{\delta_n}{n}\sigma_n\circ L_n^{[-1]}(t),\quad t\in[-1,1].$$

Instead of the asymptotic (3.5), we have uniformly for $x = \cos\theta \in [-1+\varepsilon,1-\varepsilon]$ [5, p. 403]

$$a_n^{1/2}p_m\left(L_n^{[-1]}(x)\right)W\left(L_n^{[-1]}(x)\right)\left(1-x^2\right)^{1/4}$$
$$= \sqrt{\frac{2}{\pi}}\cos\left(\left(m-n+\frac{1}{2}\right)\theta + n\pi\int_x^1\sigma_n^* - \frac{\pi}{4}\right) + o(1). \tag{4.1}$$

Instead of the asymptotic (3.7) for the recurrence coefficients, we have [5, p. 26]

$$\lim_{n\to\infty}\frac{A_n}{\delta_n} = \frac{1}{2}. \tag{4.2}$$

By proceeding as in Section 3, it is straightforward to see that

$$\lim_{n\to\infty}\frac{p_{2n}^2\left(L_{2n+1}^{[-1]}(x_n)\right)+p_{2n+1}^2\left(L_{2n+1}^{[-1]}(x_n)\right)}{A_{2n}^{-1}+A_{2n+1}^{-1}}W^2\left(L_{2n+1}^{[-1]}(x_n)\right) = \frac{1}{2\pi}, \tag{4.3}$$

for any sequence $\{x_n\}$ with limit 0. Setting $y_n = L_{2n+1}^{[-1]}(x_n)$, we see that this becomes

$$\lim_{n\to\infty}\frac{p_{2n}^2(y_n)+p_{2n+1}^2(y_n)}{A_{2n}^{-1}+A_{2n+1}^{-1}}W^2(y_n) = \frac{1}{2\pi}, \tag{4.4}$$

for any sequence $\{y_n\}$ with

$$\lim_{n\to\infty}L_{2n+1}(y_n) = \lim_{n\to\infty}\frac{y_n-\beta_{2n+1}}{\delta_{2n+1}} = 0. \tag{4.5}$$

Unfortunately, it is more problematic to establish an analogue of (3.14). The asymptotic (3.8) holds uniformly for $x \in [a_{-\alpha n}, a_{\alpha n}]$, for any fixed $\alpha \in (0,1)$. For this to be compatible with (4.5), we need $\beta_n \in [a_{-\alpha n}, a_{\alpha n}]$, for some $\alpha > 0$, so in particular, we cannot have a very asymmetric weight. Rakhmanov's representation for σ_n now takes the form

$$\sigma_n(x) = \frac{1}{\pi}\int_{|b(x)|}^{n} \frac{1}{\sqrt{(x-a_{-s})(a_s - x)}}ds = \frac{1}{\pi}\int_{|b(x)|}^{n} \frac{1}{\sqrt{\delta_s^2 - (x-\beta_s)^2}}ds.$$

Provided both for some $r > 0$,

$$|y_n - \beta_n| \le r, \quad n \ge 1, \tag{4.6}$$

and

$$|a_{-n}|/a_n \to 1 \text{ as } n \to \infty, \tag{4.7}$$

we can prove, much as in the last section, that

$$\lim_{n\to\infty} \left| 1 - \frac{\sigma_n(\beta_n)}{\sigma_n(y_n)} \right| = 0.$$

Of course (4.7) is a very severe asymptotic symmetry requirement. With the aid of (4.7), we can prove much as in the previous section, that

$$\sigma_n(\beta_n) = \frac{1}{\pi}\int_0^n \frac{1}{\delta_s} ds\,(1+o(1)).$$

$$\left(\sum_{k=0}^{n-1} A_k^{-1}\right) / \left(2\int_0^n \frac{1}{\delta_s}ds\right) = 1 + o(1).$$

Thus for $W \in \mathcal{F}(C^2)$ satisfying the additional condition (4.7), and sequences $\{y_n\}$ satisfying (4.5), we obtain

$$\frac{\sum_{k=0}^{n-1} p_k^2(y_n) W^2(y_n)}{\left(\sum_{k=0}^{n-1} A_k^{-1}\right)} = \frac{1}{2\pi}(1+o(1)),$$

and hence

$$\lim_{n\to\infty} \frac{W^2(y_n)\sum_{k=0}^{n} p_k^2(y_n)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{W^2(y_n)\left(p_{2n}^2(y_n) + p_{2n+1}^2(y_n)\right)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi}. \tag{4.8}$$

It would be interesting to see if the severe symmetry condition (4.6) can be weakened.

References

1. V. M. Badkov, *The Asymptotic Behavior of Orthogonal Polynomials*, Math USSR, Sbornik, 37 (1980), 39–51
2. P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, X. Zhou, *A Riemann-Hilbert Approach to Asymptotic Questions for Orthogonal Polynomials*, J. Comp. Appl. Math., 133(2001), 47–63.
3. G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.
4. A. Ignjatovic, *Asymptotic Behavior of Some Families of Orthonormal Polynomials and an Associated Hilbert Space*, manuscript: arXiv:1410.2661 [math.CA].
5. Eli Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, 2001.
6. A. Mate, P. Nevai, V. Totik, *Extensions of Szegő's Theory of Orthogonal Polynomials, III*, Constr. Approx., 3(1987), 73–96.
7. A. Mate, P. Nevai, V. Totik, *Szegő's Extremum Problem on the Unit Circle*, Annals of Math., 134(1991), 433–453.
8. H. N. Mhaskar and E. B. Saff, *Extremal Problems for Polynomials with Exponential Weights*, Trans. Amer. Math. Soc., 285(1984), 203–234.
9. H. N. Mhaskar and E. B. Saff, *Where does the L_p Norm of a Weighted Polynomial Live?*, Trans. Amer. Math. Soc., 303(1987), 109–124.
10. B. Simon, *Szegő's Theorem and its descendants*, Princeton University Press, Princeton, 2011.
11. G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, Providence, 1939.