GREEN EQUILIBRIUM MEASURES AND REPRESENTATIONS OF AN EXTERNAL FIELD

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Abstract. We establish a representation for external fields involving Green potentials. This is the analogue of the representation of Rakhmanov and Buyarov involving logarithmic potentials. We also establish related results, and present an example.

1. Introduction

Let Q be convex on $\mathbb{R}$, with

$$\min_{\mathbb{R}} Q = 0,$$

and with Q growing at ∞ faster than $\log|x|$. Then Q admits the representation

$$Q(x) = \int_0^\infty g_{S_\tau}(x)\, d\tau, x \in \mathbb{R}, \tag{1.1}$$

where $\{S_\tau\}$ is a suitable increasing sequence of compact intervals and g_{S_τ} denotes the Green function for $\mathbb{C}\backslash S_\tau$ with pole at ∞. This representation was discovered by Rakhmanov [14], and it turned out to be indispensable in the study of orthogonal polynomials for the weight $W = \exp(-Q)$, and in several other contexts [2], [6], [7]. Actually (1.1) was proved in [14] for a special class of convex Q. The general result was announced in [3].

Inspired by that paper, the authors proved (1.1) in [9], using results of Totik [17] on equilibrium measures for the family of weights $\left\{w^\lambda\right\}_{\lambda>0}$. This was then applied in studying orthogonal properties for non-even weights. A far reaching generalisation of (1.1) appeared in a recent paper of Buyarov and Rakhmanov [4]. They proved that (1.1) holds (for $x \in \cup_\tau S_\tau \subseteq \mathbb{R}$), for example, for any continuous function Q, and beyond. Note that (1.1) may be rewritten as

$$Q(x) = \int_0^\infty \left\{\log\frac{1}{\mathrm{cap} S_\tau} - U^{\omega_\tau}(x)\right\} d\tau, \tag{1.2}$$

where

$$U^{\omega_\tau}(x) := \int \log\frac{1}{|x-s|}\, d\omega_\tau(s)$$

is the (logarithmic) equilibrium potential for the set S_τ.

Since the study of rational functions is intimately connected with Green potentials, there is good reason to believe that an analogue of (1.2) for Green potentials will be useful for problems involving rational functions, just as (1.2) is useful for problems involving polynomials. For a wide class of functions Q on a set E (that is

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not necessarily a real interval) in a domain $G \subseteq \mathbb{C}$, we show that there is a suitable increasing family of compact sets $S_\tau \subseteq E, \tau > 0$, such that for $t > 0$ and $z \in S_t$,

$$Q(z) = \int_0^t \left\{ \frac{1}{\mathrm{cap}_G S_\tau} - V^{\omega_\tau^G}(z) \right\} d\tau. \tag{1.3}$$

Here $\mathrm{cap}_G S_\tau$ denotes the Green capacity for the set S_τ, and if $g(z,\xi)$ denotes the Green's function for G with pole at ξ,

$$V^{\omega_\tau^G}(z) := \int_{S_\tau} g(z,\xi)\, d\omega_\tau^G(\xi)$$

denotes the Green potential for the Green equilibrium measure ω_τ^G for S_τ. We emphasise that in the sequel the symbol V is associated with Green (and not logarithmic) potentials.

Since $\{S_\tau\}_{\tau>0}$ is increasing, so that the integrand in (1.3) is 0 for $\tau > t$, one also deduces from (1.3) that

$$Q(z) = \int_0^\infty \left\{ \frac{1}{\mathrm{cap}_G S_\tau} - V^{\omega_\tau^G}(z) \right\} d\tau, z \in \cup_{\tau>0} S_\tau. \tag{1.4}$$

But what is a suitable $\{S_\tau\}_{\tau>0}$? This is easy to explain. We have

$$\int_0^t V^{\omega_\tau^G}(z)\, d\tau = \int_0^t \left[\int_{S_\tau} g(z,\xi)\, d\omega_\tau^G(\xi) \right] d\tau.$$

Hence if we define the measure μ_t on S_t by

$$\mu_t := \int_0^t \omega_\tau^G d\tau, \tag{1.5}$$

we obtain, by Fubini, that

$$\int_0^t V^{\omega_\tau^G}(z)\, d\tau = \int_{S_t} g(z,\xi)\, d\mu_t(\xi) = V^{\mu_t}(z), \tag{1.6}$$

where V^{μ_t} is the Green potential of μ_t. We also see from (1.5) that

$$\mu_t(S_t) = \int_0^t \omega_\tau^G(S_t)\, d\tau = \int_0^t d\tau = t.$$

(Recall that ω_τ^G has mass 1 and is supported on $S_\tau \subseteq S_t$). Thus μ_t has mass t, and is supported on S_t. Now assuming that (1.3) holds, we obtain from (1.6) that

$$V^{\mu_t}(z) + Q(z) = c_t, z \in S_t, \tag{1.7}$$

where we set

$$c_t := \int_0^t \frac{d\tau}{\mathrm{cap}_G S_\tau}. \tag{1.8}$$

Moreover, assuming, for the moment, that

$$E = \bigcup_{\tau>0} S_\tau,$$

(which is not always the case), and keeping in mind that

$$V^{\omega_\tau^G}(z) \leq \frac{1}{\mathrm{cap}_G S_\tau}, z \in G, \tag{1.9}$$

we obtain from (1.4) that

$$V^{\mu_t}(z) + Q(z) \geq c_t, z \in E. \tag{1.10}$$

The relations (1.7), (1.10) imply that μ_t is the Green equilibrium measure of mass t for the external field Q. Hence if (1.3) holds, then the set S_t must coincide with the support of μ_t.

In the next section, we describe the class of functions for which (1.3) will be proved, and present the main theorem. We also recall some basic notions and results from potential theory. The rest of the paper is devoted to proofs. We could prove (1.3) using the above-mentioned results of Totik (which can be extended to deal with Green potentials), but we preferred to follow the same steps as in [4], thereby obtaining some other useful results, parallel to those proved in [4].

