Quadrature Sums and Lagrange Interpolation for General Exponential Weights

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Abstract. We obtain forward and converse quadrature sum estimates associated with zeros of orthogonal polynomials for general exponential weights. These are then applied to establish mean convergence of Lagrange interpolation at zeros of these orthogonal polynomials. The results generalize earlier ones for even weights on $(-1,1)$ or $\mathbb{R}$.

Running Title: Quadrature Sums and Lagrange Interpolation

1. Introduction and Results

The theory of orthogonal polynomials and approximation theory for exponential weights on a real interval began to develop in the 1960's and 1970's under the leadership of G. Freud and P. Nevai. They typically considered weights such as

$$W(x) := \exp\left(-|x|^{\alpha}\right), x \in \mathbb{R},$$

where $\alpha > 1$. With the introduction of potential theory in the 1980's, there were major advances in understanding the asymptotics of associated orthogonal polynomials. Potential theory afforded the opportunity to consider not only weights on the whole real line, but also weights such as

$$W(x) := \exp\left(-\left(1-x^2\right)^{-\alpha}\right), x \in (-1,1),$$

where $\alpha > 0$. Once the theory had been developed in its entirety, it became clear that one could simultaneously treat not only weights like those above, but also not necessarily even weights on a general real interval. See [3], [12], [16] for various perspectives on this type of potential theory and its applications.

One important application is to Lagrange interpolation. Mean convergence of Lagrange interpolation at zeros of orthogonal polynomials has been thoroughly investigated for even exponential weights - see, for example, the surveys [7], [11], [15], [18].

In this paper, we shall extend many of those results by also considering non-even weights on a real interval

$$I = (c,d) \text{ where } -\infty \leq c < 0 < d \leq \infty. \tag{1}$$

This is made possible by the results in a recently published monograph [4].

Before we define our class of weights, we need the notion of a quasi-increasing function. A function $g : (0,b) \to (0,\infty)$ is said to be *quasi-increasing* if there exists $C > 0$ such that

$$g(x) \leq Cg(y), 0 < x \leq y < b.$$

Of course, any increasing function is quasi-increasing. Similarly we may define the notion of a *quasi-decreasing* function. The notation

$$f(x) \sim g(x)$$

means that there are positive constants C_1, C_2 such that for the relevant range of x,

$$C_1 \leq f(x)/g(x) \leq C_2.$$

Similar notation is used for sequences and sequences of functions.

Definition 1.1 General Exponential Weights
Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$ satisfies the following properties:
(a) Q' is continuous in I and $Q(0) = 0$;
(b) Q'' exists and is positive in $I \backslash \{0\}$;
(c)

$$\lim_{t \to c+} Q(t) = \lim_{t \to d-} Q(t) = \infty;$$

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, t \neq 0$$

is quasi-increasing in $(0, d)$, and quasi-decreasing in $(c, 0)$, with

$$T(t) \geq \Lambda > 1, t \in I \backslash \{0\};$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{\mid Q'(x) \mid} \leq C_1 \frac{\mid Q'(x) \mid}{Q(x)}, \text{ a.e. } x \in I \backslash \{0\};$$

(f) There exists a compact subinterval J of the open interval I, and $C_2 > 0$ such that

$$\frac{Q''(x)}{\mid Q'(x) \mid} \geq C_2 \frac{\mid Q'(x) \mid}{Q(x)}, \text{ a.e. } x \in I \backslash J.$$

Then we write $W \in \mathcal{F}\left(C^2+\right)$.

The simplest case of the above definition is when $I = \mathbb{R}$ and

$$T \sim 1 \text{ in } \mathbb{R}.$$

This is the so called Freud case, for the last condition forces Q to be of at most polynomial growth. Moreover, T is then automatically quasi-increasing in $(0, d)$. A typical example is

$$Q(x) = Q_{\alpha,\beta}(x) = \begin{cases} x^\alpha, & x \in [0, \infty) \\ |x|^\beta, & x \in (-\infty, 0) \end{cases}$$

where $\alpha, \beta > 1$. For this choice, we see that

$$T(x) = \begin{cases} \alpha, & x \in (0, \infty) \\ \beta, & x \in (-\infty, 0) \end{cases}.$$

A more general example satisfying the above conditions is

$$Q(x) = Q_{\ell,k,\alpha,\beta}(x) = \begin{cases} \exp_\ell(x^\alpha) - \exp_\ell(0), & x \in [0, \infty) \\ \exp_k(|x|^\beta) - \exp_k(0), & x \in (-\infty, 0) \end{cases}$$

where $\alpha, \beta > 1$ and $k, \ell \geq 0$. Here we set $\exp_0(x) := x$ and for $\ell \geq 1$,

$$\exp_\ell(x) = \underbrace{\exp(\exp(\exp ... \exp(x))))}_{\ell \text{ times}}$$

is the ℓth iterated exponential.

An example on the finite interval $I = (-1, 1)$ is

$$Q(x) = Q^{(\ell,k,\alpha,\beta)}(x) = \begin{cases} \exp_\ell((1-x^2)^{-\alpha}) - \exp_\ell(1), & x \in [0, 1) \\ \exp_k((1-x^2)^{-\beta}) - \exp_k(1), & x \in (-1, 0) \end{cases}$$

where $\alpha, \beta > 0$ and $k, \ell \geq 0$.

Associated with the weight W^2 (note that we write the weight as a square), we can define orthonormal polynomials

$$p_n(x) = p_n(W^2, x) = \gamma_n x^n + ..., \gamma_n > 0,$$

satisfying

$$\int_I p_n p_m W^2 = \delta_{mn}.$$

We denote the zeros of p_n by

$$c < x_{nn} < x_{n-1,n} < ... < x_{1n} < d.$$

The Lagrange interpolation polynomial to a function $f : I \to \mathbb{R}$ at $\{x_{jn}\}_{j=1}^n$ is denoted by $L_n[f]$. Thus, if $\mathcal{P}_n$ denotes the polynomials of degree $\leq n$, then $L_n[f] \in \mathcal{P}_{n-1}$ satisfies

$$L_n[f](x_{jn}) = f(x_{jn}), 1 \leq j \leq n.$$

The Gauss quadrature rule for W^2 has the form

$$\int_I PW^2 = \sum_{j=1}^n \lambda_{jn} P(x_{jn}), P \in \mathcal{P}_{2n-1},$$

where the Christoffel numbers λ_{jn} are positive.

In analysis of exponential weights, an important role is played by the Mhaskar-Rakhmanov-Saff numbers $a_{\pm u}$, which for $u \in (0, \infty)$ satisfy

$$c < a_{-u} < 0 < a_u < d$$

and are the unique roots of the equations

$$u = \frac{1}{\pi} \int_{a_{-u}}^{a_u} \frac{xQ'(x)}{\sqrt{(x - a_{-u})(a_u - x)}} dx;$$

$$0 = \frac{1}{\pi} \int_{a_{-u}}^{a_u} \frac{Q'(x)}{\sqrt{(x - a_{-u})(a_u - x)}} dx.$$

It is not obvious that $a_{\pm u}$ exist or are uniquely defined, but this follows from potential theory for external fields [3], [4], [16]. Moreover, it is known that

$$\lim_{u \to \infty} a_{-u} = c; \lim_{u \to \infty} a_u = d.$$

In the special case where Q is even, the uniqueness of $a_{\pm u}$ forces

$$a_{-u} = -a_u, u > 0.$$

One of the features that motivates their importance is the Mhaskar-Saff identity [10]

$$\| PW \|_{L_\infty(I)} = \| PW \|_{L_\infty[a_{-n},a_n]}, P \in \mathcal{P}_n.$$

Another is that they describe how the smallest and largest zeros x_{nn}, x_{1n} of p_n behave. For $u > 0$, let

$$\delta_u := \frac{1}{2}\left(a_u + |a_{-u}|\right),$$

and

$$\eta_{\pm u} = \left(uT\left(a_{\pm u}\right)\sqrt{\frac{|a_{\pm u}|}{\delta_u}} \right)^{-2/3}. \tag{2}$$

Then [4]

$$\begin{aligned} 1 - \frac{x_{1n}}{a_n} &\sim \eta_n \to 0, n \to \infty; \\ 1 - \frac{x_{nn}}{a_{-n}} &\sim \eta_{-n} \to 0, n \to \infty. \end{aligned}$$

The reader will recall that in approximation theory for the interval $[-1,1]$, for example in Jackson-Bernstein theorems and Markov-Bernstein inequalities, an important role is played by the function

$$\frac{\sqrt{1-x^2}}{n} + n^{-2}, x \in [-1,1].$$

As an analogue of the latter, but with a different scaling, we shall use

$$h_n(x) := \left(|x - a_{-n}| + |a_{-n}|\,\eta_{-n}\right)\left(|x - a_n| + a_n\eta_n\right), x \in I. \tag{3}$$

We can now state our main result, which provides forward and converse quadrature sum estimates for weighted polynomials:

Theorem 1.2
Let $W \in \mathcal{F}\left(C^2+\right)$ and $1 < p < \infty$.
(I) Let

$$\frac{1}{4} - \frac{1}{p} < \Delta < \frac{5}{4} - \frac{1}{p}. \tag{4}$$

Then for $n \geq 1$ and $P \in \mathcal{P}_{n-1}$,

$$\| PWh_n^\Delta \|_{L_p(I)} \leq C \left(\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left|PWh_n^\Delta\right|^p (x_{kn}) \right)^{1/p}. \tag{5}$$

Here C is independent of P and n.
(II) Let $\Delta \in \mathbb{R}$. Then

$$\left(\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left|PWh_n^\Delta\right|^p (x_{kn}) \right)^{1/p} \leq C \| PWh_n^\Delta \|_{L_p(I)}. \tag{6}$$

Here C is independent of P and n.

The upper bound on Δ in (4) is possibly not sharp, but this is largely irrelevant to this paper: it is the lower bound on Δ in (4), which is sharp. We note that if we define for some small enough (but fixed) $\varepsilon > 0$

$$x_{0n} := x_{1n}(1 + \varepsilon\eta_n); x_{n+1,n} := x_{nn}\left(1 + \varepsilon\eta_{-n}\right), \tag{7}$$

then uniformly in j and n,

$$\lambda_{jn} W^{-2}(x_{jn}) \sim x_{j-1,n} - x_{jn}$$

while still

$$a_{-n} < x_{n+1,n} < x_{nn} < ... < x_{1n} < x_{0n} < a_n$$

so one could replace the weighted Christoffel numbers by the spacing between successive zeros.

