Marcinkiwiecz-Zygmund Type Inequalities for all Arcs of the Circle

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Abstract

We establish Marcinkiewicz-Zygmund Inequalities of the form

$$\sum_{j=1}^{n} |P(\theta_j)|^p (\theta_j - \theta_{j-1}) \leq C \int_{\alpha}^{\beta} |P(\theta)|^p \, d\theta,$$

valid for all trigonometric polynomials P of degree $\leq m$, and for $\alpha = \theta_0 < \theta_1 < ... < \theta_n = \beta$, under appropriate spacing conditions. The emphasis is on uniformity in the length of the interval $\beta - \alpha$, irrespective of whether it is close to 0 or 2π. We also establish weighted versions involving doubling weights.

1 Introduction

The classical Marcinkiewicz-Zygmund inequality has the form

$$C_1 \int_0^{2\pi} |P(\theta)|^p \, d\theta \leq \frac{1}{n} \sum_{k=0}^{2n} \left| P\left(\frac{k}{2n+1} 2\pi\right) \right|^p \leq C_2 \int_0^{2\pi} |P(\theta)|^p \, d\theta, \quad (1)$$

valid for all trigonometric polynomials P of degree n. Here $1 < p < \infty$ and C_1 and C_2 are independent of P and n. This inequality is useful in studying convergence of Lagrange interpolation, orthogonal expansions and discretization of integrals. It has also been extended in many directions.

For example, Mastroianni and Totik [6, p. 46] established a version of the right-hand inequality involving doubling weights:

$$\frac{1}{n}\sum_{k=0}^{2n} W_n\left(\frac{k}{2n+1}2\pi\right)\left|P\left(\frac{k}{2n+1}2\pi\right)\right|^p \leq C\int_0^{2\pi}|P(\theta)|^p\, W(\theta)\, d\theta. \quad (2)$$

Here W is a doubling weight. That is, there is a constant $L > 0$ such that if I is any interval, and $2I$ is the concentric interval with double the length, then

$$\int_{2I} W \leq L\int_I W.$$

The smallest such L, independent of I, is called the doubling constant. Moreover,

$$W_n(\theta) = n\int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} W. \quad (3)$$

Generalized Jacobi weights are doubling weights, and so are many others. Thus Mastroianni and Totik greatly extended the scope of earlier inequalities. They also allowed non-equally spaced points and trigonometric polynomials of degree $\leq Cn$. See [7] and [4] for surveys of Marcinkiewicz-Zygmund Inequalities.

The large sieve of number theory is closely related to (2). One formulation of it is [2, p. 208]

$$\sum_{k=1}^{m}|P(\alpha_j)|^p\,\varepsilon(\alpha_j) \leq C\tau\int_\alpha^\beta |P(\theta)|^p\, d\theta, \quad (4)$$

with C independent of $m, n, P, p, \alpha, \beta, \{\alpha_j\}$. Here P is a trigonometric polynomial of degree$\leq n$,

$$\varepsilon(\theta) = \frac{1}{pn+1}\left[\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)\right| + \left(\frac{\beta-\alpha}{pn+1}\right)^2\right]^{1/2}$$

while

$$0 \leq \alpha \leq \alpha_1 < \alpha_2 < ... < \alpha_m \leq \beta \leq 2\pi,$$

$0 < p < \infty$ and $m \geq 1$. The parameter τ is a measure of the number of α_j in small intervals, given by

$$\tau = \max_{\theta \in [\alpha,\beta]} \left| \{ j : \alpha_j \in [\theta - \varepsilon(\theta), \theta + \varepsilon(\theta)] \} \right|.$$

We note that in [2] P could be a "generalized trigonometric polynomial", not just an ordinary trigonometric polynomial. The key achievement there was independence of the size of $[\alpha, \beta]$ as $\beta - \alpha$ shrinks to 0. The one drawback of the result is that as $[\alpha, \beta]$ approaches $[0, 2\pi]$, we do not recover the usual Marcinkiewicz inequality for $[0, 2\pi]$, for the case of equally spaced points. For the full interval $[0, 2\pi]$, this shortcoming can be repaired by two application of (4), but for intervals $[\alpha, \beta]$ close to $[0, 2\pi]$, it is not clear how to derive a uniform result.

