OLD AND NEW GERONIMUS TYPE IDENTITIES FOR REAL ORTHOGONAL POLYNOMIALS

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Abstract. Let μ be a positive measure on the real line, with orthogonal polynomials $\{p_n\}$ and leading coefficients $\{\gamma_n\}$. The Geronimus type identity

$$\frac{1}{\pi}\left|\operatorname{Im} z\right| \int_{-\infty}^{\infty} \frac{P(t)}{\left|z p_n(t) - p_{n-1}(t)\right|^2} dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t)\ d\mu(t),$$

valid for all polynomials P of degree $\leq 2n-2$ has known analogues within the theory of orthogonal rational functions, though apparently unknown in the theory of orthogonal polynomials. We present new proofs of this and its generalization,

$$\int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}}{p_n}(t)\right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \left(\int P(t)\ d\mu(t)\right),$$

valid for any $h \in L_1(\mathbb{R})$.

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1. Introduction[1]

In the theory of orthogonal polynomials on the unit circle, Geronimus' identity [5, p. 198] plays an important role. Recall that if ν is a finite positive Borel measure on the unit circle, with infinitely many points in its support, and orthonormal polynomials $\{\Phi_n\}$, so that

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n\left(e^{i\theta}\right) \overline{\Phi_m\left(e^{i\theta}\right)} d\nu(\theta) = \delta_{mn},$$

then Geronimus' identity asserts that

$$\int_0^{2\pi} \frac{P\left(e^{i\theta}\right)}{\left|\Phi_n\left(e^{i\theta}\right)\right|^2} d\theta = \int P\left(e^{i\theta}\right)\ d\nu\left(e^{i\theta}\right),$$

for all polynomials P of degree $\leq n$. By symmetry, this extends to P that is a Laurent polynomial. Geronimus' identity is very useful in asymptotics for orthogonal polynomials on the unit circle, see the books of Freud [5] and Simon [12].

As far as the author was aware, there was no known analogue for orthogonal polynomials on the real line. At least, none is mentioned in the classical textbooks on orthogonal polynomials. While using the theory of de Branges spaces in the context of universality limits for random matrices, the author discovered such an identity.

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Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \ldots,\ \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

The nth reproducing kernel for μ is

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n-1} p_k(x) p_k(y) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x-y}, \end{aligned}$$

by the Christoffel-Darboux formula. The nth Christoffel function is

$$\lambda_n(x) = \frac{1}{K_n(x,x)}.$$

We give the numerator in the Christoffel-Darboux formula its own symbol, namely,

$$L_n(x,y) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y) \right) \tag{1.1}$$

and for non-real a, let

$$E_{n,a}(z) = \sqrt{\frac{2\pi}{|L_n(a,\bar{a})|}} L_n(\bar{a}, z). \tag{1.2}$$

In [9], we used the theory of de Branges spaces [1] to show that for $\operatorname{Im} a > 0$, and all polynomials P of degree $\leq 2n-2$, we have

$$\int_{-\infty}^{\infty} \frac{P(t)}{|E_{n,a}(t)|^2} dt = \int P(t)\ d\mu(t). \tag{1.3}$$

In a subsequent paper [10], we gave a self contained elementary proof, and deduced results on weak convergence, discrepancy, and Gauss quadrature. We soon found out that there is an earlier real line analogue, due to Barry Simon [13, Theorem 2.1, p. 5], namely

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(t)}{\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 p_n^2(t) + p_{n-1}^2(t)} dt = \int P(t)\ d\mu(t).$$

Simon calls this a *Carmona type formula* because of its analogy to identities in the theory of Schrodinger operators [3]. He also refers to earlier work of Krutikov and Remling [7]. This Carmona formula is the special case of (1.3) with $(p_{n-1}/p_n)(\bar{a}) = \pm i\gamma_{n-1}/\gamma_n$.