For Freud weights, more precise results are possible, and one may replace the factor h_n by a fixed power of $1+|x|$ independent of n [9]. However, in the general case above, the factor h_n seems to be natural.

Following is our second result, which helps to justify part of the restriction on Δ in Theorem 1.2.

Theorem 1.3
Let $W \in \mathcal{F}(C^2+)$, $1 < p < \infty$ and $\Delta \in \mathbb{R}$. The following are equivalent:
(a) There exists C independent of f and n such that for $n \geq 1$, and measurable $f : I \to \mathbb{R}$,

$$\| L_n[f] W h_n^{\Delta} \|_{L_p(I)} / \delta_n^{2\Delta + \frac{1}{p}} \leq C \| fW \|_{L_\infty(I)} . \tag{8}$$

(b)

$$\Delta > \frac{1}{4} - \frac{1}{p}. \tag{9}$$

The disadvantage of the above result is that the weighting factor $h_n^{\Delta}/\delta_n^{2\Delta+\frac{1}{p}}$ in the left-hand side of (8) depends on n. In analogous questions for generalized Jacobi weights on $[-1, 1]$, one can effectively take $h_n(x) = 1 - |x|$, but not here. To avoid weighting factors that depend on n, we consider separately $p < 4$ and $p \geq 4$: for the former case, we do not really need a weighting factor.

Theorem 1.4
Let $W \in \mathcal{F}(C^2+)$ and $1 < p < 4$. Let $f : I \to \mathbb{R}$ be Riemann integrable in each compact subinterval of I. Assume moreover, that if $d = \infty$, we have for some $\alpha > \frac{1}{p}$,

$$\lim_{x \to \infty} (fW)(x) (1 + |x|)^{\alpha} = 0, \tag{10}$$

while if $d < \infty$, for some $\alpha < \frac{1}{p}$,

$$\lim_{x \to d-} (fW)(x) (d - x)^{\alpha} = 0. \tag{11}$$

Assume analogous behaviour at c. Then

$$\lim_{n \to \infty} \| (L_n[f] - f) W \|_{L_p(I)} = 0. \tag{12}$$

For $p \geq 4$, the asymmetry of the weight plays a far greater role. We begin with the case where the asymmetry is not severe:

Theorem 1.5
Let $W \in \mathcal{F}(C^2+)$, $p \geq 4, \Delta \in \mathbb{R}$. Assume moreover, that

$$a_n \sim |a_{-n}|, n \geq 1. \tag{13}$$

Let

$$\Delta > \frac{1}{4} - \frac{1}{p}. \tag{14}$$

Let $f : I \to \mathbb{R}$ be Riemann integrable in each compact subinterval of I. Assume that if $d = \infty$, (10) holds with some $\alpha > \frac{1}{p}$, while if $d < \infty$, (11) holds with some $\alpha < \frac{1}{p}$. Assume, moreover, analogous behaviour at c. Then

$$\lim_{n\to\infty} \| \left(L_n[f] - f\right) W \left[1 + Q^{2/3}T\right]^{-\Delta} \|_{L_p(I)} = 0. \tag{15}$$

We note that the weighting factor $1 + Q^{2/3}T$ is exactly the same as that used in [5] for even exponential weights on $[-1, 1]$, and Theorem 1.5 is an extensive generalisation of the sufficiency part of Theorem 1.5 from [5]. There it was also shown how necessary is the factor $1 + Q^{2/3}T$, and that $\Delta \geq \frac{1}{4} - \frac{1}{p}$ is necessary for (15), with strict inequality if $p = 4$. We are certain that the necessity extends to this case.

In the case that I is a bounded interval, (13) is satisfied trivially, since

$$|a_{\pm n}| \sim 1, n \geq 1.$$

This relation is also satisfied if $I = \mathbb{R}$ and the growth of Q on the positive and negative real axis is of similar order. Next, we formulate a result for $p \geq 4$ and the general asymmetric case:

Theorem 1.6
Let $W \in \mathcal{F}\left(C^2+\right)$, $p \geq 4, \Delta \in \mathbb{R}$. Let

$$\Delta > \frac{1}{4} - \frac{1}{p}.$$

Let $f : I \to \mathbb{R}$ be Riemann integrable in each compact subinterval of I. Assume that if $d = \infty$, (10) holds with some $\alpha > \Delta + \frac{1}{p}$, while if $d < \infty$, (11) holds with some $\alpha < \frac{1}{p}$. Assume moreover, analogous behaviour at c. Then

$$\lim_{n\to\infty} \| \left(L_n[f] - f\right) W \left[1 + Q^{2/3}T\right]^{-\Delta} \|_{L_p(I)} = 0.$$

We see that in Theorem 1.6, the extra restriction is the more severe bound on α if d (or c) is infinite. We could relax this, but then seem to need to replace $1 + Q^{2/3}T$ by a more implicit factor that reflects the asymmetry of the weight.

This paper is organised as follows: in Section 2, we state extra notation, and state some technical lemmas. In Section 3, we prove a restricted range inequality and a Markov-Bernstein inequality building on those of [4]. In Section 4, we prove Theorem 1.2(I), and in Section 5, we prove Theorem 1.2(II). Then we prove the remaining results in Section 6.

2. Technical Estimates

Let us begin by introducing more notation. Throughout, $C, C_1, C_2, ...$ denote positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda), C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. We let

$$\delta_n := \frac{1}{2}\left(a_n + |a_{-n}|\right); \beta_n := \frac{1}{2}\left(a_n + a_{-n}\right)$$

so that

$$[a_{-n}, a_n] = [\beta_n - \delta_n, \beta_n + \delta_n].$$

For $s \geq 0$, we also set

$$J_n(s) := [a_{-n}\left(1 - s\eta_{-n}\right), a_n(1 - s\eta_n)],$$

where $\eta_{\pm n}$ are defined by (2). Given any fixed such s, we note that $J_n(s)$ is non-empty for n large enough. We let

$$L_n(x) := \frac{x - \beta_n}{\delta_n}$$

denote the linear map of $[a_{-n}, a_n]$ onto $[-1, 1]$, and let

$$L_n^{[-1]}(t) := \beta_n + \delta_n t$$

denote the inverse map. We let x_{0n} and $x_{n+1,n}$ be defined by (7). It will also be useful to have the numbers

$$\eta^*_{\pm n} := \frac{|a_{\pm n}|}{\delta_n}\eta_{\pm n} = \frac{|a_{\pm n}|}{\delta_n}\left(nT(a_{\pm n})\sqrt{\frac{|a_{\pm n}|}{\delta_n}}\right)^{-2/3}. \tag{16}$$

In describing spacing of zeros and related quantities, the function

$$\phi_n(x) := \frac{|x - a_{-2n}|\,|x - a_{2n}|}{n\sqrt{\left(|x - a_{-n}| + |a_{-n}|\,\eta_{-n}\right)\left(|x - a_n| + a_n\eta_n\right)}}, x \in I \tag{17}$$

plays an important role.

The Lagrange interpolation polynomial $L_n[f]$ admits the representation

$$L_n[f] = \sum_{j=1}^{n} f(x_{jn})\ell_{jn}(x)$$

where the *fundamental polynomials* ℓ_{jn} in turn admit the representation

$$\ell_{jn}(x) = \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})}.$$

In the sequel, we assume that $W \in \mathcal{F}\left(C^2+\right)$ without further mention. First we record all our estimates relating specifically to orthogonal polynomials:

Lemma 2.1
(a) There exists n_0 such that for $n \geq n_0$,

$$1 - \frac{x_{1n}}{a_n} \sim \eta_n; 1 - \frac{x_{nn}}{a_{-n}} \sim \eta_{-n}. \tag{18}$$

(b) Uniformly for $n \geq 1$ and $1 \leq j \leq n$, and $x \in [x_{j+1,n}, x_{j-1,n}]$,

$$h_n(x) \sim h_n(x_{j,n}); \phi_n(x) \sim \phi_n(x_{jn}); \tag{19}$$

and

$$1 + |x| \sim 1 + |x_{jn}|; |a_{\pm n} - x| \sim |a_{\pm n} - x_{jn}|. \tag{20}$$

(c) Uniformly for $n \geq 1$ and $1 \leq j \leq n$,

$$\lambda_{jn}W^{-2}(x_{jn}) \sim |x_{j\pm 1,n} - x_{jn}| \sim \phi_n(x_{jn}). \tag{21}$$

(d) Uniformly for $n \geq 1$ and $1 \leq j \leq n$,

$$\frac{1}{|p_n' W|(x_{jn})} \sim (x_{jn} - x_{j+1,n}) h_n(x_{jn})^{1/4}. \tag{22}$$

(e) Uniformly for $n \geq 1$ and $1 \leq j \leq n$ and $x \in I$,

$$|\ell_{jn}(x)| \, W^{-1}(x_{jn}) W(x) \sim (x_{jn} - x_{j+1,n}) h_n(x_{jn})^{1/4} \left| \frac{p_n(x) W(x)}{x - x_{jn}} \right|. \tag{23}$$

(f) Uniformly for $n \geq 1$ and $1 \leq j \leq n$ and $x \in I$,

$$|\ell_{jn}(x)| \, W^{-1}(x_{jn}) W(x) \leq C. \tag{24}$$

(g) Uniformly for $n \geq 1$ and $1 \leq j \leq n-1$ and $x \in [x_{j+1,n}, x_{jn}]$,

$$\ell_{jn}(x) W^{-1}(x_{jn}) W(x) + \ell_{j+1,n}(x) W^{-1}(x_{j+1,n}) W(x) \sim 1. \tag{25}$$

(h) Uniformly for $n \geq 1$ and $x \in I$,

$$|p_n W|(x) \leq C h_n(x)^{-1/4}. \tag{26}$$

(i) Uniformly for $n \geq 1$ and $1 \leq j \leq n-1$ and $x \in (x_{j+1,n}, x_{jn})$,

$$|p_n W|(x) \sim \frac{h_n(x_{jn})^{-1/4}}{x_{jn} - x_{j+1,n}} \min\{|x - x_{jn}|, |x - x_{j+1,n}|\}. \tag{27}$$

Proof
(a) This is Theorem 1.19(f) in [4, p.23].
(b) The relation

$$\phi_n(x) \sim \phi_n(x_{jn})$$

follows from Theorem 5.7(I)(b) in [4, pp. 125-126], in view of the spacing between successive zeros given in (c). In the course of the proof there, it is also effectively shown that

$$h_n(x) \sim h_n(x_{jn}) ; |a_{\pm n} - x| \sim |a_{\pm n} - x_{jn}|.$$

The proof that $1 + |x| \sim 1 + |x_{jn}|$ is somewhat easier.
(c) This follows from Corollary 1.14(a) in [4, p. 20] and Theorem 1.19(e) in [4, p. 23] and also (b) above.
(d) This is Theorem 1.19(a) in [4, p. 22].
(e) This is a consequence of (d) and the formula for ℓ_{jn}.
(f), (g) These are Theorem 13.3 in [4, p. 361].
(h) This follows from Theorems 1.17 and 1.18 in [4, p. 22].
(i) This is Theorem 1.19(d) in [4, p. 23], combined with (c) above. □

Next we record estimates involving Q and a_u.