In this paper, we present a version of (4), which will have the correct form for all choices of $[\alpha, \beta]$ - whether $\beta - \alpha$ is very small or close to 2π. We can do this using a Bernstein inequality of the authors, which is sharp in order for all arcs on the circle, or equivalently, all subintervals of $[0, 2\pi]$ [3]. The drawback, however, is that we obtain inequalities only for $p > 1$, and for trigonometric polynomials, not generalized trigonometric polynomials. We prove:

Theorem 1
Let $0 \leq \alpha < \beta \leq 2\pi$ and for $n \geq 1$, define

$$\varepsilon_n(\theta) = \frac{1}{n} \left[\frac{\left| \sin\left(\frac{\theta-\alpha}{2}\right) \sin\left(\frac{\theta-\beta}{2}\right) \right| + \left(\frac{\beta-\alpha}{n}\right)^2}{\left| \cos\left(\frac{\theta + \frac{\alpha+\beta}{2}}{2}\right) \right|^2 + \left(\frac{1}{n}\right)^2} \right]^{1/2}, \theta \in [\alpha, \beta]. \tag{5}$$

Let $K \geq 1$, $m \geq 1, 1 \leq p < \infty$ and

$$\alpha = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{m+1} = \beta \tag{6}$$

satisfy

$$\theta_{j+1} - \theta_j \leq K \varepsilon_n(\theta_j), j = 0, 1, 2, ..., m. \tag{7}$$

Then for all trigonometric polynomials P of degree $\leq Kn$,

$$\sum_{j=0}^{m} |P(\theta_j)|^p (\theta_{j+1} - \theta_j) \leq C \int_\alpha^\beta |P|^p, \tag{8}$$

where C is independent of $n, m, \alpha, \beta, \{\theta_j\}$ and P.

The essential feature is the uniformity in $[\alpha, \beta]$, irrespective of whether $\beta - \alpha$ is small or close to 2π. Thus as $[\alpha, \beta]$ approaches $[0, 2\pi]$, we see that

$$\varepsilon_n(\theta) \to \frac{1}{n}\left[\frac{\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{2\pi}{n}\right)^2}{\left(\sin\frac{\theta}{2}\right)^2 + \frac{1}{n^2}}\right]$$

and the right-hand side lies between $\frac{1}{n}$ and $\frac{1}{n}(2\pi)^2$, so we recover the form of the classical Marcinkiewicz-Zygmund inequality. On the other hand for any α, β, θ we have

$$\varepsilon_n(\theta) \geq \frac{1}{2n}\left[\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)\right| + \left(\frac{\beta-\alpha}{n}\right)^2\right]^{1/2}. \tag{9}$$

and so the large sieve inequality (4) is implied by Theorem 1, with appropriate change of notation.

We shall also prove a result involving doubling weights, by using an inequality of Erdelyi [1]. For simplicity we formulate it only on intervals of the form $[-\omega, \omega]$, where $\omega < \frac{1}{2}$, to conform with Erdelyi. Thus its chief use is on "small intervals". We also introduce the notation

$$\delta_n(\theta) = \frac{1}{n}\left[\left|\sin\left(\frac{\theta-\omega}{2}\right)\sin\left(\frac{\theta+\omega}{2}\right)\right| + \left(\frac{2\omega}{n}\right)^2\right]^{1/2} \tag{10}$$

and

$$W_{\delta_n}(\theta) = \frac{2}{\delta_n(\theta)}\int_{\theta-\delta_n(\theta)}^{\theta+\delta_n(\theta)} W(y)\,dy. \tag{11}$$

This is an extension of the notation W_n used by Mastroianni, Totik, Erdelyi and others, from constant increment $\frac{1}{n}$ to a variable increment $\delta_n(\theta)$.

Theorem 2
Let $0 \leq \omega < \frac{1}{2}$. Let $W : [-\omega, \omega] \to \mathbb{R}$ be such that $W(\omega\cos t)$ is a doubling weight on $[0, \pi]$. Let $K \geq 1$, $m \geq 1, 1 \leq p < \infty$ and

$$-\omega = \theta_0 < \theta_1 < \theta_2 < ... < \theta_m = \omega \tag{12}$$

satisfy

$$\theta_{j+1} - \theta_j \leq K\delta_n(\theta_j), j = 0, 1, 2, ..., m. \tag{13}$$

Then for all trigonometric polynomials P of degree$\leq Kn$,

$$\sum_{j=0}^{m} W_{\delta_n}\left(\theta_j\right)\left|P\left(\theta_j\right)\right|^p\left(\theta_{j+1}-\theta_j\right) \leq C \int_\alpha^\beta |P|^p W,$$

where C is independent of $n, m, \alpha, \beta, \{\theta_j\}$ and P.