A more explicit, and pleasing, form of (1.3) was established in [11]: if $\operatorname{Im}(z) \neq 0$,

$$\frac{1}{\pi} |\operatorname{Im} z| \int_{-\infty}^{\infty} \frac{P(t)}{|z p_n(t) - p_{n-1}(t)|^2} dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t)\ d\mu(t). \tag{1.4}$$

Of course, orthogonal polynomials is an old subject, and it is surprising to find a new general identity. So the author performed an extensive literature search, including in the theory of orthogonal rational functions. However, it was only at the 2010 Jaen conference, that the author became aware that a close cousin of these identities is known within the theory of orthogonal rational functions - thank you to Adhemar Bultheel, for the gentle discussion, and information. Let

$$\mathcal{P}(t,w)=\frac{\operatorname{Im} w}{|t-w|^2}$$

denote the Poisson kernel for the upper half-plane. In their groundbreaking monograph [2, Thm. 6.3.2, p. 136; Thm. 6.4.3, p. 145], Bultheel, Gonzalez-Vera, Hendriksen and Njåstad, showed that the inner products generated by the measures $\frac{d\mu(t)}{1+t^2}$ and $\frac{\mathcal{P}(t,w)\mathcal{K}_n(w,\bar{w})}{|K_n(t,\bar{w})|^2}dt$ on the real line, are identical on a suitable finite dimensional space of rational functions. Here $\mathcal{K}_n$ is a related, but different, reproducing kernel to the one in this paper. On choosing appropriate values of the poles, parameters and measures, one could in principle recover (1.3), and hence (1.4). Formally, the results in [2, Thm. 6.3.2, p. 136; Thm. 6.4.3, p. 145] do not include the case of all poles at ∞, so (1.4) is apparently not there - but an elaboration of the methods there, does yield (1.4). It is instructive that for orthogonal polynomials on the unit circle, Bultheel et al refer to analogues within systems theory [4], as well as within classical orthogonal polynomials.

Observe that in (1.4),

$$\frac{\operatorname{Im} z}{|zp_n(t)-p_{n-1}(t)|^2}=\frac{1}{p_n(t)^2}\mathcal{P}\left(\frac{p_{n-1}(t)}{p_n(t)},z\right).$$

Using this, and classical results on boundary behavior of Poisson integrals for the upper-half plane, the author proved [11] that

$$\int_{-\infty}^{\infty}\frac{P(t)}{p_n(t)^2}h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)dt=\frac{\gamma_{n-1}}{\gamma_n}\left(\int_{-\infty}^{\infty}h(t)\,dt\right)\left(\int P(t)\ d\mu(t)\right). \tag{1.5}$$

for all polynomials P of degree $\leq 2n-2$ and any $h\in L_1(\mathbb{R})$. When, for example,

$$h(x)=\frac{\log x^{-2}}{1-x^2},$$

one obtains an entropy type integral

$$\frac{2}{\pi^2}\int_{-\infty}^{\infty}P(t)\frac{\ln|p_{n-1}(t)|-\ln|p_n(t)|}{p_{n-1}(t)^2-p_n(t)^2}dt=\frac{\gamma_{n-1}}{\gamma_n}\int P(t)\ d\mu(t). \tag{1.6}$$

We note that this circle of ideas also leads to explicit formulae for orthogonal polynomials associated with a reciprocal polynomial weight [8].

In Sections 2 and 3, we present respectively new proofs of (1.4) and (1.5). In Section 4, we outline the proof from [10], inspired by de Branges spaces, which has some points of contact with that used by Bultheel et al.

2. A New Proof of (1.4)

Fix a polynomial P of degree $\leq 2n-2$. For $\operatorname{Im}(z)>0$, let

$$G(z)=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{P(t)}{p_n(t)}\frac{1}{p_{n-1}(t)-zp_n(t)}dt.$$

Here PV stands for Cauchy principal value integral, because of the non-integrable singularities at the zeros $\{x_{jn}\}$ of p_n. Thus

$$G(z)=\frac{1}{\pi}\lim_{\substack{\varepsilon_j\to 0+,\\ 1\le j\le n}}\int_{(-\infty,\infty)\setminus\bigcup_{j=1}^{n}(x_{jn}-\varepsilon_j,x_{jn}+\varepsilon_j)}\frac{P(t)}{p_n(t)}\frac{1}{p_{n-1}(t)-zp_n(t)}dt.$$

This limit exists because of the smoothness of the integrand after the extraction of the singularities. Note that the factor $p_{n-1}(t)-zp_n(t)$ cannot vanish when $\operatorname{Im}(z)\neq 0$, as p_{n-1} and p_n have no common zeros.