Lemma 2.2
(a) For $u > 0$,

$$Q(a_{\pm u}) \sim u \sqrt{\frac{|a_{\pm u}|}{\delta_u T(a_{\pm u})}}; \tag{28}$$

$$Q'(a_{\pm u}) \sim u \sqrt{\frac{T(a_{\pm u})}{|a_{\pm u}| \, \delta_u}}. \tag{29}$$

(b) Let $\alpha, \beta > 0$. Then uniformly for $j = 0, 1$, and $u > 0$,

$$T(a_{\alpha u}) \sim T(a_{\beta u}); Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}); \eta_{\alpha u} \sim \eta_{\beta u}. \tag{30}$$

(c) There exist $C, \varepsilon > 0$ such that for $n \geq 1$,

$$\frac{\delta_n T(a_n)}{a_n n^2} \leq C n^{-\varepsilon} \tag{31}$$

and

$$T(a_n)\eta_n \leq C n^{-\varepsilon}. \tag{32}$$

(d) There exists $C > 0$ such that for $\frac{1}{2} \leq \frac{u}{v} \leq 2$,

$$\left|1 - \frac{a_u}{a_v}\right| \sim \frac{1}{T(a_u)}\left|1 - \frac{u}{v}\right|. \tag{33}$$

Moreover, if $\alpha > 0$, there exists $C > 0$ such that for $u \geq C$,

$$\left|1 - \frac{a_{\alpha u}}{a_u}\right| \sim \frac{1}{T(a_u)}. \tag{34}$$

Proof
(a) This is part of Lemma 3.4 in [4, p. 69].
(b) The first two $\sim$ relations are part of Lemma 3.5(b) in [4, p. 72]. The third $\sim$ relation follows easily from the first two.
(c) This is Lemma 3.7 in [4, p.76].
(d) This is part of Lemma 3.11 in [4, p.81]. $\square$

Next, we record a restricted range inequality and a Markov-Bernstein inequality:

Lemma 2.3
Let $0 < p \leq \infty$ and $s > 0$.
(a) There exist C, n_0 such that for $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$\| PW \|_{L_p(I)} \leq C \| PW \|_{L_p\left(a_{-n}\left(1 - s\eta_{-n}\right), a_n(1 - s\eta_n)\right)}. \tag{35}$$

(b) For $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\| (PW)' \phi_n \|_{L_p(I)} \leq C \| PW \|_{L_p(I)}. \tag{36}$$

Proof
(a) This is Theorem 1.9(a) in [4, p. 15].
(b) This is Theorem 1.15 in [4, p. 21]. $\square$

Next, we record a lower bound for integrals involving the orthogonal polynomials p_n:

Lemma 2.4
Let $0 < p < \infty, 0 < A < B < \infty$. Let $\xi : I \to (0, \infty)$ be a function with the following property: uniformly for $n \geq 1, 1 \leq j \leq n$,

$$A \leq \frac{\xi(x)}{\xi(x_{jn})} \leq B, x \in [x_{j+1,n}, x_{jn}]. \tag{37}$$

For $n \geq 1$, let $\mathcal{I}_n$ be a subinterval of (x_{nn}, x_{1n}) containing at least two zeros of p_n. Then

$$\| p_n W \xi \|_{L_p(\mathcal{I}_n)} \geq C \| h_n^{-1/4} \xi \|_{L_p(\mathcal{I}_n)}. \tag{38}$$

The constant C is independent of $n, \mathcal{I}_n, \xi$ but depends on A, B in (37).
Proof
We note first that if $1 \leq j \leq n-1$, Lemma 2.1(i) and (37) give

$$\int_{x_{j+1,n}}^{x_{jn}} |p_n W \xi|^p \sim \left(\frac{h_n(x_{jn})^{-1/4}}{x_{jn} - x_{j+1,n}} \right)^p \xi(x_{jn})^p \int_{x_{j+1,n}}^{x_{jn}} \min\{|x - x_{jn}|, |x - x_{j+1,n}|\}^p dx$$

$$\sim h_n(x_{jn})^{-p/4} \xi(x_{jn})^p (x_{jn} - x_{j+1,n}) \sim \int_{x_{j+1,n}}^{x_{jn}} h_n^{-p/4} \xi^p$$

by Lemma 2.1(b) and (37). Adding over those j for which $[x_{j+1,n}, x_{jn}] \subset \mathcal{I}_n$ gives the result: note that terms over adjacent intervals are of the same size up to $\sim$. Thus if the endpoints of I_n do not coincide with zeros of p_n, the small intervals around these endpoints are of the same size as an adjacent $[x_{j+1,n}, x_{jn}] \subset \mathcal{I}_n$. Of course, as $\mathcal{I}_n$ contains at least two zeros, there is such an adjacent interval.□

Our final technical lemma concerns the size of ϕ_n for different n:

Lemma 2.5
Let $A > 0$. For $n \geq 1$, let

$$m := m(n) \leq A/\sqrt{\eta_n^*} \tag{39}$$

and let

$$\ell := \ell(n) := n + m.$$

Then uniformly in n and $x \in K_n := [\beta_n, a_\ell]$, we have

$$\phi_n(x) \sim \phi_\ell(x); \tag{40}$$
$$h_n(x) \sim h_\ell(x) \tag{41}$$

Proof
Note first that from Lemma 2.2(c), and the definition (16) of η_n^*,

$$m/n \leq C \left(\frac{\delta_n T(a_n)}{a_n n^2} \right)^{1/3} \to 0, n \to \infty.$$

Then Lemma 2.2(d) shows that

$$\begin{aligned} & |a_\ell / a_n - 1| \\ = \; & O\left(\frac{m}{T(a_n) n} \right) = O\left(\frac{1}{nT(a_n)} \sqrt{\frac{\delta_n}{a_n}} \right)^{2/3} = O(\eta_n). \end{aligned} \tag{42}$$

Similarly,

$$|a_{-\ell}/a_{-n} - 1| \to 0, n \to \infty.$$

Then for n large enough and $x \in K_n$, we have

$$\begin{aligned} |x - a_{-2\ell}| &\sim |x - a_{-2n}| \sim \delta_n; \\ |x - a_{-\ell}| + |a_{-\ell}| \eta_{-\ell} &\sim |x - a_{-n}| + |a_{-n}| \eta_{-n} \sim \delta_n. \end{aligned} \tag{43}$$

Recall the definition of ϕ_n at (17). We see that

$$\frac{\phi_n(x)}{\phi_\ell(x)} \sim \left| \frac{x - a_{2n}}{x - a_{2\ell}} \right| \frac{\sqrt{|x - a_\ell| + a_\ell \eta_\ell}}{\sqrt{|x - a_n| + a_n \eta_n}}. \tag{44}$$

Here as at (42), Lemma 2.2(d) gives uniformly for $x \in K_n$,

$$\begin{aligned} \left|\frac{x-a_{2n}}{x-a_{2\ell}}-1\right| &= \left|\frac{a_{2\ell}-a_{2n}}{x-a_{2\ell}}\right| \\ &\leq C\frac{a_n\eta_n}{a_{2\ell}-a_\ell} \\ &\leq C\eta_n T\left(a_n\right)=o\left(1\right). \end{aligned}$$

Here we used (34) in the second last line, and then we used (32). Next,

$$\left|\frac{|x-a_\ell|+a_\ell\eta_\ell}{|x-a_n|+a_n\eta_n}-1\right| \leq \frac{|a_n-a_\ell|+a_\ell\eta_\ell+a_n\eta_n}{a_n\eta_n} \leq C,$$

by (42). A similar inequality holds if we reverse the roles of the numerator and denominator in the left-hand side of this last line. Then (40) of the lemma follows from (44) and these last two steps. In a somewhat easier manner, since

$$\frac{h_n\left(x\right)}{h_\ell\left(x\right)} \sim \frac{|x-a_n|+a_n\eta_n}{|x-a_\ell|+a_\ell\eta_\ell},$$

we also obtain (41). $\square$

3. Two Inequalities

In this section, we shall slightly extend a restricted range inequality, and Markov-Bernstein inequality from [4], by inserting a power of h_n into the weight. First we state the restricted range inequality, which involves the interval

$$J_n\left(s\right) := \left[a_{-n}\left(1-s\eta_{-n}\right), a_n\left(1-s\eta_n\right)\right], s \geq 0.$$

For a given s, this will be non-empty for large enough n.

Lemma 3.1
Let $0<p\leq\infty$ and $\Delta\in\mathbb{R}$. Let $s>0$. Then there exists n_0 such that for $n\geq n_0$ and $P\in\mathcal{P}_n$,

$$\| PWh_n^{\Delta} \|_{L_p(I)} \leq C \| PWh_n^{\Delta} \|_{L_p(J_n(s))}. \tag{45}$$

Next, we state our Markov-Bernstein inequality:

Lemma 3.2
Let $0<p\leq\infty$ and $\Delta\in\mathbb{R}$. Then for $n\geq 1$ and $P\in\mathcal{P}_n$,

$$\| \left(PW\right)' h_n^{\Delta}\phi_n \|_{L_p[a_{-n},a_n]} \leq C \| PWh_n^{\Delta} \|_{L_p(I)}. \tag{46}$$

We first establish:

Proposition 3.3
Suppose that for each fixed positive integer A, and for each fixed non-negative integer B,

and for n large enough, we have polynomials S_m of degree $m = m(n) \sim 1/\sqrt{\eta_n^*}$ such that if

$$\begin{aligned} \ell &: = \ell(n) = n + Am(n) + B; \\ K_n &: = [\beta_n, a_\ell]; \end{aligned} \tag{47}$$

then
(i)

$$S_m \le C_1 h_n^\Delta \text{ in } [a_{-\ell}, a_\ell]; \tag{48}$$

(ii)

$$S_m \ge C_2 h_n^\Delta \text{ in } [\beta_n, \infty); \tag{49}$$

(iii)

$$|S_m' \phi_n| \le C_3 h_n^\Delta \text{ in } K_n. \tag{50}$$

Moreover, suppose that similar polynomials exist when we replace K_n by $[a_{-\ell}, \beta_n]$ *and so on. Then the conclusions of Lemma 3.1 and Lemma 3.2 follow.*