The proofs are presented in the next two sections.

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2 The Proof of Theorem 1

We use Nevai's method [8] for establishing such inequalities together with the Bernstein inequality

$$\int_\alpha^\beta \left|P'\varepsilon_n\right|^p \leq C \int_\alpha^\beta |P|^p, \tag{14}$$

valid for all trigonometric polynomials P of degree $\leq n$ [3, p. 345]. We also note the inequality [3, p.355, Lemma 3.1(c)] that

$$\left|e^{i\theta}-e^{i\phi}\right| \leq \frac{1}{28}\varepsilon_n(\phi) \Rightarrow \frac{1}{2} \leq \frac{\varepsilon_n(\theta)}{\varepsilon_n(\phi)} \leq \frac{3}{2}.$$

(The notation there is a little different). A little reflection then shows that, given $K \geq 1$, there exists $L > 1$ such that

$$|\theta-\phi| \leq \min\left\{\frac{\pi}{2}, K\varepsilon_n(\phi)\right\} \Rightarrow \frac{1}{L} \leq \frac{\varepsilon_n(\theta)}{\varepsilon_n(\phi)} \leq L. \tag{15}$$

Here $L > 1$ depends on on K (not on n, θ, ϕ).

Proof of Theorem 1
Let us assume (6) and (7). Fix $0 \leq j \leq m$ and choose $s \in [\theta_j, \theta_{j+1}]$ such that

$$|P(s)| = \min_{[\theta_j,\theta_{j+1}]} |P|.$$

Then

$$\left|P\left(\theta_j\right)\right|^p=\left|P\left(s\right)\right|^p+\int_s^{\theta_j}\frac{d}{d\theta}\left|P\left(\theta\right)\right|^p d\theta, \tag{16}$$

so

$$\begin{aligned}
&\left|P\left(\theta_j\right)\right|^p\left(\theta_{j+1}-\theta_j\right)\\
\leq\ &\left(\min_{[\theta_j,\theta_{j+1}]}\left|P\right|^p\right)\left(\theta_{j+1}-\theta_j\right)+K\varepsilon_n\left(\theta_j\right)\int_{\theta_j}^{\theta_{j+1}}p\left|P\right|^{p-1}\left|P'\right|\\
\leq\ &\int_{\theta_j}^{\theta_{j+1}}\left|P\right|^p+C\int_{\theta_j}^{\theta_{j+1}}\left|P\right|^{p-1}\left|P'\right|\varepsilon_n,
\end{aligned}$$

by first (7) and then (15), with C independent of $P,n,j,....$ Adding over j, followed by Hölder's inequality and our Bernstein inequality (14) give

$$\begin{aligned}
&\sum_{j=0}^{m}\left|P\left(\theta_j\right)\right|^p\left(\theta_{j+1}-\theta_j\right)\\
\leq\ &\int_\alpha^\beta\left|P\right|^p+C\int_\alpha^\beta\left|P\right|^{p-1}\left|P'\right|\varepsilon_n\\
\leq\ &\int_\alpha^\beta\left|P\right|^p+C\left(\int_\alpha^\beta\left|P\right|^p\right)^{\frac{p-1}{p}}\left(\int_\alpha^\beta\left|P'\right|^p\varepsilon_n^p\right)^{\frac{1}{p}}\\
\leq\ &C\int_\alpha^\beta\left|P\right|^p,
\end{aligned}$$

as desired. ■

3 Proof of Theorem 2

We begin by presenting some background:

(I) A transformation of $[-\pi,\pi]$ onto $[-\omega,\omega]$.
Let us define, as did Erdelyi, a transformation

$$L\left(t\right)=\arcsin\left[\left(\sin\omega\right)\left(\cos t\right)\right], t\in\left[-\pi,\pi\right].$$