The function $G(z)$ is also analytic for z in the upper-half plane, and for such z,

$$G'(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{P(t)}{(p_{n-1}(t)-zp_n(t))^2}dt, \tag{2.1}$$

an ordinary Lebesgue integral. Indeed,

$$\begin{aligned}&\frac{G(z+h)-G(z)}{h}\\ =\ &\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{P(t)}{(p_{n-1}(t)-zp_n(t))(p_{n-1}(t)-(z+h)p_n(t))}dt.\end{aligned}$$

We can use Lebesgue's Dominated Convergence Theorem to justify the limit as $h\to 0$. All one needs is a bound on the integrand independent of h, that holds for small enough $|h|$. Assume $|h|<\frac{1}{2}|\operatorname{Im} z|$. Now outside small neighborhoods of the zeros of p_n, we can bound $|p_{n-1}(t)-(z+h)p_n(t)|$ below by $\frac{1}{2}|\operatorname{Im} z||p_n(t)|$, and in those neighborhoods of these zeros, we can use instead the lower bound $\frac{1}{2}|p_{n-1}(t)|$, at least for small enough $|h|$.

We next claim that for all $\operatorname{Im}(z)>0$,

$$G'(z)=0. \tag{2.2}$$

To see this, fix z with $\operatorname{Im}(z)>0$. We first show that the polynomial in t, $p_{n-1}(t)-zp_n(t)$ has no zeros in the upper half-plane. Indeed, as the zeros of p_n and p_{n-1} interlace, all residues in the partial fraction representation of $\frac{p_{n-1}}{p_n}$ are positive. More precisely,

$$\frac{p_{n-1}}{p_n}(t)=\sum_{j=1}^{n}\frac{c_j}{t-x_{jn}},$$

with all $c_j>0$. (It is well known that $c_j=\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn})$, but we shall not need this.) So if $p_{n-1}(t)-zp_n(t)=0$, we have

$$0<\operatorname{Im} z=\operatorname{Im}\left(\sum_{j=1}^{n}\frac{c_j}{t-x_{jn}}\right)=\operatorname{Im}(\bar{t})\sum_{j=1}^{n}\frac{c_j}{|t-x_{jn}|^2},$$

and necessarily t lies in the lower-half plane. Then, as a function of t, $\frac{P(t)}{(p_{n-1}-zp_n(t))^2}$ is analytic in the upper-half plane, and is $O(t^{-2})$ as $|t|\to\infty$, so Cauchy's integral theorem (or, the residue theorem) shows that $G'(z)=0$.

It follows that for some constant C, we have

$$\operatorname{Im} G(z)=C \text{ for } \operatorname{Im}(z)>0.$$

That is,

$$\frac{\operatorname{Im} z}{\pi}\int_{-\infty}^{\infty}\frac{P(t)}{\left|zp_n(t)-p_{n-1}(t)\right|^2}dt=C\text{ for }\operatorname{Im}(z)>0. \tag{2.3}$$

We evaluate C by computing

$$\begin{aligned} C &= \lim_{y\to\infty}\operatorname{Im}G(iy) \\ &= \lim_{y\to\infty}\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{P(t)}{p_{n-1}^2(t)+y^2p_n^2(t)}dt. \end{aligned} \tag{2.4}$$

Write

$$-\infty=x_{n+1,n}<x_{n,n}<...<x_{1n}<x_{0n}=\infty,$$

and let

$$I_j=(x_{j+1,n},x_{jn}),\ 0\le j\le n;$$

$$\psi_j(t)=\frac{p_{n-1}(t)}{p_n(t)},t\in I_j.$$

Note that for $1\le j\le n$, ψ_j is strictly decreasing in I_j, from ∞ to $-\infty$, so has an inverse $\psi_j^{[-1]}:(-\infty,\infty)\to I_j$. For $j=0$, instead ψ_j is strictly decreasing from ∞ to 0, so $\psi_0^{[-1]}:(0,\infty)\to I_0$. For $j=n$, instead ψ_j is strictly decreasing from 0 to $-\infty$, so $\psi_n^{[-1]}:(-\infty,0)\to I_n$. Also,

$$\begin{aligned} \psi_j'(t) &= -\frac{\left(p_n'p_{n-1}-p_{n-1}'p_n\right)(t)}{p_n^2(t)} \\ &= -\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}K_n(t,t)/p_n^2(t), \end{aligned}$$

recall the confluent form of the Christoffel-Darboux formula,

$$K_n(t,t)=\frac{\gamma_{n-1}}{\gamma_n}\left(p_n'p_{n-1}-p_{n-1}'p_n\right)(t).$$