Proof

Step 1: The conclusion of Lemma 3.1 follows

Let $t > 0$. We have from (ii),

$$\begin{aligned} \| \quad & PWh_n^\Delta \|_{L_p[\beta_n, d)} \le C_2^{-1} \| PS_m W \|_{L_p[\beta_n, d)} \\ \le \quad & C_2^{-1} \| PS_m W \|_{L_p(I)} . \end{aligned}$$

Using our restricted range inequality Lemma 2.3(a), and the fact that PS_m has degree $n + m(n) \le \ell$, we continue this as

$$\begin{aligned} \le \quad & C_2^{-1} C \| PS_m W \|_{L_p(J_\ell(t))} \\ \le \quad & C_2^{-1} C C_1 \| P h_n^\Delta W \|_{L_p(J_\ell(t))}, \end{aligned}$$

by (i). A similar inequality holds over the interval $(c, \beta_n]$ and then we obtain

$$\| PWh_n^\Delta \|_{L_p(I)} \le C \| PWh_n^\Delta \|_{L_p(J_\ell(t))} .$$

If we can show that given $s > 0$, there exists $t > 0$ and n_0 such that for $n \ge n_0$, we have

$$J_\ell(t) \subseteq J_n(s),$$

then we obtain (45). Let $s > 0$. We shall show that $\exists t > 0$ such that for large enough n,

$$a_\ell (1 - t\eta_\ell) \le a_n (1 - s\eta_n). \tag{51}$$

A similar inequality holds for $a_{-\ell}, a_{-n}$, and then the desired inclusion follows. Now

$$\left| \frac{a_\ell}{a_n} - 1 \right| \le \frac{Cm}{T(a_n) n} \le C\eta_n,$$

as at (42). Since $\eta_n \sim \eta_\ell$, we can find $t > 0$ for which (51) holds.

Step 2: The conclusion of Lemma 3.2 follows

We have from (ii) and then Lemma 2.5,

$$\begin{aligned} \| \quad & (PW)' h_n^\Delta \phi_n \|_{L_p[\beta_n, a_\ell]} \le C_2^{-1} \| (PW)' S_m \phi_n \|_{L_p[\beta_n, a_\ell]} \\ = \quad & C_2^{-1} \| \left[(PS_m W)' - PWS_m' \right] \phi_n \|_{L_p[\beta_n, a_\ell]} \\ \le \quad & C_3 \left(\| (PS_m W)' \phi_\ell \|_{L_p[\beta_n, a_\ell]} + \| PWS_m' \phi_n \|_{L_p[\beta_n, a_\ell]} \right) \\ \le \quad & C_4 \left(\| PS_m W \|_{L_p[a_{-\ell}, a_\ell]} + \| PWS_m' \phi_n \|_{L_p[\beta_n, a_\ell]} \right) \end{aligned}$$

by the Markov-Bernstein inequality and restricted range inequalities in Lemma 2.3. Using (i) and (iii) above we continue this as

$$\leq C_4 \parallel PWh_n^{\Delta} \parallel_{L_p(I)} .$$

A similar inequality holds over $[a_{-\ell}, \beta_n]$, so we deduce that

$$\parallel (PW)' \, h_n^{\Delta} \phi_n \parallel_{L_p[a_{-\ell}, a_\ell]} \leq C_5 \parallel PWh_n^{\Delta} \parallel_{L_p(I)} .$$

Since $[a_{-\ell}, a_\ell]$ contains $[a_{-n}, a_n]$, the result follows. $\square$

We now turn to the construction of the polynomials S_m. We first show that it suffices to consider $\Delta \in \left(-\frac{1}{2}, 0\right)$:

Proposition 3.4
It suffices to construct the polynomials S_m for $\Delta \in \left(-\frac{1}{2}, 0\right)$.
Proof
Step 1: Then we may construct the polynomials for all $\Delta \leq 0$
For $\Delta = 0$, we can choose $S_m \equiv 1$. Given $\Delta < 0$, we can write

$$\Delta = \Delta_1 r,$$

where $\Delta_1 \in \left(-\frac{1}{2}, 0\right)$ and r is a positive integer. Assume that we have polynomials S_{m,Δ_1} which satisfy the properties (i), (ii), (iii) in Proposition 3.3 with Δ replaced by Δ_1. We then set

$$S_{m,\Delta} := S_{m,\Delta_1}^r.$$

As r is fixed, $S_{m,\Delta}$ does have degree $\sim 1/\sqrt{\eta_n^*}$. Next, we see that both (i) and (ii) follow directly for Δ from that for Δ_1 if we replace A by Ar. (It is here that we need the parameter A in the definition (47) of ℓ). Finally, in K_n,

$$\begin{aligned} \left|S'_{m,\Delta} \phi_n\right| &= r \left|S'_{m,\Delta_1} \phi_n\right| \left|S_{m,\Delta_1}\right|^{r-1} \\ &\leq C h_n^{\Delta_1} h_n^{\Delta_1 (r-1)} = C h_n^{\Delta}, \end{aligned}$$

by (i), (iii) for S_{m,Δ_1}.
Step 2: Then we may construct the polynomials for all $\Delta > 0$
Given $\Delta > 0$, we may write

$$\Delta = \Delta_1 + 2r,$$

where r is a positive integer and $\Delta_1 \in (-2, 0)$. We set

$$f_n(x) := \left[\left((x - a_{-n})^2 + \left(a_{-n}\eta_{-n}\right)^2 \right) \left((x - a_n)^2 + \left(a_n \eta_n\right)^2 \right) \right]$$

and

$$S_{m,\Delta} := S_{m,\Delta_1} f_n^r,$$

a polynomial of degree equal to that of S_{m,Δ_1} plus $4r$. Then as r is fixed, the degree restrictions are satisfied. Since uniformly in $x \in \mathbb{R}$ and $n \geq 1$, we see that

$$f_n(x) \sim h_n(x)^2 ,$$

it is easy to see that (i), (ii) for $S_{m,\Delta}$ follow from those for S_{m,Δ_1}. Next, in K_n,

$$\begin{aligned} \left|S'_{m,\Delta}(x) \phi_n(x)\right| &\leq \left|S'_{m,\Delta_1}(x) \phi_n(x)\right| f_n(x)^r + r \left|S_{m,\Delta_1}(x) \phi_n(x)\right| f_n(x)^{r-1} \left|f'_n(x)\right| \\ &\leq C h_n(x)^{\Delta_1 + 2r} + C h_n(x)^{\Delta_1 + 2r} \phi_n(x) \left|f'_n(x) / f_n(x)\right| , \end{aligned}$$

by (iii) and (i) for S_{m,Δ_1}. (Recall that $K_n \subset [a_{-\ell}, a_\ell]$). If we can show that

$$\phi_n(x) \left| f_n'(x) / f_n(x) \right| \le C \text{ in } K_n,$$

then we obtain (iii) for Δ. Now we see that in K_n,

$$\begin{aligned} \left| f_n'(x) / f_n(x) \right| &= \left| \frac{2(x - a_{-n})}{(x - a_{-n})^2 + (a_{-n}\eta_{-n})^2} + \frac{2(x - a_n)}{(x - a_n)^2 + (a_n\eta_n)^2} \right| \\ &\le C \left(\frac{1}{\delta_n} + \frac{1}{|x - a_n| + a_n\eta_n} \right) \le \frac{C_1}{|x - a_n| + a_n\eta_n}. \end{aligned}$$

Moreover, using (43) and Lemma 2.2(d),

$$\phi_n(x) \sim \frac{\sqrt{\delta_n}}{n} \frac{|x - a_{2n}|}{\sqrt{|x - a_n| + a_n\eta_n}} \le C \frac{\sqrt{\delta_n}}{n} \frac{|x - a_n| + a_n/T(a_n)}{\sqrt{|x - a_n| + a_n\eta_n}}$$

so

$$\phi_n(x) \left| f_n'(x) / f_n(x) \right| \le C \frac{\sqrt{\delta_n}}{n} \frac{|x - a_n| + a_n/T(a_n)}{(|x - a_n| + a_n\eta_n)^{3/2}}.$$

Since for large n, η_n is much smaller than $1/T(a_n)$, (recall (32)) a little calculus shows that this last right-hand side is largest when $|x - a_n|$ is smallest, so we deduce that

$$\phi_n(x) \left| f_n'(x) / f_n(x) \right| \le C \frac{\sqrt{\delta_n}}{n} \frac{a_n/T(a_n)}{(a_n\eta_n)^{3/2}} = C,$$

by definition of η_n. $\square$

We next map $[a_{-\ell}, a_\ell]$ to an interval slightly larger than $[-1, 1]$. Recall that the linear transformation

$$t = L_n(x) = \frac{x - \beta_n}{\delta_n} \Leftrightarrow x = L_n^{[-1]}(t) = \delta_n t + \beta_n$$

maps $[a_{-n}, a_n]$ onto $[-1, 1]$. We shall use the function

$$h_n^*(t) := \left(|1 + t| + \eta_{-n}^* \right) \left(|1 - t| + \eta_n^* \right), \tag{52}$$

which may be thought of as h_n transformed to the interval $[-1, 1]$.