It maps $[0,\pi]$ (and $[-\pi,0]$) onto $[-\omega,\omega]$. Observe that

$$\sin L\left(t\right)=\left(\sin\omega\right)\left(\cos t\right)$$

and since $L(t) \in [-\omega, \omega] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$L'(t) = -\frac{(\sin\omega)(\sin t)}{\cos L(t)} \sim -(\sin\omega)(\sin t),$$

uniformly in $\omega \in \left[0, \frac{1}{2}\right]$ and $t \in [-\pi, \pi]$. The notation $\sim$ is in the sense standard in orthogonal polynomials: the ratio of the two sides is bounded above and below by positive constants independent of ω and t. (There are trivial modifications if both sides vanish). Similar notation will be used for sequences and sequences of functions. We also then have

$$\begin{aligned}
|L'(t)| &\sim (\sin\omega)\sqrt{1 - \left(\frac{\sin L(t)}{\sin\omega}\right)^2} \\
&= \sqrt{|\sin(L(t) - \omega)\sin(L(t) + \omega)|} \\
&\sim \sqrt{\left|\sin\left(\frac{L(t) - \omega}{2}\right)\sin\left(\frac{L(t) + \omega}{2}\right)\right|} \\
&\leq Cn\delta_n(L(t)),
\end{aligned} \tag{17}$$

recall (10). Moreover, we have

$$|L'(t)| \sim n\delta_n(L(t)) \tag{18}$$

uniformly in ω and t such that

$$\omega - |L(t)| \geq \frac{\omega}{n^2}. \tag{19}$$

(II) Transform the $\{\theta_j\}$ into $\{t_j\}$.

Since L is strictly increasing on $[-\pi, 0]$, it has a strictly increasing inverse $L^{[-1]}$ that maps $[-\omega, \omega]$ onto $[0, \pi]$. So given $\{\theta_j\}$ as in (12), we can define

$$t_j = L^{[-1]}(\theta_j) \Leftrightarrow \theta_j = L(t_j).$$

We shall frequently use the fact that given $C_1 \geq 1$, there exists $C_2 > 0$ such that

$$|\theta - \phi| \leq C_1\delta_n(\theta) \Rightarrow \frac{1}{C_2} \leq \frac{\delta_n(\theta)}{\delta_n(\phi)} \leq C_2. \tag{20}$$

For a proof of this, see [5, p. 12], and apply the inequality there several times. This and (18) also show that L' does not grow by faster than a constant multiple in correspondingly small intervals. Using the mean value theorem, we see that for some ξ between t_j and t_{j+1},

$$\begin{aligned} \theta_{j+1} - \theta_j &= L'(\xi)(t_{j+1} - t_j) \\ &\leq Cn\delta_n(\theta_j)(t_{j+1} - t_j), \end{aligned} \tag{21}$$

and moreover,

$$\theta_{j+1} - \theta_j \sim L'(t_j)(t_{j+1} - t_j) \sim n\delta_n(\theta_j)(t_{j+1} - t_j) \tag{22}$$

uniformly in j (and m, n, α, β) such that (20) holds for $t = t_j$. From our spacing restriction (13) on the $\{\theta_j\}$, we deduce that uniformly in j (and $m, n, ...$),

$$t_{j+1} - t_j \leq \frac{C}{n}. \tag{23}$$

(One needs a minor modification to this argument for t_j close to $\pm\omega$, violating (19)).

(III) **The relation between W_{δ_n} and $W_{n,\omega}$**

Let us define, as did Erdelyi,

$$W_\omega(t) = W(L(t))$$

and

$$W_{\omega,n}(t) = n\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} W_\omega.$$

Erdelyi notes that $W_{n,\omega}$ is a doubling weight with constant independent of n, depending only on the doubling constant of $W(\omega\cos t)$. Moreover, because of the spacing (23) on the $\{t_j\}$, we have uniformly in j

$$W_{\omega,n}(t) \sim W_{\omega,n}(t_j), t \in [t_j, t_{j+1}].$$

Next, from (11),

$$W_{\delta_n}(\theta_j) = \frac{2}{\delta_n(\theta)} \int_{L^{[-1]}(\theta_j - \delta_n(\theta_j))}^{L^{[-1]}(\theta_j + \delta_n(\theta_j))} W(L(t)) L'(t)\, dt.$$