Now let $1\le j\le n-1$, and $g(t)=P(t)/K_n(t,t)$, and make the substitution $s=\psi_j(t)=p_{n-1}(t)/p_n(t)$. We see that

$$\begin{aligned} T_j &:= \frac{y}{\pi}\int_{I_j}\frac{P(t)}{p_{n-1}^2(t)+y^2p_n^2(t)}dt \\ &= \frac{-y}{\pi}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\int_{I_j}\frac{P(t)}{K_n(t,t)}\frac{\psi_j'(t)}{y^2+\psi_j^2(t)}dt \\ &= \frac{1}{\pi}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\int_{-\infty}^{\infty}g\left(\psi_j^{[-1]}(s)\right)\frac{y}{y^2+s^2}ds. \end{aligned} \tag{2.5}$$

We continue this as

$$T_j=\frac{1}{\pi}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\int_{-\infty}^{\infty}g\left(\psi_j^{[-1]}(ty)\right)\frac{1}{1+t^2}dt.$$

Now

$$\lim_{y\to\infty}\psi_j^{[-1]}(ty)=\begin{cases} x_{j+1,n}, & t>0, \\ x_{jn}, & t<0 \end{cases}$$

and g is bounded on the real line, so Lebesgue's Dominated Convergence Theorem gives

$$\begin{aligned}\lim_{y\to\infty} T_j &= \frac{1}{\pi}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\left[g\left(x_{j+1,n}\right)\int_{-\infty}^{0}\frac{dt}{1+t^2}+g\left(x_{jn}\right)\int_{0}^{\infty}\frac{dt}{1+t^2}\right]\\ &= \frac{1}{2}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\left[g\left(x_{j+1,n}\right)+g\left(x_{jn}\right)\right].\end{aligned}$$

Similarly,

$$\lim_{y\to\infty} T_0=\lim_{y\to\infty}\frac{1}{\pi}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\int_0^{\infty} g\left(\psi_0^{[-1]}(ty)\right)\frac{1}{1+t^2}dt=\frac{1}{2}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)g\left(x_{1n}\right),$$

and

$$\lim_{y\to\infty} T_n=\lim_{y\to\infty}\frac{1}{\pi}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\int_{-\infty}^{0} g\left(\psi_n^{[-1]}(ty)\right)\frac{1}{1+t^2}dt=\frac{1}{2}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)g\left(x_{nn}\right).$$

Recalling (2.4), (2.5), and adding over $0\le j\le n$ gives

$$\begin{aligned}&\lim_{y\to\infty}\operatorname{Im} G(iy)\\ &= \lim_{y\to\infty}\sum_{j=0}^{n}T_j\\ &= \frac{\gamma_{n-1}}{\gamma_n}\sum_{j=1}^{n}g\left(x_{jn}\right)\\ &= \frac{\gamma_{n-1}}{\gamma_n}\sum_{j=1}^{n}P\left(x_{jn}\right)/K_n\left(x_{jn},x_{jn}\right)=\frac{\gamma_{n-1}}{\gamma_n}\int P\,d\mu,\end{aligned}$$

by the classical Gauss quadrature formula. This, (2.3) and (2.4), give the result.■

3. A New Proof of (1.5)

We start by recalling that given $s\neq 0$, there are n simple zeros $\{t_j\}$ of $p_{n-1}(t)-sp_n(t)$, and a corresponding Gauss quadrature

$$\sum_{j=1}^{n}\lambda_n\left(t_j\right)P\left(t_j\right)=\int P\,d\mu,$$

valid for all polynomials P of degree $\le 2n-2$. Here, recall,

$$\lambda_n(t)=1/K_n(t,t)$$

is the nth Christoffel function. The $\{t_j\}$ interlace the zeros of p_n. See [5, p.19 ff.]. If $\{\psi_j\}$ are as in the previous section, we see that $\{t_j\}_j=\left\{\psi_j^{[-1]}(s)\right\}_j$. Thus we can write the quadrature formula as

$$\sum_{j=0}^{n}\left(\lambda_nP\right)\left(\psi_j^{[-1]}(s)\right)=\int P\,d\mu.$$