Proposition 3.5
Let $\Delta \in \left(-\frac{1}{2}, 0\right)$. Suppose that there exists $C_0 > 0$ such that for each $s > 0$, we have polynomials R_m of degree $m = m(n) \le C_0/\sqrt{\eta_n^*}$ with also $m \sim 1/\sqrt{\eta_n^*}$ such that
(i')

$$R_m(t) \le C_1 \left(|1 - t| + \eta_n^* \right)^\Delta \text{ in } [-2, 1 + s\eta_n^*]; \tag{53}$$

(ii')

$$R_m(t) \ge C_2 \left(|1 - t| + \eta_n^* \right)^\Delta \text{ in } [0, \infty). \tag{54}$$

Then there exist polynomials S_m satisfying the conclusions of Proposition 3.3.
Proof
Assuming the $\{R_m\}$ exist, we set

$$S_m(x) := \delta_n^{2\Delta} R_m(L_n(x)).$$

Now if $t = L_n(x)$, then a straightforward substitution shows that

$$h_n(x) = \delta_n^2 \left(|1 + t| + \eta_{-n}^* \right) \left(|1 - t| + \eta_n^* \right) = \delta_n^2 h_n^*(t).$$

Hence for $t = L_n(x) \in [-2,2]$,

$$h_n(x) \leq \delta_n^2 \left(3 + \eta_{-n}^*\right)\left(|1-t| + \eta_n^*\right).$$

Then as $\Delta < 0$, (i') gives

$$\begin{aligned} S_m(x) &\leq C_1 \delta_n^{2\Delta}\left(|1-t| + \eta_n^*\right)^{\Delta} \\ &\leq C_2 h_n(x)^{\Delta}, \end{aligned} \tag{55}$$

for $t = L_n(x) \in [-2, 1 + s\eta_n^*]$. Now let $\ell := \ell(n)$ be given by (47). Then

$$\begin{aligned} L_n(a_\ell) - 1 &= L_n(a_\ell) - L_n(a_n) \\ &= \frac{a_\ell - a_n}{\delta_n} \\ &= O\left(\frac{a_n}{\delta_n T(a_n)} \frac{m}{n}\right), \end{aligned}$$

by Lemma 2.2(d). Then (42) and the definition of η_n^* show that for some $s > 0$,

$$L_n(a_\ell) \leq 1 + s\frac{a_n}{\delta_n}\eta_n = 1 + s\eta_n^*.$$

Next,

$$\begin{aligned} L_n(a_{-\ell}) + 1 &= L_n(a_{-\ell}) - L_n(a_{-n}) \\ &= \frac{a_{-\ell} - a_{-n}}{\delta_n} \\ &= O\left(\frac{|a_{-\ell}|}{\delta_\ell} \frac{m}{n}\right) = o(1), \end{aligned}$$

by Lemma 2.2(d) again. Then for n large enough,

$$L_n[a_{-\ell}, a_\ell] \subseteq [-2, 1 + s\eta_n^*].$$

Then we obtain (48) of Proposition 3.3 from (55). Next, in $[0,\infty)$, we have $|1+t| \geq 1$, so (ii') gives

$$\begin{aligned} R_m(t) &\geq C_2\left(|1-t| + \eta_n^*\right)^{\Delta} \\ &\geq C_2 h_n^*(t)^{\Delta} \end{aligned}$$

and then, as

$$L_n(\beta_n) = 0,$$

we have in $[\beta_n, \infty)$,

$$S_m(x) \geq C h_n(x)^{\Delta},$$

so we have (49) of Proposition 3.3. We turn to (50), and for this we use Dzadyk's inequality. Let

$$R_m^*(t) := R_m\left(t(1 + s\eta_n^*)\right).$$

Then using the above inequalities and the fact that $\Delta < 0$, we see that for $t \in [-1,1]$,

$$|R_m^*(t)| \leq C\left(1 - t^2 + \eta_n^*\right)^{\Delta} \leq C_1\left(1 - t^2 + m^{-2}\right)^{\Delta}.$$

By Dzadyk's inequality (see [2, Thm. 2.3, pp. 241-2] or [17, p. 285])

$$|R_m^{*\prime}(t)| \leq Cm\left(1 - t^2 + m^{-2}\right)^{\Delta - 1/2}, t \in [-1,1].$$

Then also

$$|R_m'(t)| \le Cm\left(1-t^2+m^{-2}\right)^{\Delta-1/2}, t \in [-1,1].$$

Moreover, for $x \in [\beta_n, a_\ell] \Rightarrow t \in [0, 1+s\eta_n^*]$, as in the proof of Proposition 3.4,

$$\begin{aligned} \phi_n\left(L_n^{[-1]}(t)\right) &= \phi_n(x) \\ &\sim \frac{\sqrt{\delta_n}}{n}\frac{|x-a_{2n}|}{\sqrt{|x-a_n|+a_n\eta_n}} \\ &\sim \frac{\delta_n}{n}\frac{|1-t|+a_n/\left(T(a_n)\delta_n\right)}{\sqrt{|1-t|+\eta_n^*}}. \end{aligned}$$

Then with $t = L_n(x) \in [0, 1+s\eta_n^*] \supseteq L_n[\beta_n, a_\ell]$,

$$\begin{aligned} |S_m'\phi_n|(x)/h_n(x)^\Delta &= |R_m'(t)|\,\delta_n^{-1}\phi_n\left(L_n^{[-1]}(t)\right)/h_n^*(t)^\Delta \\ &\le Cm\left(\frac{1-t^2+\eta_n^*}{h_n^*(t)}\right)^\Delta \frac{1}{n}\frac{|1-t|+a_n/\left(T(a_n)\delta_n\right)}{1-t+\eta_n^*} \\ &\le C\frac{m}{n}\frac{1}{T(a_n)\eta_n} \le C, \end{aligned}$$

recall (42). So we have all the conclusions of Proposition 3.3 for $\Delta \in \left(-\frac{1}{2}, 0\right)$. $\square$

Finally, we can construct polynomials satisfying (i') and (ii'), using Christoffel functions for Jacobi weights:

Proposition 3.6
Let $\Delta \in \left(-\frac{1}{2}, 0\right)$. Then for large enough n, there exist polynomials R_m of degree $m = m(n) \sim 1/\sqrt{\eta_n^*}$ satisfying the conclusions of Proposition 3.5.
Proof
Let

$$\tau := -\left(\Delta+\frac{1}{2}\right) \Leftrightarrow -\left(\tau+\frac{1}{2}\right) = \Delta.$$

Then $\tau \in (-\frac{1}{2}, 0)$. We use the Christoffel function $\lambda_k(x)$ for the Jacobi weight

$$u(x) := (1+x)^{-1/2}(1-x)^\tau, x \in (-1,1).$$

For $k \ge 1$, $\lambda_k^{-1}(x)$ is a polynomial of degree $2k-2$ and it is known [13, p. 108] that

$$k^{-1}\lambda_k^{-1}(x) \sim \left(|1-x|+k^{-2}\right)^{-\tau-1/2} = \left(|1-x|+k^{-2}\right)^\Delta, \tag{56}$$

uniformly for $x \in [-1,1], k \ge 1$. Since $k^{-1}\lambda_k^{-1}(x)$ is increasing in $(1,\infty)$, while the last right-hand side is decreasing there, we also obtain

$$k^{-1}\lambda_k^{-1}(x) \ge C\left(|1-x|+k^{-2}\right)^\Delta \text{ in } (1,\infty). \tag{57}$$

We now choose

$$k := m(n) := \text{greatest integer} \le \frac{1}{2\sqrt{\eta_n^*}}$$

and for fixed $s > 0$,

$$R_m(t) := k^{-1}\lambda_k^{-1}\left(\frac{t+1}{2(1+s\eta_n^*)}\right),$$

so that R_m has degree $2k-2 \leq 1/\sqrt{\eta_n^*}-2$ with $\sim$ for large enough n. Since the degree is independent of s, we have satisfied the degree restrictions in Proposition 3.5. Next for $t \in [-2, 1+s\eta_n^*]$,

$$\frac{t+1}{2(1+s\eta_n^*)} \in \left(-\frac{1}{2}, 1\right),$$

so (56) gives

$$\begin{aligned} R_m(t) &\sim \left(\left|1-\frac{t+1}{2(1+s\eta_n^*)}\right|+\eta_n^*\right)^{\Delta} \\ &\sim (|1-t|+\eta_n^*)^{\Delta}. \end{aligned}$$

Thus we have (53) in a stronger form. Similarly we may deduce (54) from (57). □

4. THE PROOF OF THEOREM 1.2(I)

We shall deduce this from a result in [6]. To avoid conflicts of notation with that of this paper, we slightly change the notation there.

Theorem 4.1
Let $1 < p < \infty$, $n \geq 1$ and let $\{t_j\}_{j=1}^n$ satisfy

$$-1 \leq t_1 < t_2 < ... < t_n \leq 1.$$

Set $t_j := -1, j \leq 0$ and $t_j := 1, j > n$.
(I) Let $b \in \left[\frac{1}{2}, 1\right]$, $\beta \in [0, \frac{1}{2}]$ and

$$-\frac{1}{p} < \sigma < 1-\frac{1}{p}. \tag{58}$$

(II) Let

$$\omega(t) := \left(\left|1-\left|\frac{t}{b}\right|\right|+\beta\right)^{\sigma}. \tag{59}$$

Let $\nu : [-1, 1] \to [0, \infty)$ be measurable and let $\pi_n(t)$ be a polynomial of degree n whose zeros are $\{t_j\}_{j=1}^n$, normalized by the condition

$$|\pi_n \nu| \leq \omega \text{ in } [-1, 1]. \tag{60}$$

(III) Let

$$\Delta_j := t_{j+1} - t_{j-1}, 1 \leq j \leq n. \tag{61}$$

Assume that there exists $\alpha > 0$ such that for $1 \leq j, k \leq n$ with $|j-k| \geq 1$,

$$|t_j - t_k| \geq \alpha |j-k|^{1/3} [1 + \log|j-k|]^{2/3} \Delta_j. \tag{62}$$

(IV) Assume moreover, that for some $\tau > 0$, and $1 \leq j \leq n$,

$$\left|1-\left|\frac{t_j}{b}\right|\right|+\beta \geq \tau\Delta_j. \tag{63}$$

Then for $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \sum_{j=1}^{n} |P(t_j)|^p \left\{\int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p + \frac{\Delta_j \omega(t_j)^p}{[\Delta_j |\pi_n'(t_j)|]^p}\right\}. \tag{64}$$

The integer K depends only on L, α, and the constant C depends on $L, \alpha, \sigma, \tau, p$ but is independent of $\nu, \omega, \{t_j\}_{j=1}^n, b, \beta, n, P$.

Proof

See [6, Thm. 1.7, p. 583]. □

The Proof of Theorem 1.2(I)

Step 1: Choice of $\{t_j\}, \pi_n, \nu, \omega, b, \beta$

We shall apply the theorem above with

$$\begin{aligned} t_j &:= L_n(x_{jn}), 0 \leq j \leq n; \\ \Delta_j &:= t_{j-1} - t_{j+1}, 1 \leq j \leq n. \end{aligned} \tag{65}$$

(We are reversing the order of the $\{t_j\}$. Of course t_j depends on n, but we do not display this dependence). As our polynomial π_n whose zeros are $\{t_j\}_{j=1}^n$, we may choose

$$\pi_n(t) = \delta_n^{1/2} p_n\left(L_n^{[-1]}(t)\right)/B, \tag{66}$$

where B is a fixed large enough positive number. Moreover, for Δ satisfying (4), we write

$$\sigma := \Delta - \frac{1}{4}. \tag{67}$$

Then (58) is satisfied. In ω, we choose $b = 1, \beta = 0$, so that

$$\omega(t) = (1 - |t|)^\sigma \tag{68}$$

and we choose

$$\nu(t) := W\left(L_n^{[-1]}(t)\right)(1 - |t|)^\Delta. \tag{69}$$

Step 2: We verify (60)

From our bound (26) on p_n, we have

$$\begin{aligned} |\pi_n \nu|(t) &\leq CB^{-1}\delta_n^{1/2} h_n\left(L_n^{[-1]}(t)\right)^{-1/4}(1 - |t|)^\Delta \\ &\leq CB^{-1}(1 - |t|)^{-1/4+\Delta} \leq \omega(t), \end{aligned}$$

if B is large enough.