Here

$$L^{[-1]}(\theta_j \pm \delta_n(\theta_j)) = L^{[-1]}(\theta_j) \pm \delta_n(\theta_j) \frac{dL^{[-1]}}{d\theta}(\xi) = t_j \pm \frac{\delta_n(\theta_j)}{L'(L^{[-1]}(\xi))},$$

where ξ is between θ_j and $\theta_j \pm \delta_n(\theta_j)$. Using (18), (20) and (22), we see that

$$L^{[-1]}(\theta_j \pm \delta_n(\theta_j)) = t_j + O\left(\frac{1}{n}\right),$$

uniformly in j. From (18) and (20) and the doubling properties, we obtain

$$W_{\delta_n}(\theta_j) \sim n \int_{t_{j-\frac{1}{n}}}^{t_j+\frac{1}{n}} W_\omega = W_{n,\omega}(t_j) \sim W_{n,\omega}(t), t \in [t_j, t_{j+1}]. \tag{24}$$

We are now ready for the

Proof of Theorem 2

As in the proof of Theorem 1, we obtain

$$\begin{aligned} |P(\theta_j)|^p &= |P(L(t_j))|^p \\ &\leq \min_{[t_j, t_{j+1}]} |P \circ L|^p + \int_{t_j}^{t_{j+1}} p\,|P \circ L|^{p-1} |P' \circ L|\, L'. \end{aligned}$$

Now we use the spacing (13), (20), (22), and the fact that $W_{n,\omega}$, δ_n, and L' do not change much in small intervals (in the form (18), (20), (24)) to deduce that

$$\begin{aligned} & W_{\delta_n}(\theta_j)\, |P(\theta_j)|^p (\theta_{j+1} - \theta_j) \\ \leq\ & C \int_{t_j}^{t_{j+1}} |P(L(t))|^p\, W_{n,\omega}(t)\, L'(t)\, dt + C \int_{t_j}^{t_{j+1}} |P \circ L|^{p-1} |(P'\delta_n) \circ L|\, W_{n,\omega} L', \end{aligned}$$

with C independent of $P, j, n, m, \ldots$. Add over j :

$$\begin{aligned} & \sum_{j=0}^{m} W_{\delta_n}(\theta_j)\, |P(\theta_j)|^p (\theta_{j+1} - \theta_j) \\ \leq\ & C \int_{-\pi}^{\pi} |P(L(t))|^p\, W_{n,\omega}(t)\, L'(t)\, dt + C \int_{-\pi}^{\pi} |P \circ L|^p |(P'\delta_n) \circ L|\, W_{n,\omega} L' \\ \leq\ & C \int_{-\pi}^{\pi} |P(L(t))|^p\, W_{n,\omega}(t)\, L'(t)\, dt \\ & + C \left(\int_{-\pi}^{\pi} |P \circ L|^p\, W_{n,\omega} L' \right)^{1-\frac{1}{p}} \left(\int_{-\pi}^{\pi} |(P'\delta_n) \circ L|^p\, W_{n,\omega} L' \right)^{1/p}, \end{aligned}$$

by Hölder's inequality. Using Erdelyi's Bernstein inequality in the form (2.9) in [1, p. 334], with appropriate changes of notation, we have

$$\int_{-\pi}^{\pi} |(P'\delta_n) \circ L|^p W_{n,\omega} L' \leq C \left(\int_{-\pi}^{\pi} |P \circ L|^p W_{n,\omega} L' \right)$$

and hence we have shown that

$$\sum_{j=0}^{m} W_{\delta_n}(\theta_j) |P(\theta_j)|^p (\theta_{j+1} - \theta_j) \leq C \int_{-\pi}^{\pi} |P(L(t))|^p W_{n,\omega}(t) L'(t)\, dt.$$

Using Theorem 2.1 in [1, p. 331], we can replace $W_{n,\omega}$ in the last right-hand side by W_ω, so we can continue this as

$$\begin{aligned} \sum_{j=0}^{m} W_{\delta_n}(\theta_j) |P(\theta_j)|^p (\theta_{j+1} - \theta_j) &\leq C \int_{-\pi}^{\pi} |P(L(t))|^p W_\omega(t) L'(t)\, dt \\ &= C \int_{-\omega}^{\omega} |P|^p W. \end{aligned}$$

■

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