2. Preliminaries and Main Theorem

Let G be any domain in $\overline{\mathbb{C}}$, whose boundary ∂G has positive capacity, and let $g(z,\xi)$ denote the Green function for G with pole at ξ. So g is characterized by the following properties:
(i) As a function of z, with ξ fixed, $g(z,\xi)$ is non-negative, subharmonic in $\overline{\mathbb{C}}\setminus\{\xi\}$ and harmonic in $G\setminus\{\xi\}$;
(ii) $g(z,\xi) + \log|z-\xi|$ remains bounded as $z \to \xi$;
(iii) $g(z,\xi) = 0$ for q.e. $z \in \partial G$ where q.e. (quasi-everywhere) means except for a set of capacity 0.
Given a finite positive measure μ on G, we recall that its Green potential V^μ is defined by

$$V^\mu(z) = \int g(z,\xi)\, d\mu(\xi), z \in G.$$

The support of μ will be denoted by S_μ, and we always assume that S_μ is a compact subset of G. Such a V^μ is l.s.c. (lower semi-continuous) and superharmonic in G. Also

$$\lim_{z\to x\in\partial G} V^\mu(z) = 0 \text{ for q.e. } x \in \partial G. \tag{2.1}$$

Hence by the minimum principle, $V^\mu > 0$ in G (but may attain the value ∞).

Furthermore, V^μ is continuous in the fine topology (this is the weakest topology making all potentials continuous). This implies that for any $z_0 \in G$ and any $\varepsilon > 0$, the set

$$\{z \in G : |V^\mu(z) - V^\mu(z_0)| \geq \varepsilon\} \tag{2.2}$$

(with obvious adjustment for the case $V^\mu(z_0) = \infty$) is *thin* at z_0. All these notions and facts can be found in, for example, [16, Chapters 1, 2] or [8, Theorem 5.11]. When using the fine topology, we shall say so. Thus, unless otherwise mentioned, all limits and topological notions are with respect to the usual Euclidean topology.

Given a function $Q : E \to (-\infty,\infty]$, we say that Q *is admissible on* E if the following properties hold for E and Q:

(A.1) E is closed in G.
(That is, E is closed relative to G).

(A.2) E is not thin at any of its points.
(Such an E is called regular).

(A.3) E has empty interior, and for any compact $K \subseteq E$, the complement $G \backslash K$ is connected.

(A.4) Q is l.s.c. on E and the set $\{z \in E : Q(z) < \infty\}$ has positive capacity.
(In particular, $cap(E) > 0$, though this follows from (A.2) as well).

(A.5) For any $z_0 \in E$ with $Q(z_0)$ finite and for any $\varepsilon > 0$, the set

$$\{z \in E : |Q(z) - Q(z_0)| < \varepsilon\} \tag{2.3}$$

is not thin at z_0.
(Since Q is l.s.c., this also gives

$$\liminf_{z \to z_0} Q(z) = Q(z_0)).$$

(A.6) If $z_0 \in \partial G$ or $z_0 = \infty$ is a limit point of E, then

$$\lim_{z \to z_0, z \in E} Q(z) = \infty.$$

Note that (A.1), (A.4) and (A.6) imply that for any $N > 0$, the set $\{z \in E : Q(z) \leq N\}$ is a compact subset of G.

Remarks
(a) If Q is admissible on E, then $Q + V^{\mu}$ is also admissible, as follows from the properties of V^{μ} above.
(b) All of the above are satisfied if, for example, E is a smooth arc, possibly unbounded, and Q is piecewise continuous on E, satisfying (A.6).
(c) For some of our results, we do not need all of (A.1) to (A.6), and shall point this out where relevant.

Next, we need well known results on Green equilibrium potentials: let

$$M_t := M_t(E) := \{\mu : S_\mu \subseteq E \text{ and } \mu(E) = t\}.$$

For $\mu \in M_t$, consider its energy integral

$$\begin{aligned} I(\mu) \quad &: \quad = I(\mu, Q) \\ &: \quad = \int\int [g(z,\xi) + Q(z) + Q(\xi)]\, d\mu(z)\ d\mu(\xi) \\ &= \int V^{\mu} d\mu + 2t \int Q d\mu. \end{aligned}$$

Theorem 2.1
Assume (A.1), (A.4) and (A.6).
(a) There exists a unique $\mu_t \in M_t$ such that

$$I_t := I_t(\mu_t) := \inf\{I(\mu) : \mu \in M_t\}. \tag{2.4}$$

Moreover, I_t is finite, and μ_t has finite energy:

$$(0 \leq) \int V^{\mu_t} d\mu_t < \infty.$$

(b) The support S_{μ_t} of μ_t is a compact subset of G, and more precisely for some N,

$$S_{\mu_t} \subseteq \{z \in E : Q(z) \leq N\}.$$

(c) Setting

$$c_t := c_t(Q) := t^{-1}\left[I_t - \int Q d\mu_t\right],$$

we have

$$V^{\mu_t}(z) + Q(z) \geq c_t, \text{ q.e. } z \in E; \tag{2.5}$$

$$V^{\mu_t}(z) + Q(z) \leq c_t, \text{ all } z \in S_{\mu_t}. \tag{2.6}$$

This measure μ_t is called the equilibrium measure of mass t for the external field Q, and c_t is called the equilibrium (or extremal) constant.

Remark

Since $t^{-1}Q$ satisfies the same conditions as does Q, it suffices to prove the theorem for $t = 1$. For this case, it appears in [16, Theorem II.5.10], but under two additional restrictions. First, instead of (A.6), it is assumed in [16] that $Q(z) - \log|z| \to \infty$ as $z \to \infty$ (if E is unbounded), while we only assumed that $Q(z) \to \infty$ in this case. Second, no assumption on Q is made in [16], if E has limit points on ∂G. This is due to the (tacit) agreement that the phrase "closed subset $E \subset G$" used there, actually means that the closure of E in $\mathbb{C}$ still belongs to G (otherwise the result is incorrect, if Q is bounded near ∂G). Yet the proof of Theorem 1 requires only minor modifications of that in [16], so we only indicate two places where (A.6) comes into play.