Step 3: We verify (62)

Now Lemma 2.1(b) and (c) show that uniformly in j and n,

$$\int_{x_{j+1,n}}^{x_{jn}} \frac{dx}{\phi_n(x)} \sim \frac{x_{jn} - x_{j+1,n}}{\phi_n(x_{jn})} \sim 1.$$

Then for $j \neq k$,

$$\left|\int_{x_{jn}}^{x_{kn}} \frac{dx}{\phi_n(x)}\right| \sim |k - j|.$$

The constants in $\sim$ are independent of j, k, n. Suppose for example that $x_{jn}, x_{kn} \geq \beta_n$. Since also $x_{jn}, x_{kn} \leq a_n(1 - \varepsilon\eta_n)$ for some $\varepsilon > 0$, we see that in the integral,

$$\begin{aligned} \phi_n(x) &\sim \frac{\sqrt{\delta_n}}{n}\frac{|x - a_{2n}|}{\sqrt{|x - a_n|}} \\ &\sim \frac{\sqrt{\delta_n}}{n}\frac{a_n - x + a_n/T(a_n)}{\sqrt{a_n - x}}, \end{aligned} \tag{70}$$

as in the proof of Proposition 3.4. Then this and the substitution $a_n - x = ya_n/T(a_n)$ gives

$$
\begin{aligned}
|k-j| &\leq C\frac{n}{\sqrt{\delta_n}}\left|\int_{x_{jn}}^{x_{kn}} \frac{\sqrt{a_n - x}}{a_n - x + a_n/T(a_n)}dx\right| \\
&= C\frac{n}{\sqrt{\delta_n}}\sqrt{\frac{a_n}{T(a_n)}}\left|\int_{(1-x_{jn}/a_n)T(a_n)}^{(1-x_{kn}/a_n)T(a_n)} \frac{\sqrt{y}}{y+1}dy\right| \qquad (71)\\
&\leq C\frac{n}{\sqrt{\delta_n}}\sqrt{\frac{a_n}{T(a_n)}}\left|\int_{(1-x_{jn}/a_n)T(a_n)}^{(1-x_{kn}/a_n)T(a_n)} \frac{1}{\sqrt{y}}dy\right| \\
&\leq C\frac{n}{\sqrt{\delta_n}}\sqrt{\frac{a_n}{T(a_n)}}\left|\sqrt{(1-x_{kn}/a_n)T(a_n)} - \sqrt{(1-x_{jn}/a_n)T(a_n)}\right| \\
&= C\frac{n}{\sqrt{\delta_n}}\left|\frac{x_{jn}-x_{kn}}{\sqrt{a_n - x_{kn}} + \sqrt{a_n - x_{jn}}}\right|.
\end{aligned}
$$

So,

$$
|x_{jn} - x_{kn}| \geq C\,|k-j|\,\frac{\sqrt{\delta_n}}{n}\left(\sqrt{a_n - x_{kn}} + \sqrt{a_n - x_{jn}}\right). \qquad (72)
$$

If

$$
a_n - x_{jn} \geq a_n/T(a_n), \qquad (73)
$$

then

$$
\begin{aligned}
a_{2n} - x_{jn} &= a_{2n} - a_n + a_n - x_{jn} \\
&\sim a_n/T(a_n) + a_n - x_{jn} \sim a_n - x_{jn}
\end{aligned}
$$

(recall (34)) so

$$
x_{j-1,n} - x_{j+1,n} \sim \phi_n(x_{jn}) \sim \frac{\sqrt{\delta_n}}{n}\frac{|x_{jn} - a_{2n}|}{\sqrt{a_n - x_{jn}}} \sim \frac{\sqrt{\delta_n}}{n}\sqrt{a_n - x_{jn}}. \qquad (74)
$$

Hence (72) gives

$$
\frac{|x_{jn} - x_{kn}|}{x_{j-1,n} - x_{j+1,n}} \geq C\,|k-j|. \qquad (75)
$$

If (73) fails, we return to (71) to obtain

$$
\begin{aligned}
|k-j| &\leq C\frac{n}{\sqrt{\delta_n}}\sqrt{\frac{a_n}{T(a_n)}}\left|\int_{(1-x_{jn}/a_n)T(a_n)}^{(1-x_{kn}/a_n)T(a_n)} \sqrt{y}dy\right| \\
&\leq C\frac{n}{\sqrt{\delta_n}}\sqrt{\frac{a_n}{T(a_n)}}\,|(x_{kn} - x_{jn})\,T(a_n)/a_n|\left|\sqrt{(1-x_{kn}/a_n)T(a_n)} + \sqrt{(1-x_{jn}/a_n)T(a_n)}\right| \\
&= C(a_n\eta_n)^{-3/2}\,|x_{kn} - x_{jn}|\left[\sqrt{a_n - x_{kn}} + \sqrt{a_n - x_{jn}}\right]. \qquad (76)
\end{aligned}
$$

Here we have used the fact that $\sqrt{y}$ is increasing in $(0,\infty)$. Since (73) fails, we also obtain from the second $\sim$ in (74), (which is still valid),

$$
x_{j-1,n} - x_{j+1,n} \leq C\frac{\sqrt{\delta_n}}{n}\frac{a_n/T(a_n)}{\sqrt{a_n - x_{jn}}} = C\frac{(a_n\eta_n)^{3/2}}{\sqrt{a_n - x_{jn}}}.
$$

Then provided

$$
\sqrt{a_n - x_{kn}} \leq 2\sqrt{a_n - x_{jn}}, \qquad (77)
$$

(76) gives

$$|k-j| \le C \frac{|x_{kn}-x_{jn}|}{x_{j-1,n}-x_{j+1,n}}.$$

If (77) fails, then

$$\begin{aligned} |x_{kn}-x_{jn}| &= |(a_n - x_{kn}) - (a_n - x_{jn})| \\ &\ge \frac{3}{4}(a_n - x_{kn}) \end{aligned}$$

so (76) gives

$$|k-j| \le C (a_n \eta_n)^{-3/2} |x_{jn}-x_{kn}|^{3/2}.$$

If we can show that

$$x_{j-1,n}-x_{j+1,n} \le C a_n \eta_n, \tag{78}$$

then the last inequality gives

$$|k-j| \le C \left(\frac{|x_{jn}-x_{kn}|}{x_{j-1,n}-x_{j+1,n}} \right)^{3/2},$$

whence

$$\frac{|x_{jn}-x_{kn}|}{x_{j-1,n}-x_{j+1,n}} \ge C |k-j|^{2/3}. \tag{79}$$

To show (78), we recall that since $x_{jn} \ge \beta_n$ and as (73) fails, the second $\sim$ in (74) gives

$$\begin{aligned} x_{j-1,n}-x_{j+1,n} &\sim \frac{\sqrt{\delta_n}}{n} \frac{|x_{jn}-a_{2n}|}{\sqrt{a_n - x_{jn}}} \\ &\sim \frac{\sqrt{\delta_n}}{n} \frac{a_n/T(a_n)}{\sqrt{a_n - x_{jn}}} \\ &\le C \frac{\sqrt{\delta_n}}{n} \frac{a_n/T(a_n)}{\sqrt{a_n \eta_n}} = C a_n \eta_n. \end{aligned}$$

In summary, we have shown that for all $x_{jn}, x_{kn} \ge \beta_n$, (79) holds (for $|k-j| \ge |k-j|^{2/3}$). Similarly, we may establish this when $x_{jn}, x_{kn} \le \beta_n$. The case where x_{jn} and x_{kn} lie on opposite sides of the midpoint β_n of $[a_{-n}, a_n]$ follows from the other two cases: one chooses a pair of zeros that bracket β_n and then applies the relevant result to the pairs of zeros on each side of β_n. Thus (79) holds in all cases. Since

$$\frac{|t_j - t_k|}{t_{j-1}-t_{j+1}} = \frac{|x_{jn}-x_{kn}|}{x_{j-1,n}-x_{j+1,n}},$$

we obtain a stronger form of (62). Of course the constant is independent of n, j, k, and that is crucial.

Step 4: We verify (63)

Because of our choice $b = 1, \beta = 0$, we must show that for some τ independent of j and n,

$$|1 - |t_j|| \ge \tau (t_{j-1} - t_{j+1}).$$

Note that all $x_{jn} < a_n$ (even for $j = 0$) and hence all $t_j < 1$. If $t_j \ge 0$, this last inequality is implied by

$$|1 - t_j| \ge \tau (1 - t_{j+1}). \tag{80}$$

Since Lemma 2.1(b) shows that uniformly in j and n,

$$a_n - x_{jn} \sim a_n - x_{j\pm1,n},$$

we obtain

$$1 - t_j \sim 1 - t_{j\pm 1}$$

and so (80) follows. The case $t_j < 0$ is similar.
Step 5: Completion of the proof of (5)
We have the estimate (64) and must translate it from $[-1, 1]$ to $[a_{-n}, a_n]$. But first we must bound the fundamental polynomials $\left\{\ell^*_{jn}\right\}_{j=1}^n$ for the points $\{t_j\}_{j=1}^n$ on $(-1, 1)$. We see that

$$\ell^*_{jn}(t) = \ell_{jn}\left(L_n^{[-1]}(t)\right),$$

where $\{\ell_{jn}\}_{j=1}^n$ are the fundamental polynomials for the points $\{x_{jn}\}_{j=1}^n$. Then using our Lemma 2.1(f), we see that for $t \in I$ and uniformly in j and n,

$$\begin{aligned} \left|\ell^*_{jn}(t)\,\nu(t)\right| &= |\ell_{jn} W|\left(L_n^{[-1]}(t)\right)(1 - |t|)^{\Delta} \\ &\leq CW(x_{jn})(1 - |t|)^{\Delta}. \end{aligned}$$