Here if $s>0$, the term for $j=n$ is dropped as there is no root of $\psi_n(t)=s$ in $(-\infty,x_{nn})$. Similarly, if $s<0$, the term for $j=0$ is dropped. Now we let

$h \in L_1(\mathbb{R})$, multiply by $h(s)$ and integrate to obtain

$$\sum_{j=0}^{n} \int_{-\infty}^{\infty} (\lambda_n P)\left(\psi_j^{[-1]}(s)\right) h(s)\, ds = \left(\int_{-\infty}^{\infty} h(s)\, ds\right)\left(\int P(t)\ d\mu(t)\right). \tag{3.1}$$

The integrand is treated as 0 for $j = 0$ and $s < 0$ (and for $j = n$ and $s > 0$). Now let $1 \le j \le n-1$. We make the substitution $s = \psi_j(t)$ and recall that $\psi_j^{[-1]}$ maps $(-\infty, \infty)$ onto I_j, while

$$\begin{aligned} \psi_j'(t) &= -\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} K_n(t,t)/p_n^2(t) \\ &= -\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} \lambda_n^{-1}(t)/p_n^2(t). \end{aligned}$$

We then see that

$$\begin{aligned} &\int_{-\infty}^{\infty} (\lambda_n P)\left(\psi_j^{[-1]}(s)\right) h(s)\, ds \\ = &\ -\int_{I_j} (\lambda_n P)(t)\, h\left(\psi_j(t)\right) \psi_j'(t)\, dt \\ = &\ \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} \int_{I_j} P(t)\, h\left(\frac{p_{n-1}}{p_n}(t)\right) \frac{dt}{p_n^2(t)}. \end{aligned}$$

For $j = 0$, and $j = n$, we make the appropriate adjustments, taking account of our convention of 0 integrand on half of $(-\infty, \infty)$. Substituting into (3.1) gives

$$\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} \sum_{j=0}^{n} \int_{I_j} P(t)\, h\left(\frac{p_{n-1}}{p_n}(t)\right) \frac{dt}{p_n^2(t)} = \left(\int_{-\infty}^{\infty} h(s)\, ds\right)\left(\int P(t)\ d\mu(t)\right),$$

or

$$\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} \int_{-\infty}^{\infty} P(t)\, h\left(\frac{p_{n-1}}{p_n}(t)\right) \frac{dt}{p_n^2(t)} = \left(\int_{-\infty}^{\infty} h(s)\, ds\right)\left(\int P(t)\ d\mu(t)\right),$$

which yields (1.5). Of course, (1.4) follows with a special choice of h. This proof is simpler than that given in [11], but uses slightly deeper results on orthogonal polynomials. ■

4. A de Branges style proof of (1.4)

By multiplying out the right-hand side, cancelling, and then refactorizing, we see that for any complex α, β, v, z,

$$L_n(z,v)\, L_n(\alpha,\beta) = L_n(\alpha,z)\, L_n(\beta,v) - L_n(\beta,z)\, L_n(\alpha,v).$$

Now let $\operatorname{Im} a > 0$, and set $\alpha = a, \beta = \bar{a}$, so

$$L_n(z,v) = \frac{1}{L_n(a,\bar{a})}(L_n(a,z)\, L_n(\bar{a},v) - L_n(\bar{a},z)\, L_n(a,v)).$$

Since p_n is real on the real axis, and

$$L_n(a,\bar{a}) = 2i \operatorname{Im}(a)\, K_n(a,\bar{a}) = i\left|L_n(a,\bar{a})\right|,$$

we obtain

$$\begin{aligned} K_n(z,v) &= \frac{L_n(z,v)}{z-v} \\ &= \frac{i}{|L_n(a,\bar{a})|}\frac{L_n(\bar{a},z)\overline{L_n(\bar{a},\bar{v})}-\overline{L_n(\bar{a},\bar{z})}L_n(\bar{a},v)}{z-v} \\ &= \frac{i}{2\pi}\frac{E_{n,a}(z)E^*_{n,a}(v)-E^*_{n,a}(z)E_{n,a}(v)}{z-v} \end{aligned} \tag{4.1}$$