Proof

(a) Being l.s.c., and since $Q > -\infty$ on E, Q is bounded below on compact subsets of E. Then (A.6) ensures that Q is bounded below on the whole of E (and of course attains its minimum on E). Since $V^{\mu} \geq 0$, it follows that the infimum in (2.4) is $> -\infty$. That it cannot be ∞, is proved by standard methods, using (A.4). Denote this infimum by I_1 (that is, I_t with $t = 1$).

(b) Let

$$E_N := \{z \in E : Q(z) \leq N\}.$$

According to (A.6), E_N is compact and we use (A.6) again to show that for N large enough,

$$I_1 = \inf\{I(\mu) : \mu \in M_t, S_\mu \subseteq E_N\}. \tag{2.7}$$

Once we have this, the rest of the proof is exactly the same as indicated in [16, pp. 28-29, p.132]. To prove (2.7), it is enough in turn, to show that for N large enough,

$$g(z,\xi) + Q(z) + Q(\xi) > 1 + I_1, (z,\xi) \notin E_N \times E_N \tag{2.8}$$

(see [16, pp. 29-30] for deduction of (2.7) from (2.8)). But (2.8) is obvious for large N, since $g \geq 0, Q$ is bounded from below, and either $Q(z)$ or $Q(\xi)$ is larger than N. $\square$

Under additional assumptions on E and Q, one can strengthen (c) of Theorem 2.1:

Theorem 2.2
Assume (A.1), (A.2) and (A.4) - (A.6), that is, we only drop the geometrical condition (A.3) on E. Then (2.5) can be refined to

$$V^{\mu_t}(z) + Q(z) \geq c_t, \text{ all } z \in E, \tag{2.9}$$

so that (2.6) becomes

$$V^{\mu_t}(z) + Q(z) = c_t \text{ all } z \in S_{\mu_t}. \tag{2.10}$$

Moreover, Q is continuous on S_{μ_t}, and V^{μ_t} is continuous and bounded on G.
Proof
This is standard. By (2.5), the exceptional set

$$E' := \{z \in E : V^{\mu_t}(z) + Q(z) < c_t\}$$

has capacity 0, so it is thin at every point of E. Then the continuity of V^{μ_t} in the fine topology (see (2.2)) together with (A.2), (A.5) ensures, for any $z_0 \in E$, the existence of $\{z_n\} \subseteq E \backslash E'$ such that

$$(V^{\mu_t} + Q)(z_n) \to (V^{\mu_t} + Q)(z_0), n \to \infty.$$

Then (2.9) follows from (2.5). Since V^{μ_t} is l.s.c., while $c_t - Q$ is u.s.c. (upper semi-continuous), (2.10) shows that V^{μ_t} and Q are continuous on S_{μ_t}. Then V^{μ_t} is continuous in G (cf. [16, Thm. II.3.5] and recall that the Green potential of μ_t differs from the logarithmic potential by a harmonic function). The boundedness of V^{μ_t} in G follows by the maximum principle for Green potentials (cf. [16, Cor. II. 5.9]). $\square$

Finally, recall that for the case $Q \equiv 0$, the following classical result holds:

Theorem 2.3
Let K be a compact subset of G, with $\mathrm{cap}(K) > 0$. There exists a unique probability measure ω_K^G, supported on K and such that for some constant $c > 0$,

$$V^{\omega_K^G}(z) = c \text{ q.e. } z \in K; \tag{2.11}$$

$$V^{\omega_K^G}(z) \leq c \text{ all } z \in G. \tag{2.12}$$

We call ω_K^G the Green equilibrium measure for K. Furthermore if $E := K$ also satisfies (A.3), then

$$\mathrm{cap}\left(K \backslash S_{\omega_K^G}\right) = 0 \tag{2.13}$$

and the Green equilibrium measures formed for K and for $S_{\omega_K^G}$ coincide. Also, (2.11) holds at every regular point of K; if K is regular, then $S_{\omega_K^G} = K$ and $V^{\omega_K^G}$ is continuous in G.

The *Green capacity* of K (relative to G) is defined by

$$\mathrm{cap}_G(K) = c^{-1}, \tag{2.14}$$

where, of course, c is as in (2.11-12).

Next, for a measure σ supported on E, we set

$$c(\sigma) := c(\sigma, Q) := \min_{E} (V^{\sigma} + Q), \tag{2.15}$$

and

$$S^{\sigma} := S^{\sigma}(Q) := \{z \in E : V^{\sigma}(z) + Q(z) = c(\sigma)\}. \tag{2.16}$$

Notice that S^{σ} is a compact subset of G (by (A.1), (A.4) and (A.6)). We see from these definitions, that the equilibrium conditions (2.9), (2.10) are equivalent to the inclusion

$$S_{\sigma} \subseteq S^{\sigma}. \tag{2.17}$$

Hence, under the assumptions of Theorem 2.2, (2.17) holds with $\sigma = \mu_t$ (and with $c(\sigma)$ in (2.15) equal to c_t). Moreover, μ_t is the only measure in M_t that satisfies (2.17) (see [16, Theorem II.5.12]). Now we can formulate the main result. It will be convenient to use the abbreviations

$$\begin{aligned} S_t \quad &: \; = S_{\mu_t};\; S^t := S^{\mu_t};\; V^t := V^{\mu_t}; \\ \omega_t \quad &: \; = \omega^G_{S_{\mu_t}};\; \omega^t := \omega^G_{S^{\mu_t}}, \end{aligned} \tag{2.18}$$

and recall that c_t coincides with $c(\mu_t)$. Thus ω_t is the (unweighted, classical) Green equilibrium measure for the support $S_{\mu_t} = S_t$ of μ_t; and ω^t plays the same role for the set S^t where $V^{\mu_t} + Q = V^t + Q$ attains its minimum. Also, as μ_t is not affected if we replace Q by $Q + Const$, we assume that

$$\min_{E} Q = 0. \tag{2.19}$$

Theorem 2.4
Let Q be admissible on E and satisfy (2.19).
(a) The family $\{S_t\}_{t>0}$ is an increasing family of sets. Moreover, if we set

$$S_0 := \{z \in E : Q(z) = 0\},$$

then

$$S_0 = \bigcap_{t>0} S_t. \tag{2.20}$$

(b) There holds

$$S_t = \overline{\bigcup_{\tau<t} S_{\tau}} \subseteq \bigcap_{\tau>t} S_{\tau} = S^t, t > 0,$$

and there exists a countable set $N \subset (0, \infty)$ such that

$$cap\left(S^t \backslash S_t\right) = 0, t \notin N.$$

(c) The equilibrium measure μ_t and the extremal constant c_t have the representations

$$\mu_t = \int_0^t \omega_{\tau} d\tau;\; c_t = \int_0^t \frac{1}{\mathrm{cap}_G S_{\tau}} d\tau. \tag{2.21}$$