Next, using Lemma 2.1(b), (c), translated to the $\{t_j\}$, we see that for some C independent of j, n,

$$\int_{t_{j-K}}^{t_{j+K+1}} \left|\ell^*_{jn}\nu\right|^p \leq CW^p(x_{jn})(1 - |t_j|)^{\Delta p}(t_{j-1} - t_{j+1}).$$

Next,

$$\pi_n'(t_j) = \delta_n^{3/2} p_n'(x_{jn})/B$$

so Lemma 2.1(c), (d) give

$$(t_{j-1} - t_{j+1})\,W(x_{jn})\,|\pi_n'(t_j)| \sim \left(1 - t_j^2\right)^{-1/4}$$

and then (recall the notation (65) and (67), (68))

$$\frac{\Delta_j \omega(t_j)^p}{\left[\Delta_j\,|\pi_n'(t_j)|\right]^p} \sim W^p(x_{jn})(1 - |t_j|)^{p\Delta}(t_{j-1} - t_{j+1}).$$

Thus (64) gives for any $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} \left|P(t) W\left(L_n^{[-1]}(t)\right)\left(1 - t^2\right)^{\Delta}\right|^p dt \leq C \sum_{j=1}^{n} |P(t_j)\,W(x_{jn})|^p \left(1 - t_j^2\right)^{p\Delta}(t_{j-1} - t_{j+1}).$$

Applying this to $P \circ L_n^{[-1]}$ and then making the substitution $t = L_n(x)$ and using Lemma 2.1(c) gives

$$\begin{aligned} &\int_{a_{-n}}^{a_n} \left|(PW)(x)\left[|x - a_{-n}|\,|a_n - x|\right]^{\Delta}\right|^p dx \\ \leq\ & C \sum_{j=1}^{n} \left|(PW)(x_{jn})\left[|x_{jn} - a_{-n}|\,|a_n - x_{jn}|\right]^{\Delta}\right|^p (x_{j-1,n} - x_{j+1,n}) \\ \leq\ & C \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \left|(PW)(x_{jn})\left[|x_{jn} - a_{-n}|\,|a_n - x_{jn}|\right]^{\Delta}\right|^p. \end{aligned}$$

Now for fixed $\varepsilon > 0$ and $x \in \left[a_{-n}\left(1 - \varepsilon\eta_{-n}\right), a_n(1 - \varepsilon\eta_n)\right]$,

$$|x - a_{-n}|\,|a_n - x| \sim h_n(x).$$

In particular this holds for $x = x_{jn}, 1 \leq j \leq n$ by Lemma 2.1(a), provided ε is small enough. We deduce that

$$\int_{a_{-n}(1-\varepsilon\eta_{-n})}^{a_n(1-\varepsilon\eta_n)} \left|(PW)(x)\, h_n(x)^{\Delta}\right|^p dx \leq C \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \left|(PW)(x_{jn})\, h_n(x_{jn})^{\Delta}\right|^p .$$

The restricted range inequality Lemma 3.1 then gives (5). □

5. The Proof of Theorem 1.2(II)

The method of proof is due to P. Nevai [13]. Given a polynomial P of degree $\leq n$, and $1 \leq j \leq n$, the fundamental theorem of calculus gives

$$\begin{aligned} & |PW|^p(x_{jn}) \\ \leq\ & \min_{[x_{jn}, x_{j-1,n}]} |PW|^p + \int_{x_{jn}}^{x_{j-1,n}} p\, |PW|^{p-1} \left|(PW)'\right| . \end{aligned}$$

In view of the $\sim$ relations in Lemma 2.1(b), (c), we see that we may insert a factor of $h_n^{\Delta p}(x_{jn})$ and $\lambda_{jn} W^{-2}(x_{jn})$ or $x_{j-1,n} - x_{jn}$:

$$\begin{aligned} & \lambda_{jn} W^{-2}(x_{jn}) \left|PW h_n^{\Delta}\right|^p (x_{jn}) \\ \leq\ & C\left(x_{j-1,n} - x_{jn}\right) \left|PW h_n^{\Delta}\right|^p (x_{jn}) \\ \leq\ & C \int_{x_{jn}}^{x_{j-1,n}} \left|PW h_n^{\Delta}\right|^p + C \int_{x_{jn}}^{x_{j-1,n}} |PW|^{p-1} \left|(PW)'\right| h_n^{\Delta p} \phi_n . \end{aligned}$$

Here C is independent of n, j, P. Adding over j, and using our knowledge of the location of the zeros gives

$$\begin{aligned} & \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \left|PW h_n^{\Delta}\right|^p (x_{jn}) \\ \leq\ & C \int_{a_{-n}}^{a_n} \left|PW h_n^{\Delta}\right|^p + C \int_{a_{-n}}^{a_n} |PW|^{p-1} \left|(PW)'\right| h_n^{\Delta p} \phi_n . \end{aligned} \tag{81}$$

Applying Hölder's inequality to the second term in the last right-hand side gives

$$\begin{aligned} & \int_{a_{-n}}^{a_n} \left|PW h_n^{\Delta}\right|^{p-1} \left|(PW)' h_n^{\Delta} \phi_n\right| \\ \leq\ & \left(\int_{a_{-n}}^{a_n} \left|PW h_n^{\Delta}\right|^p\right)^{1-\frac{1}{p}} \left(\int_{a_{-n}}^{a_n} \left|(PW)' h_n^{\Delta} \phi_n\right|^p\right)^{\frac{1}{p}} \\ \leq\ & C \int_{a_{-n}}^{a_n} \left|PW h_n^{\Delta}\right|^p , \end{aligned}$$

by our Markov-Bernstein inequality Lemma 3.2. Then (81) gives the desired inequality

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \left|PW h_n^{\Delta}\right|^p (x_{jn}) \leq C \int_{a_{-n}}^{a_n} \left|PW h_n^{\Delta}\right|^p .$$

□

6. The Proof of Theorems 1.3 to 1.6

We begin with the

Proof of (b)⇒(a) of Theorem 1.3
Assume (9). We may write

$$\Delta = \Delta_1 + r,$$

where Δ_1 satisfies (4) and $r \geq 0$. Then Theorem 1.2(I) with $P = L_n[f]$, our restricted range inequality Lemma 3.1, and the fact that $h_n \leq C\delta_n^2$ in $[a_{-n}, a_n]$ give

$$\begin{aligned}
\| \quad & L_n[f]Wh_n^{\Delta} \|_{L_p(I)} \leq C \| L_n[f]Wh_n^{\Delta_1+r} \|_{L_p[a_{-n},a_n]} \\
\leq \quad & C\delta_n^{2r} \| L_n[f]Wh_n^{\Delta_1} \|_{L_p[a_{-n},a_n]} \\
\leq \quad & C\delta_n^{2r}\left(\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left|fWh_n^{\Delta_1}\right|^p (x_{kn})\right)^{1/p} \\
\leq \quad & C\delta_n^{2r} \| fW \|_{L_\infty(I)} \left(\sum_{k=1}^{n} (x_{k-1,n} - x_{kn}) \left|h_n^{\Delta_1}\right|^p (x_{kn})\right)^{1/p} \\
\leq \quad & C\delta_n^{2r} \| fW \|_{L_\infty(I)} \left(\int_{a_{-n}}^{a_n} h_n^{\Delta_1 p}\right)^{1/p}.
\end{aligned} \tag{82}$$

Here we have used Lemma 2.1(b), (c). Now

$$\Delta_1 p > \frac{p}{4} - 1 > -1$$

so we may continue (82) as

$$\leq C\delta_n^{2r+2\Delta_1+\frac{1}{p}} \| fW \|_{L_\infty(I)} \left(\int_{-1}^{1} \left[\left(1+t+\eta_{-n}^*\right)\left(1-t+\eta_n^*\right)\right]^{\Delta_1 p} dt\right)^{1/p}$$

and we have (8). □

In the proof of the necessity part of Theorem 1.3, we use the following:

Lemma 5.1
For $n \geq 1$, let $f_n : I \to \mathbb{R}$, with $f_n = 0$ in (β_n, d) and

$$f_n(x_{jn}) = W^{-1}(x_{jn}) sign\left(p_n'(x_{jn})\right), x_{jn} \in (c, \beta_n). \tag{83}$$

Then there exists n_0 such that for $n \geq n_0$ and $x \in [\beta_n, d)$,

$$|L_n[f_n](x)| \geq C\delta_n^{1/2} |p_n(x)|. \tag{84}$$

Proof
We have for $x \geq \beta_n$, by (83) and then Lemma 2.1(d),

$$|L_n[f_n](x)| = |p_n(x)| \sum_{x_{jn}\in(c,\beta_n)} \frac{1}{|p_n'W|(x_{jn})(x - x_{jn})}$$

$$\sim |p_n(x)| \sum_{x_{jn}\in(c,\beta_n)} \frac{(x_{jn} - x_{j+1,n})h_n(x_{jn})^{1/4}}{x - x_{jn}}$$

$$\begin{aligned} &\geq \quad C\frac{|p_n(x)|}{\delta_n}\int_{a_{-n}}^{\beta_n} h_n\left(y\right)^{1/4}dy \\ &\geq \quad C\left|p_n(x)\right|\delta_n^{1/2}\int_{-1}^{0}\left(1-t^2\right)^{1/4}dt, \end{aligned}$$

Here we have used Lemma 2.1(b), (c) in the second last line, and the substitution $y = L_n^{[-1]}(t)$ in the last line.□

Proof of the Necessity part of Theorem 1.3
Assume (8). Construct f_n as in Lemma 5.1 so that f_n also satisfies

$$\| f_n W \|_{L_\infty(I)} = 1.$$

(We may also assume that f_n is continuous, but that is irrelevant to the proof). Then for some C_1 independent of n,

$$1 = \| f_n W \|_{L_\infty(I)} \geq C\delta_n^{-2\Delta-1/p} \| L_n[f_n] W h_n^{\Delta} \|_{L_p(I)}$$

$$\geq C\delta_n^{-2\Delta-1/p}\left(\int_{\beta_n}^{a_n}\left[\delta_n^{1/2}\left|p_n W h_n^{\Delta}\right|(x)\right]^p dx\right)^{1/p}.$$

