Orthogonal Polynomials for weights $x^{2\rho}\exp\left(-2Q\left(x\right)\right)$

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Abstract

Let $\rho > -\frac{1}{2}$ and Q be an increasing function on $(0, d)$, where $0 < d \leq \infty$. We discuss bounds on orthogonal polynomials and related quantities associated with the weight $x^{2\rho}\exp\left(-2Q\left(x\right)\right)$ on I. For example, the results apply to the weights

$$w\left(x\right) = x^{2\rho}\exp\left(-x^{\alpha}\right) \text{ on } (0, \infty)$$

where $\alpha > \frac{1}{2}$, or

$$w\left(x\right) = x^{2\rho}\exp\left(-\left(1-x\right)^{-\alpha}\right) \text{ on } (0, 1)$$

where $\alpha > 0$.

1 Introduction

Let $I = (0, d)$, $0 < d \leq \infty$ and $\rho > -\frac{1}{2}$. Let $Q : I \to [0, \infty)$ be increasing and

$$\begin{aligned} W(x) &= \exp(-Q(x)); \\ W_\rho(x) &= x^\rho \exp(-Q(x)). \end{aligned}$$

Assume that all the power moments

$$\int_I x^j W^2(x)\, dx, j = 0, 1, 2, ...$$

converge. Then we can define orthonormal polynomials (p_n) such that

$$\int_I p_n p_m W_\rho^2 = \delta_{mn}.$$

One may think of these orthonormal polynomials as generalized Laguerre polynomials.

In this paper, we announce bounds on (p_n) and related quantities, that may be thought of as one-sided analogues of our recent results [1] for even and non-even weights on an interval (c, d) where $-\infty \leq c < 0 < d \leq \infty$. Of course fairly often the passage from weights on an interval $I = (0, d)$ to their even cousins on the interval $\left(-\sqrt{d}, \sqrt{d}\right)$ is via the substitution $t = x^2$. Thus

$$x \in \left(-\sqrt{d}, \sqrt{d}\right) \Leftrightarrow t \in (0, d).$$

In particular, this is very often done in relating orthonormal Laguerre polynomials $L_n^{(-1/2)}(t)$ for the Laguerre weight $t^{1/2}e^{-t}, t \in (0, \infty)$ and orthonormal Hermite polynomials $H_{2n}(x)$ for the weight e^{-x^2} on $(-\infty, \infty)$. They admit the identity

$$H_{2n}(x) = L_n^{(-1/2)}\left(x^2\right) = L_n^{(-1/2)}(t).$$

(Our notation here is not the standard one). However, for other Laguerre weights $t^\alpha e^{-t}, t \in (0, \infty)$, with $\alpha \neq \pm\frac{1}{2}$, there is no such simple relationship. It is for that reason we believe in the value of the study. In fact even for $L_n^{\left(\pm\frac{1}{2}\right)}$, there are features that distinguish Laguerre polynomials from Hermite polynomials: 0 plays a special role, and maxima of weighted Laguerre polynomials occur close to 0 rather than the largest zero. In any event, we cannot deduce our results for the weights $x^{2\rho}e^{-2Q(x)}$ from their even cousins.

The Mhaskar-Rakhmanov-Saff numbers a_s, $s>0$, play an important role in all quantitative analysis of orthogonal polynomials associated with exponential weights [4], [3]. In our setting, a_s is the (positive) root of the equation

$$s=\frac{2}{\pi}\int_0^1 \frac{a_s x Q'(a_s x)}{\sqrt{x(1-x)}}dx, s>0. \tag{1}$$

It is uniquely defined if, for example, $xQ'(x)$ is increasing on $(0,d)$. One of its features is the Mhaskar-Saff identity

$$\| PW \|_{L_\infty[0,d)}=\| PW \|_{L_\infty[0,a_n]},$$

valid for all polynomials P of degree $\leq n$. Typically, the largest zero of p_n is quite close to a_n.