(d) The external field Q has the representation

$$Q(z) = \int_0^{\infty} \left(\frac{1}{\mathrm{cap}_G S_t} - V^{\omega_t}(z) \right) dt, z \in \bigcup_{t>0} S_t. \tag{2.22}$$

Remarks
(a) Let

$$S_\infty := \bigcup_{t>0} S_t.$$

It follows from Theorem 2.2 that if $S_\infty \neq E$, then

$$Q(z) \geq \sup_t \left\{ c_t - V^t(z) \right\}, z \in E \backslash S_\infty, \tag{2.23}$$

and one can assign Q an arbitrary value on $E\backslash S_\infty$ (but subject to (2.23)) without affecting the family $\{S_\tau\}_{\tau>0}$. Obviously,

$$\{z : Q(z) = \infty\} \subseteq E\backslash S_\infty,$$

but it is worth noting that there may exist $z \in E\backslash S_\infty$ with $Q(z) < \infty$.
(b) The convergence of the integral for c_t in (2.21) implies that $\text{cap}_G S_\tau$ cannot approach 0 too rapidly as $\tau \to 0+$; in particular it is not possible that

$$\text{cap}_G S_\tau = O(\tau), \tau \to 0+.$$

3. Extremal Properties of c_t, S_t

We first establish

Theorem 3.1
Under the assumptions of Theorem 2.2, we have

$$c_t = c(\mu_t) = \sup\{c(\sigma) : \sigma \in M_\tau, \tau \leq t\}. \tag{3.1}$$

Moreover, if (A.3) is satisfied, that is, Q is admissible, then equality holds in (3.1) only for $\sigma = \mu_t$.
Proof
For any measure σ on E, we have, by the definition (2.15) of $c(\sigma)$, and by Theorem 2.2:

$$V^\sigma + Q - c(\sigma) \geq 0 = V^t + Q - c_t \text{ on } S_t. \tag{3.2}$$

Hence

$$V^\sigma \geq V^t + c(\sigma) - c_t \text{ on } S_t. \tag{3.3}$$

Let $\tau > 0$ and $\sigma \in M_\tau$. Then by (2.12), (2.14),

$$\int V^\sigma d\omega_t = \int V^{\omega_t} d\sigma \leq \frac{\tau}{\text{cap}_G S_t}.$$

Similarly by (2.11),

$$\int V^t d\omega_t = \int V^{\omega_t} d\mu_t = \frac{t}{\text{cap}_G S_t}.$$

(Note that although E is regular, S_t need not be regular, so that (2.11) holds q.e. in S_t. However μ_t has finite energy, hence it is C-absolutely continuous, that is, sets of capacity 0 have zero μ_t-measure). On integrating (3.3) against ω_t, we thus obtain

$$c_t - c(\sigma) \geq \frac{t - \tau}{\text{cap}_G S_t}. \tag{3.4}$$

This holds for any τ, and if $\tau \le t$, we get (3.1). Next, if $\sigma = \mu_\tau$, where $\tau > 0$, we obtain that

$$c_t - c_\tau \ge \frac{t-\tau}{\mathrm{cap}_G S_t}. \tag{3.5}$$

Reversing the roles of t and τ, we also get

$$c_t - c_\tau \le \frac{t-\tau}{\mathrm{cap}_G S_\tau}. \tag{3.6}$$

(We shall use (3.5) and (3.6) later on). Assume now that $\sigma \in M_\tau, \tau \le t$, and $c(\sigma) = c_t$. Then (3.4) shows that $\tau = t$. Also, (3.3) then becomes

$$V^\sigma \ge V^t \text{ on } S_t.$$

Integrating this against ω_t, we obtain as before

$$\frac{t}{\mathrm{cap}_G S_t} \ge \int V^\sigma d\omega_t \ge \int V^t d\omega_t = \frac{t}{\mathrm{cap}_G S_t}. \tag{3.7}$$

Hence

$$\int \left(V^\sigma - V^t\right) d\omega_t = 0,$$

and since the integrand is non-negative, the set

$$K := S_{\omega_t} \cap \left\{z : \left(V^\sigma - V^t\right)(z) > 0\right\}$$

has ω_t-measure 0. On the other hand, K being an intersection of S_{ω_t} with an open set (recall that V^t is continuous while V^σ is l.s.c.) must have positive ω_t-measure, if it is non-empty. We have thus showed that K is empty, so

$$V^t(z) = V^\sigma(z) \text{ all } z \in S_{\omega_t}. \tag{3.8}$$

Now the assumption (A.3) comes into play. It implies (via the maximum principle for harmonic functions) that strict inequality holds:

$$V^{\omega_t}(z) < \frac{1}{\mathrm{cap}_G S_t}, z \in G \backslash S_{\omega_t}.$$

Therefore, if $S_\sigma \nsubseteq S_{\omega_t}$,

$$\int V^\sigma d\omega_t = \int V^{\omega_t} d\sigma < \frac{t}{\mathrm{cap}_G S_t}$$

and we obtain a contradicition to (3.7). So $S_\sigma \subseteq S_{\omega_t}$ and then (3.8) shows that V^σ is bounded on S_σ, hence has finite energy. Since we have simultaneously (from (3.8))

$$V^\sigma \le V^t \text{ on } S_\sigma \text{ and } V^t \le V^\sigma \text{ on } S_t,$$

we conclude by the principle of domination for Green potentials (cf. [16, Theorem II.5.8]) that $V^\sigma = V^t$ in G and hence $\sigma = \mu_t$. $\square$

Next, given a compact $K \subseteq E$ of positive capacity, we set for $t > 0$,

$$F_t(K) := F_t(K, Q) := -\frac{t}{\mathrm{cap}_G K} - \int Q d\omega_K. \tag{3.9}$$

This functional was introduced in [10] and it is an analogue of the so-called F functional of Mhaskar and Saff [16, p.194]. The latter plays an important role in the determination of the support of the equilibrium measure. The functional (3.9) plays a similar role in the context of this paper - see the example in Section 5. For the following result, we recall the notation (2.18).