Similarly, we may derive an estimate over $[a_{-n}, \beta_n]$ and combining these gives

$$\begin{aligned} C \quad &\geq \quad C\delta_n^{1/2-2\Delta-1/p} \| p_n W h_n^{\Delta} \|_{L_p[a_{-n},a_n]} \\ &\geq \quad C\delta_n^{1/2-2\Delta-1/p} \| h_n^{\Delta-1/4} \|_{L_p[x_{nn},x_{1n}]}, \end{aligned} \tag{85}$$

by Lemma 2.4. That lemma is applicable since $\xi = h_n^{\Delta}$ satisfies (37) (see Lemma 2.1(b)). Next,

$$1 - L_n\left(x_{1n}\right) = \frac{a_n - x_{1n}}{\delta_n} \sim \eta_n^*$$

with a similar relation for x_{nn}, and a substitution shows that

$$\| h_n^{\Delta-1/4} \|_{L_p[x_{nn},x_{1n}]}^p = \delta_n^{2p(\Delta-1/4)+1}\int_{-1+O\left(\eta_{-n}^*\right)}^{1-O(\eta_n^*)}\left(\left(|1+t|+\eta_{-n}^*\right)\left(|1-t|+\eta_n^*\right)\right)^{p\left(\Delta-\frac{1}{4}\right)}dt. \tag{86}$$

If (9) is violated, then

$$p\left(\Delta - \frac{1}{4}\right) \leq -1,$$

and since $\eta_{\pm n}^* \to 0, n \to \infty$, an easy estimation of the integral in (86) shows that

$$\delta_n^{1/2-2\Delta-1/p} \| h_n^{\Delta-1/4} \|_{L_p[x_{nn},x_{1n}]} \to \infty, n \to \infty,$$

contradicting (85). So (9) must be true. □

Proof of Theorem 1.4
Let f satisfy (10) or (11) according as d is infinite or finite and let P be a polynomial. Then from Theorem 1.2(I) with $\Delta = 0$, and n large enough,

$$\| \left(f - L_n[f]\right) W \|_{L_p(I)}$$

$$\leq \| \left(f - P\right) W \|_{L_p(I)} + \| L_n[P - f] W \|_{L_p(I)}$$

$$\leq \| (f-P)\,W \|_{L_p(I)} + C \left(\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f)\,W \right|^p (x_{kn}) \right)^{1/p}. \tag{87}$$

Now by our hypothesis, $W^{-2}\left|(P-f)W\right|^p$ is Riemann integrable over each compact subinterval $[a,b]$ of I, so

$$\lim_{n\to\infty} \sum_{x_{kn}\in[a,b]} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f)\,W \right|^p (x_{kn}) = \int_a^b \left| (P-f)W \right|^p. \tag{88}$$

This follows from the fact that the left-hand side is a Riemann-Stieltjes sum. See [19, p.50, Thm. 3.41.1 ff.]. Next if $d=\infty$, our hypothesis asserts that for some $\alpha > 1/p$,

$$\lim_{x\to\infty} (fW)(x)\,(1+|x|)^{\alpha} = 0,$$

so given $\varepsilon > 0$, we may assume that b is so large that

$$\left|(P-f)W\right|(x) \leq \varepsilon\,(1+|x|)^{-\alpha}, x \geq b.$$

(Note that P is fixed in this and the weight W decays much faster than any polynomial can grow). Then

$$\begin{aligned} & \sum_{x_{kn}\geq b} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f)\,W \right|^p (x_{kn}) \\ \leq\ & C\varepsilon^p \sum_{x_{kn}\geq b} \frac{x_{k-1,n}-x_{k+1,n}}{(1+|x_{kn}|)^{\alpha p}} \\ \leq\ & C\varepsilon^p \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{\alpha p}}, \end{aligned}$$

with C independent of n, b, ε. As usual this follows using Lemma 2.1(b), (c). If $d<\infty$, our hypothesis asserts that for some $\alpha < \frac{1}{p}$,

$$\lim_{x\to d-} (fW)(x)\,(d-x)^{\alpha} = 0.$$

Again, given $\varepsilon > 0$, we may assume that $b > 0$ is so close to d that

$$\left|(P-f)W\right|(x) \leq \varepsilon\,(d-x)^{-\alpha}, x \in (b,d).$$

Then

$$\begin{aligned} & \sum_{x_{kn}\geq b} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f)\,W \right|^p (x_{kn}) \\ \leq\ & C\varepsilon^p \sum_{x_{kn}\geq b} \frac{x_{k-1,n}-x_{k+1,n}}{(d-x_{kn})^{\alpha p}} \\ \leq\ & C\varepsilon^p \int_0^d \frac{dx}{(d-x)^{\alpha p}}, \end{aligned}$$

with C independent of n, b, ε. As usual this follows using Lemma 2.1(b), (c). Thus in all cases, we may make sure that the sum involving $x_{jn} \geq b$ is small, and similarly we may handle the sum over $x_{jn} \leq a$ for a close to c. It follows from these considerations and (87) and (88) that

$$\limsup_{n\to\infty} \| (f-L_n[f])\,W \|_{L_p(I)} \leq C \| (f-P)\,W \|_{L_p(I)}$$

with C independent of P. Since W decays sufficiently rapidly near $\pm\infty$ if d or c are infinite, we may choose a polynomial P for which this last right-hand side is as small as we please. Then the result follows. $\square$

In the proof of Theorems 1.5 and 1.6, we shall use:

Lemma 5.2
Let

$$F(x) := 1 + Q^{2/3}(x)T(x). \tag{89}$$

Then for $n \geq 1$ and $x \in I$,

$$\frac{h_n(x)}{a_n\,|a_{-n}|}F(x) \geq C. \tag{90}$$

Proof
Now we may consider only $x \geq 0$. Since

$$\frac{h_n(x)}{a_n\,|a_{-n}|} = \left(1 + \frac{x}{|a_{-n}|} + \eta_{-n}\right)\left(\left|1 - \frac{x}{a_n}\right| + \eta_n\right),$$

we need only bound below $\left(\left|1 - \frac{x}{a_n}\right| + a_n\eta_n\right) F\left(x\right)$ by some $C > 0$. We consider three ranges of $x \geq 0$.
(I) $x \in [0, a_{n/2}]$
Write $x = a_r$. Then

$$\begin{aligned} &\left(\left|1 - \frac{x}{a_n}\right| + \eta_n\right) \\ \geq\ \ & 1 - \frac{a_r}{a_n} \geq 1 - \frac{a_r}{a_{2r}} \sim \frac{1}{T(x)} \end{aligned}$$

by Lemma 2.2(d). Then

$$\left(\left|1 - \frac{x}{a_n}\right| + \eta_n\right) F(x) \geq C\left[\frac{1}{T(x)} + Q^{2/3}(x)\right] \geq C.$$

(II) $x \in [a_{n/2}, a_{2n}]$
Here Lemma 2.2(a) and the definition of η_n give

$$F(x) \sim Q^{2/3}(a_n)T(a_n) \sim (n\sqrt{\frac{a_n}{\delta_n T\left(a_n\right)}})^{2/3}T\left(a_n\right) = \eta_n^{-1}. \tag{91}$$

Then

$$\left(\left|1 - \frac{x}{a_n}\right| + \eta_n\right) F(x) \geq C\eta_n F\left(x\right) \geq C.$$

(III) $x \in [a_{2n}, d)$
As both F and $\left|1 - \frac{x}{a_n}\right| + \eta_n$ are increasing over this range of x, the desired lower bound follows from the previous range of x. $\square$

Proof of Theorem 1.5
Let P be a polynomial and f satisfy the hypotheses of Theorem 1.5. We proceed similarly to Theorem 1.4. Note that $\Delta > 0$ follows from (14). We also note that if the conclusion of Theorem 1.5 holds for a given Δ, then it holds for any larger Δ, so we may assume

that Δ is small enough to satisfy (4). We shall also use our hypothesis $a_n \sim |a_{-n}|$, which implies that

$$\left|1 - \frac{x}{a_{\pm n}}\right| \leq C \text{ in } [a_{-n}, a_n]$$

and hence

$$\frac{h_n}{a_n |a_{-n}|} \leq C \text{ in } [a_{-n}, a_n]. \tag{92}$$

Let n be larger than the degree of P. Using Lemma 5.2, followed by Theorem 1.2(I), gives

$$\begin{aligned}
&\| (f - L_n[f]) WF^{-\Delta} \|_{L_p(I)} \\
&\leq C \left[\| (f-P) WF^{-\Delta} \|_{L_p(I)} + \| L_n[P-f] W \left(\frac{h_n}{a_n |a_{-n}|}\right)^{\Delta} \|_{L_p(I)} \right] \\
&\leq C \left[\| (f-P) WF^{-\Delta} \|_{L_p(I)} + \left(\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f) W \left(\frac{h_n}{a_n |a_{-n}|}\right)^{\Delta} \right|^p (x_{kn}) \right)^{1/p} \right] \\
&\leq C \left[\| (f-P) W \|_{L_p(I)} + \left(\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f) W \right|^p (x_{kn}) \right)^{1/p} \right]
\end{aligned} \tag{93}$$

by (92). Then proceeding as in the proof of Theorem 1.4, we obtain

$$\limsup_{n\to\infty} \| (f - L_n[f]) WF^{-\Delta} \|_{L_p(I)} \leq C \| (f-P) W \|_{L_p(I)}$$

with C independent of P and the result follows.□

Proof of Theorem 1.6

Let P be a polynomial and f satisfy the hypotheses of Theorem 1.6. We proceed similarly to Theorem 1.4. As before, the estimate (93) holds. The difference is that now $h_n / (a_n |a_{-n}|)$ need not be bounded in $[a_{-n}, a_n]$. Instead, we use that for $x \in [0, a_n]$,

$$\frac{h_n(x)}{a_n |a_{-n}|} \leq \left(1 + \frac{x}{|a_{-n}|} + \eta_{-n}\right)(1 + \eta_n) \leq C (1 + |x|).$$

Similarly we may show that this holds in $[a_{-n}, 0]$. Then

$$\begin{aligned}
&\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f) W \left(\frac{h_n}{a_n |a_{-n}|}\right)^{\Delta} \right|^p (x_{kn}) \\
\leq \quad & C \sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f)(x_{kn}) W(x_{kn}) (1 + |x_{kn}|)^{\Delta} \right|^p.
\end{aligned}$$

Now if $d = \infty$, we assumed that for some $\varepsilon > 0$,

$$\lim_{x\to\infty} |f(x)| W(x) (1+x)^{\Delta + 1/p + \varepsilon} = 0,$$

with a similar limit if $c = -\infty$. We may show as in Theorem 1.4 that

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn}) \left| (P-f) W \left(\frac{h_n}{a_n |a_{-n}|}\right)^{\Delta} \right|^p (x_{kn}) \\
\leq \quad & C \| (f-P)(x) W(x) (1+|x|)^{\Delta} \|_{L_p(I)}.
\end{aligned}$$

Again this may be made arbitrarily small and so the proof may be completed as before.
□

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