where $g^*(z)=\overline{g(\bar{z})}$, and we have used the definition (1.2) of $E_{n,a}$. Next, we claim that for polynomials S of degree $\leq n-1$,

$$S(z)=\int_{-\infty}^{\infty}\frac{S(t)K_n(t,z)}{|E_{n,a}(t)|^2}dt. \tag{4.2}$$

To prove this, let us assume that $\mathrm{Im}(a)>0, \mathrm{Im}(z)>0$. The right-hand side equals

$$\int_{-\infty}^{\infty}\frac{S(t)K_n(t,z)}{E_{n,a}(t)E^*_{n,a}(t)}dt$$

$$= \frac{i}{2\pi}\begin{pmatrix} E^*_{n,a}(z)\int_{-\infty}^{\infty}\frac{S(t)}{E^*_{n,a}(t)(t-z)}dt \\ -E_{n,a}(z)\int_{-\infty}^{\infty}\frac{S(t)}{E_{n,a}(t)(t-z)}dt \end{pmatrix}. \tag{4.3}$$

Here all zeros of $E_{n,a}$ lie in the lower half plane. See [10, Lemma 2.2(c)] for a full proof, but note that it follows from the fact that $K_n(z,\bar{a})$ has all its zeros there, and in turn this is a consequence of the Christoffel-Darboux formula. Then, in the first integral in (4.3), all zeros of $E^*_{n,a}$ lie in the upper-half plane, so $\frac{S(t)}{E^*_{n,a}(t)(t-z)}$ is analytic as a function of t in the lower-half plane, and $O(t^{-2})$ at ∞. Thus the first integral in the right-hand side of (4.3) is 0. For the second integral, $\frac{S(t)}{E_{n,a}(t)(t-z)}$ is analytic as a function of t in the upper-half plane, except for a simple pole at $t=z$. We deduce that

$$\int_{-\infty}^{\infty}\frac{S(t)K_n(t,z)}{E_{n,a}(t)E^*_{n,a}(t)}dt$$

$$=\frac{i}{2\pi}\left(0-E_{n,a}(z)2\pi i\frac{S(z)}{E_{n,a}(z)}\right)=S(z),$$

Thus we have (4.2). We now turn to the proof of (1.4). Write $P=RS$ where

$\deg(R), \deg(S) \le n-1$. Then

$$\begin{aligned}\int P\, d\mu &= \int (RS)(z)\, d\mu(z) \\ &= \int R(z)\left[\int_{-\infty}^{\infty} S(t)\frac{K_n(t,z)}{|E_{n,a}(t)|^2}dt\right] d\mu(z) \\ &= \int_{-\infty}^{\infty} S(t)\frac{1}{|E_{n,a}(t)|^2}\left[\int R(z)\, K_n(t,z)\, d\mu(z)\right] dt \\ &= \int_{-\infty}^{\infty} S(t)\frac{1}{|E_{n,a}(t)|^2} R(t)\, dt \\ &= \int_{-\infty}^{\infty} \frac{P(t)}{|E_{n,a}(t)|^2} dt.\end{aligned}$$

This gives (1.3). Finally, fix $z \in \mathbb{C}\backslash\mathbb{R}$, and choose $a \in \mathbb{C}$ such that

$$p_{n-1}(\bar{a}) = z p_n(\bar{a}).$$

There are n choices for a, counting multiplicity. Then from (1.1), we see that

$$L_n(\bar{a}, t) = -\frac{\gamma_{n-1}}{\gamma_n} p_n(\bar{a})\left(z p_n(t) - p_{n-1}(t)\right)$$

and

$$L_n(a, \bar{a}) = 2i\frac{\gamma_{n-1}}{\gamma_n}\operatorname{Im}(z)\, |p_n(a)|^2.$$

Hence

$$\begin{aligned}|E_{n,a}(t)|^2 &= \frac{2\pi}{|L_n(a,\bar{a})|}|L_n(\bar{a},t)|^2 \\ &= \frac{\pi}{|\operatorname{Im} z|}\frac{\gamma_{n-1}}{\gamma_n}|z p_n(t) - p_{n-1}(t)|^2.\end{aligned}$$

Substituting into (1.3) gives (1.4). ■

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