Theorem 3.2
Let $K \subseteq E$ be compact, with cap$(K) > 0$. Under the assumptions of Theorem 2.2, there holds

$$F_t(K) \leq F_t(S_t) = -c_t. \tag{3.10}$$

Moreover, if (A.3) is also satisfied, then equality occurs in (3.10) iff

$$S_t \subseteq S_{\omega_K} \subseteq S^t. \tag{3.11}$$

Proof
This is very similar to the proof of Theorem 3.1. On integrating

$$V^t + Q \geq c_t \text{ in } E$$

against ω_K, we obtain

$$c_t \leq \int \left(V^t + Q\right) d\omega_K \leq \frac{t}{\text{cap}_G K} + \int Q d\omega_K = -F_t(K),$$

with equalities if $K = S_t$. So we have (3.10). Moreover, equalities can occur iff

$$V^t + Q = c_t, \ \omega_K \text{ a.e.}, \tag{3.12}$$

and

$$V^{\omega_K} = \frac{1}{\text{cap}_G K}, \quad \mu_t \text{ a.e.} \tag{3.13}$$

Now as S_{ω_K} cannot contain isolated points (for example, by (2.12)), we see that (3.12) must hold on a dense subset of S_{ω_K}, that is this subset is contained in S^t. Since S^t is closed, we obtain the second inclusion in (3.11). Note that we did not use (A.3) here. Similarly, equality (3.13) must hold on a dense subset of S_t. Also, due to (A.3), we have

$$V^{\omega_K} < \frac{1}{\text{cap}_G K}$$

outside S_{ω_K}, so that the above dense subset of S_t is contained in S_{ω_K}. Since the latter set is closed, we conclude that $S_t \subseteq S_{\omega_K}$. $\square$

Now, for any $\varepsilon > 0$, the set

$$E_\varepsilon := \{z \in E : Q(z) \leq \varepsilon\}$$

is compact, and it has positive capacity by (A.5), while $\min_E Q = 0$ (recall (2.19)). Then (3.10) gives, with $K = E_\varepsilon$,

$$c_t \leq -F_t(E_\varepsilon) \leq \varepsilon + \frac{t}{\text{cap}_G E_\varepsilon}.$$

Here $c_t \geq 0$, since it is the minimum of the non-negative function $V^t + Q$ - recall that the Green's function $g(z, \xi)$ is non-negative. On letting first $t \to 0$ and then $\varepsilon \to 0$, we obtain

$$\lim_{t \to 0+} c_t = 0. \tag{3.14}$$

Next, we have seen above, that the equilibrium relations (2.9), (2.10) of Theorem 2.2 can be written in the form

$$\left(S_{\mu_t} =\right) S_t \subseteq S^t \left(= S^{\mu_t}\right).$$

It is easy to construct Q for which strict inclusion occurs. Then $V^t + Q$ may attain its minimum on E also outside S_t. This can never happen for other $\sigma \in M_t$. More precisely, we have

Theorem 3.3
Let Q be admissible on E. For any measure $\sigma \in M_\tau$ with $\tau \leq t$ and $\sigma \neq \mu_t$, we have

$$S^\sigma \subseteq S_t. \tag{3.15}$$

Proof
Consider the function

$$u(z) := [V^\sigma(z) - c(\sigma)] - \left[V^t(z) - c_t\right],$$

which is superharmonic in $G \backslash S_t$ and bounded below ($V^\sigma \geq 0$ while V^t is bounded). Furthermore, we have by (2.1), for q.e. $x \in \partial G$,

$$\lim_{z \to x} u(z) = 0 + c_t - c(\sigma) \geq 0,$$

the last inequality following by Theorem 3.1. Next, as V^t is continuous and V^σ is l.s.c., we obtain for $x \in S_t$,

$$\liminf_{z \to x, z \in G \backslash S_t} u(z) \geq u(x) \geq 0$$

(recall (3.2)). Since $\sigma \neq \mu_t$, u is non-constant and the minimum principle for superharmonic functions yields

$$u(z) > 0, z \in G \backslash S_t.$$

(We need (A.3) here). Since $u \leq 0$ on S^σ (recall (2.15), (2.9)), we obtain (3.15). $\square$

We conclude this section with a concavity property of the functions c_t and $c_t - V^t(z)$, with z fixed.

Theorem 3.4
Assume the conditions of Theorem 2.2 and fix $z \in G$. Then the functions c_t and $c_t - V^t(z)$ are concave functions of t.
Proof
Let $t = \alpha t_1 + (1-\alpha) t_2$, where $\alpha \in (0,1)$ and consider the function

$$u(z) := \alpha V^{t_1}(z) + (1-\alpha) V^{t_2}(z) - V^t(z), z \in G. \tag{3.16}$$

By Theorem 2.2,

$$u(z) \geq \alpha c_{t_1} + (1-\alpha) c_{t_2} - c_t, z \in S_t, \tag{3.17}$$

so that

$$\int u d\omega_t \geq \alpha c_{t_1} + (1-\alpha) c_{t_2} - c_t.$$

On the other hand, an integration of (3.16) yields

$$\int u d\omega_t \leq [\alpha t_1 + (1-\alpha) t_2 - t] \frac{1}{\mathrm{cap}_G S_t} = 0.$$

We have used here the equilibrium relations of Theorem 2.3. Therefore

$$\alpha c_{t_1} + (1-\alpha) c_{t_2} - c_t \leq 0 \tag{3.18}$$

and the concavity of c_t follows. Now u is superharmonic in $G\backslash S_t$, and tends to 0 as $z \to z_0 \in \partial G$, at least for q.e. z_0. Also, u is continuous, bounded, and is bounded below on S_t by a non-positive constant (see (3.17), (3.18)). Hence (3.17) holds for all $z \in G$. After substituting there $u(z)$ from (3.16) and rearrangement, we obtain that $c_t - V^t(z)$ is concave. □

4. Proof of Theorem 2.4

Proof of part (a) of Theorem 2.4
We start with the proof of (2.20). Assume that $z \in S_t$, for all $t > 0$. Then as $V^t > 0$ and $Q \geq 0$ by our assumption (2.19), we obtain from (2.10) that

$$0 \leq Q(z) \leq c_t \text{ for all } t > 0.$$

Then (3.14) gives $Q(z) = 0$, that is $z \in S_0$. This proves the inclusion

$$\bigcap_{t>0} S_t \subseteq S_0.$$

For the other direction, we consider two cases.
Case 1: E is compact
Let $0 < \varepsilon < t$. Since E is regular, we have for all $z \in E$,

$$V^{\varepsilon\omega_E}(z) + Q(z) = \frac{\varepsilon}{\text{cap}_G E} + Q(z).$$

Hence the left-hand side attains its minimum on E exactly for $z \in S_0$. This means that

$$S_0 = S^{\varepsilon\omega_E} \text{ and } c(\varepsilon\omega_E) = \frac{\varepsilon}{\text{cap}_G E} \tag{4.1}$$

(recall (2.15), (2.16)). Then the inclusion

$$S_0 \subseteq S_t, t > 0, \tag{4.2}$$

follows by Theorem 3.3 (obviously $\varepsilon\omega_E \neq \mu_t$ as $\varepsilon < t$).
Case 2: E is not compact
Then $Q(z) \to \infty$ as $z \to \partial G$ (or as $z \to \infty$). Hence one can find a bounded open set G_1 with $\overline{G_1} \subset G$ such that

$$Q(z) \geq 1, z \in E \cap (G\backslash G_1).$$

We set

$$K := E \cap \overline{G_1}$$

and note that K is a compact subset of G, and every $z \in K$ that belongs to G_1 is a regular point for K. Thus, for $\varepsilon > 0$,

$$V^{\varepsilon\omega_K}(z) + Q(z) = \frac{\varepsilon}{\text{cap}_G K} + Q(z), z \in E \cap G_1,$$

while for $z \in E \cap (G\backslash G_1)$, the left-hand side is at least $Q(z)$, that is ≥ 1. It then follows, as in Case 1, that if ε is small enough, (4.1) holds with E replaced by K and we deduce (4.2) as before.

Next, we prove that the family $\{S_t\}$ is increasing in t. We shall prove a stronger statement, namely

$$S_t \subseteq S^t \subseteq S_{t+\delta} \ \forall t, \delta > 0. \tag{4.3}$$

The first inclusion is clear (recall (2.17) and the remarks thereafter) and the second follows from Theorem 3.3, if we replace t there by $t+\delta$ and take $\sigma := \mu_t$. $\square$

Proof of part (b) of Theorem 2.4
We first show that the family $\{\mu_t\}_{t>0}$ is increasing, and continuous in the weak $*$ sense. Both assertions follow from the relation

$$\mu_{t+\delta} - \mu_t \in M_\delta \ \forall t, \delta > 0. \tag{4.4}$$

The proof of (4.4) is exactly the same as in [4], but we include the proof for the reader's convenience. Let

$$Q_t := V^t + Q - c_t.$$

Then Q_t is also admissible on E (see Remark (a) after the definition of admissible Q), $Q_t \geq 0$ on E, and $Q_t = 0$ precisely on S^t. Thus

$$S_0(Q_t) = S^t.$$

Let $\mu := \mu_\delta(Q_t)$ be the equilibrium measure of mass δ for Q_t. By what was already proved, we have

$$S_t \subseteq S^t = S_0(Q_t) \subseteq S_\mu.$$

(The last inclusion follows from Theorem 2.4(a)). Hence

$$S_{\mu+\mu_t} = S_\mu,$$

so that the equilibrium relations for μ can be stated as

$$V^\mu + Q_t = const = \min_E (V^\mu + Q_t) \text{ on } S_{\mu+\mu_t}.$$

Inserting here Q_t, we arrive at

$$V^{\mu+\mu_t} + Q = const = \min_E \left(V^{\mu+\mu_t} + Q\right) \text{ on } S_{\mu+\mu_t}.$$

This means that the measure $\mu + \mu_t$ (of mass $t+\delta$) is the equilibrium measure $\mu_{t+\delta}$ for the original Q. Hence (4.4) follows. $\square$

Now let

$$S_{t-0} := \bigcup_{\tau<t} S_\tau; \ S_{t+0} := \bigcap_{\tau>t} S_\tau \tag{4.5}$$

so that (see (4.3))

$$S_{t-0} \subseteq \overline{S_{t-0}} \subseteq S_t \subseteq S^t \subseteq S_{t+0} \tag{4.6}$$

Since μ_τ converges weakly to μ_t as $\tau \to t$ (by (4.4)), we must have

$$S_t \subseteq \overline{\bigcup_{\tau<t} S_\tau} = \overline{S_{t-0}}. \tag{4.7}$$

Next, if $x \in S_{t+0}$, then $x \in S_\tau, \tau > t$, so that

$$V^\tau(x) + Q(x) = c_\tau, \tau > t. \tag{4.8}$$

By Theorem 3.4, both c_τ and $V^\tau(x)$ are concave functions of τ, therefore they are continuous, and if we let in (4.8), $\tau \to t+0$, we obtain

$$V^t(x) + Q(x) = c_t, \forall x \in S_{t+0}.$$

This shows that $S_{t+0} \subseteq S^t$, and together with (4.6), (4.7), we have the first statement of part (b), namely

$$S_t = \overline{S_{t-0}} \subseteq S_{t+0} = S^t.$$

Next, by well known properties of capacities (see [15, p. 128, Theorem 5.13 (a), (b)] for a proof for classical capacities, but the same proof works for Green capacities), we have

$$\lim_{\tau \to t \pm 0} \mathrm{cap}_G S_\tau = \mathrm{cap}_G S_{t\pm 0}. \tag{4.9}$$

Since the family $\{S_\tau\}_{\tau>0}$ is increasing, $\mathrm{cap}_G S_\tau$ is an increasing function of τ. Hence it is continuous if $t \notin N$, some countable set N. Then (4.9), (4.6) show that for $t \notin N$,

$$\mathrm{cap}_G S_{t-0} = \mathrm{cap}_G S_t = \mathrm{cap}_G S^t = \mathrm{cap}_G S_{t+0}. \tag{4.10}$$

Since $S_t \subseteq S^t$, this implies that the Green equilibrium measure formed for S^t coincides with that formed for S_t. Therefore (see (2.13) of Theorem 2.3 and recall that we are assuming (A.3) in the present proof), we have

$$\mathrm{cap}\left(S^t \backslash S_t\right) = 0$$

and this completes the proof of (b). Another consequence of (4.10) is that for $t \notin N$,

$$\omega_\tau \text{ converges weakly to } \omega_t \text{ as } \tau \to t. \tag{4.11}$$

Indeed, let $\tau_n \nearrow t, n \to \infty$. Then $\omega_{\tau_n} \to \omega_{S_{t-0}}$ in the weak $*$ sense, by Lemma 2.10 in [8, p. 154]. Moreover the proof of that lemma shows that the first equality of (4.10) ensures that $\omega_{S_{t-0}} = \omega_t$. Thus we get (4.11) provided $\tau \to t-0$. Now let $\tau_n \searrow t, n \to \infty$, and assume that ω_{τ_n} converges weakly to σ. Clearly

$$S_\sigma \subseteq S_{t+0} = S^t.$$

Also,

$$\mathrm{cap}_G S_{\tau_n} \to \mathrm{cap}_G S_t.$$

Therefore the equilibrium relations (2.11), (2.12) yield (via the lower envelope theorem and the principle of descent), that $\sigma = \omega_{S^t}$. But $\omega_{S^t} = \omega_{S_t}$, as we have already mentioned, and this completes the proof of (4.11). $\square$

Proof of parts (c), (d) of Theorem 2.4

By (3.5), (3.6) and the above properties of $\mathrm{cap}_G S_t$, we obtain that

$$\frac{d}{dt} c_t = \frac{1}{\mathrm{cap}_G S_t}, t \notin \mathcal{N}.$$

Being concave, c_t is absolutely continuous, and in view of (3.14), we conclude that

$$c_t = \int_0^t \frac{1}{\mathrm{cap}_G S_\tau} d\tau. \tag{4.12}$$

To show that μ_t and Q have the desired representations, one may proceed exactly as in [4, pp. 800-801], replacing there g_t (the Green function for S_t with pole at ∞), by $\frac{1}{\mathrm{cap}_G S_t} - V^{\omega_t}$, in the present notation.

We suggest, however, a different proof. We shall show that for $t, \delta > 0$,

$$\omega_{t+\delta|S_t} \leq \frac{1}{\delta}\left[\mu_{t+\delta|S_t} - \mu_t\right] \leq \omega_t, \tag{4.13}$$

where $\nu_{|S}$ denotes the restriction of the measure ν to S. Based on this, we complete the proof of Theorem 2.4 as follows. By (4.13), (with a similar inequality for $t-\delta$ instead of t), and (4.11), there holds

$$\frac{d\mu_t}{dt} = \omega_t, t \notin \mathcal{N}.$$

Since μ_t is absolutely continuous in t (recall (4.4)), we obtain the desired representation

$$\mu_t = \int_0^t \omega_\tau d\tau.$$

Then the equilibrium relation (2.10) gives (see (1.6) and (4.12)) that

$$Q(z) = \int_0^t \left(\frac{1}{\mathrm{cap}_G S_\tau} - V^{\omega_\tau}(z) \right) d\tau, z \in S_\tau,$$

and since $\{S_t\}$ is increasing, while $V^{\omega_\tau}(z) = \frac{1}{\mathrm{cap}_G S_\tau}$ q.e. in S_t for $\tau \geq t$, we obtain the last statement (2.22) of Theorem 2.4. $\square$

Proof of (4.13)

For the case of logarithmic potentials this result was proved by Totik (cf. [16, Theorem IV.4.9] or [17, Lemma 5.7]). The proof is basically the same for our case, but some changes are required. Also our notation is different from that in [16], so we provide the details. The main ingredient is the following analogue of Theorem IV.4.5 in [16].

Theorem

Let μ, ν be measures of compact support in G, having finite potentials. Assume that for some constant c we have

$$V^\mu(z) \leq V^\nu(z) + c \;\forall z \in G. \tag{4.14}$$

Let A be a subset of G in which equality holds in (4.14). Then

$$\nu_{|A} \leq \mu_{|A}.$$

Assuming this theorem, we proceed as follows. Since $S_t \subseteq S_{t+\delta}$, we have, by Theorems 2.2 and 2.3,

$$\left(V^t - c_t\right) + \delta \left(V^{\omega_{t+\delta}} - \frac{1}{\mathrm{cap}_G S_{t+\delta}} \right) \geq V^{t+\delta} - c_{t+\delta}, \text{ q.e. in } S_{t+\delta}. \tag{4.15}$$

Furthermore, equality holds q.e. in S_t. Therefore, if we set

$$a := c_{t+\delta} - c_t - \frac{\delta}{\mathrm{cap}_G S_{t+\delta}}$$

we can rewrite (4.15) as

$$V^{\mu_{t+\delta}} \leq V^{\mu_t + \delta\omega_{t+\delta}} + a, \text{ q.e. in } S_{t+\delta} \tag{4.16}$$

with equality q.e. in S_t. Now, (3.5) ensures that $a \geq 0$. Also $\mu_{t+\delta}$ is $C-$absolutely continuous, hence (4.16) holds $\mu_{t+\delta}$ a.e., and we conclude by the principle of domination (cf. [16, Theorem II.5.8]) that (4.16) holds everywhere in G. Since equality holds q.e. in S_t, we obtain by the above theorem that

$$(\mu_t + \delta\omega_{t+\delta})_{|S_t} \leq (\mu_{t+\delta})_{|S_t}.$$

(Note that all measures involved are $C-$absolutely continuous, hence they vanish on sets of capacity 0). So we have the first inequality in (4.13). The proof of the second is similar: we have

$$\left(V^t - c_t\right) + \delta\left(V^{\omega_t} - \frac{1}{\mathrm{cap}_G S_t}\right) \leq V^{t+\delta} - c_{t+\delta}, \text{ q.e. in } S_t$$

(actually equality holds q.e. in S_t). On setting

$$b := \frac{\delta}{\mathrm{cap}_G S_t} - (c_{t+\delta} - c_t)$$

we obtain that

$$V^{\mu_t+\delta\omega_t} \leq V^{\mu_{t+\delta}} + b, \text{ q.e. in } S_t,$$

with actual equality q.e. in S_t. Here $b \geq 0$, by (3.6). We then continue as before, and obtain

$$\left(\mu_{t+\delta}\right)_{|S_t} \leq (\mu_t + \delta\omega_t)_{|S_t}$$

and this is the second inequality in (4.13).

Thus it remains to prove the above theorem. Since the Green potentials V^μ, V^ν differ from the corresponding logarithmic ones U^μ, U^ν by a harmonic function, we see that (4.14) is equivalent to

$$U^\mu(z) \leq U^\nu(z) + u(z), \forall z \in G,$$

where $u(z)$ is harmonic in G. If $u(z)$ were a constant c say, this would be Theorem IV.4.5 in [16]. However, the only property of c used in the proof of that Theorem is, that the average of c over a circle centred at some point is independent of the radius of this circle. Since harmonic functions enjoy this property, we see that Theorem IV.4.5 actually was proved in [16] for c replaced by a harmonic function. This completes the proof.

5. An Example

Let

$$G := \{z : \mathrm{Re}\, z > 0\}; E := (0, \infty);$$

and let Q be convex. Then the convexity of Q and the convexity of the Green's function for the right-half plane guarantee that S_t is a compact interval, say,

$$S_t = [a_t, b_t] \subset (0, \infty).$$

(This follows just as for logarithmic potentials). We place a symmetry hypothesis on Q, which is akin to that of evenness when dealing with logarithmic potentials:

$$Q(x) = Q\left(x^{-1}\right), x \in (0, \infty).$$

Then the uniqueness of μ_t gives

$$a_t b_t = 1.$$

Now if $0 < a < 1$,

$$\mathrm{cap}_G\left[a, a^{-1}\right] = \mathrm{cap}_G\left[a^2, 1\right] = \frac{K'\left(a^2\right)}{\pi K\left(a^2\right)},$$

where K and K' are complete elliptic integrals:

$$K(k)=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}};K'(k)=K(k');k^2+k'^2=1.$$

Also,

$$d\omega_{[a,a^{-1}]}=\frac{1}{K'(a^2)}\frac{dx}{\sqrt{(x^2-a^2)(1-a^2x^2)}}.$$

(All these may be easily derived from Example 5.14 in [16, pp.133-134], by mapping G conformally onto the unit ball in such a way that $\left[a,a^{-1}\right]$ or $\left[a^2,1\right]$ is mapped onto $[-\alpha,\alpha]$ for some $0<\alpha<1$. One uses the conformal map to transform the equilibrium density w.r.t. the unit ball to that w.r.t. G. See [11] for a very similar situation; some of the necessary calculations appear in [1, p.121 ff.].) Thus for the set $\left[a,a^{-1}\right]$, F_t is

$$-F_t(a):=-F_t\left(\left[a,a^{-1}\right]\right)=t\frac{\pi K\left(a^2\right)}{K'\left(a^2\right)}+\frac{1}{K'\left(a^2\right)}\int_a^{a^{-1}} Q(x)\frac{dx}{\sqrt{(x^2-a^2)(1-a^2x^2)}}.$$

If we take

$$Q(x):=x+x^{-1},$$

then

$$\begin{aligned}-F_t(a) &= \frac{\pi}{K'(a^2)}\left\{tK\left(a^2\right)+\frac{1}{a}\right\}\\ &= \frac{\pi}{K'(k)}\left\{tK(k)+\frac{1}{\sqrt{k}}\right\},\end{aligned}$$

with $k:=a^2$. Differentiating with respect to k and setting $=0$ gives

$$\left[t\frac{dK}{dk}-\frac{1}{2k^{3/2}}\right]K'(k)-\left[tK(k)+\frac{1}{\sqrt{k}}\right]\frac{dK'}{dk}=0. \tag{5.1}$$

Since [5, 8.123.2, p.907]

$$\frac{dK}{dk}=\frac{E}{kk'^2}-\frac{K}{k},$$

where

$$E(k):=\int_0^1\sqrt{\frac{1-k^2x^2}{1-x^2}}dx$$

is the complete elliptic integral of the second kind, we also obtain

$$\frac{dK'}{dk}=\frac{dK}{dk'}(k')\frac{dk'}{dk}=-\frac{k}{k'}\left\{\frac{E'}{k'k^2}-\frac{K'}{k'}\right\}.$$

Then (5.1) can be rearranged to

$$\left[t\left\{\frac{E}{kk'^2}-\frac{K}{k}\right\}-\frac{1}{2k^{3/2}}\right]K'+\left(tK+\frac{1}{\sqrt{k}}\right)\left\{\frac{E'}{kk'^2}-\frac{K'k}{k'^2}\right\}=0,$$

or

$$\begin{aligned}&t\frac{1}{kk'^2}\left[EK'+E'K-KK'\right]\\ =\;& K'\frac{k'^2+2k^2}{2k^{3/2}k'^2}-\frac{E'}{k^{3/2}k'^2}=K'\frac{1+k^2}{2k^{3/2}k'^2}-\frac{E'}{k^{3/2}k'^2}.\end{aligned}$$

Since the term in [] in the left-hand side is $\pi/2$ [5, 8.122, p.907], we obtain that the defining equation for a_t is

$$\pi t = K' \frac{1+k^2}{\sqrt{k}} - 2\frac{E'}{\sqrt{k}}; k = a_t^2,$$

that is,

$$\pi t = a_t K'\left(a_t^2\right)\left(a_t^{-2} + a_t^2\right) - 2E'\left(a_t^2\right)/a_t.$$

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