An important example is

$$Q_\alpha(x)=x^\alpha, x\in[0,\infty), \tag{2}$$

for which (1) gives

$$a_s=s^{1/\alpha}\left[\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}\right]^{1/\alpha}, s>0.$$

In particular the case $\alpha=1$ is the Laguerre case. Our results apply to the case $\alpha>\frac{1}{2}$: the case $\alpha=\frac{1}{2}$ corresponds to the boundary between determinate and indeterminate moment problems, and certain quantities exhibit different behaviour or become much more difficult to estimate.

A more general example on $[0,\infty)$ to which our results apply is

$$Q_{k,\alpha}(x)=\exp_k(x^\alpha), x\in[0,\infty), \tag{3}$$

where $\alpha>\frac{1}{2}$, $k\geq 0$, and $\exp_0(x)=x$ while $\exp_k=\exp(\exp(\ldots\exp()))$ denotes the kth iterated exponential. For $k\geq 1$, a_s grows like $(\log_k s)^{1/\alpha}$, where $\log_k$ denotes the kth iterated logarithm.

As a final example, we mention one on $[0,1)$:

$$Q^{k,\alpha}(x)=\exp_k\left((1-x)^{-\alpha}\right), x\in[0,1), \tag{4}$$

where $\alpha > 0, k \geq 0$. Note that in all three examples we have mentioned, $Q(x)$ increases on $[0, d)$ and grows "sufficiently rapidly" to ∞ at d.

For a class of weights $\mathcal{L}(C^2+)$ that includes all the above examples, we have proved bounds that hold over all of I. We shall define this class at the end of this section, and for the moment just assume that $W = e^{-Q}$ belongs to this class. We need a little more notation: we set

$$T(x) = xQ'(x)/Q(x), x \in (0, d), \tag{5}$$

and

$$\eta_n = (nT(a_n))^{-2/3}, n \geq 1. \tag{6}$$

We also use $\sim$ in the customary way of orthogonal polynomials: given sequences $(f_n), (g_n)$, we write

$$f_n \sim g_n$$

if the ratio f_n/g_n is bounded above and below by positive constants independent of n.

Theorem 1

Let $\rho > -\frac{1}{2}$ and $W = e^{-Q} \in L$. Let (p_n) denote the orthonormal polynomials for the weight W_ρ^2 on $I = [0, d)$. Then for $n \geq 1$,

$$\sup_{x \in I} p_n^2(x)\left(x + \frac{a_n}{n^2}\right)^{2\rho} W^2(x)\left[\left(x + \frac{a_n}{n^2}\right)(|a_n - x| + \eta_n)\right]^{1/2} \sim 1 \tag{7}$$

and

$$\sup_{x \in I} p_n^2(x)\left(x + \frac{a_n}{n^2}\right)^{2\rho} W^2(x) \sim \left(\frac{n}{a_n}\right)^{1/2}. \tag{8}$$

An interesting feature here, which already occurs for Laguerre weights, is that the sup in (8) is attained for x close to 0 rather than for x close to a_n. This illustrates the special nature of 0 for "one-sided" weights. One of our auxiliary results is a restricted range inequality, which may be thought of as an L_p analogue of the Mhaskar-Saff identity:

Theorem 2

Let $0 < p < \infty, t > 0$ and $\rho > -\frac{1}{p}$. Then for all polynomials P of degree $\leq t - \rho - \frac{3}{2p}$,

$$\| PW_\rho \|_{L_p[a_t,d)} < \| PW_\rho \|_{L_p[0,a_t]}. \tag{9}$$

For $p = \infty$, this holds with $<$ replaced by $\leq$.

From this, and estimates on Christoffel functions, we can derive bounds on the zeros

$$x_{nn} < x_{n-1,n} < x_{n-2,n} < \ldots < x_{1n}$$

of the orthogonal polynomial p_n:

Theorem 3
We have

$$x_{1n} < a_{n+\rho+\frac{1}{4}}. \tag{10}$$

Moreover,

$$x_{nn} \sim \frac{a_n}{n^2} \tag{11}$$

and

$$1 - \frac{x_{1n}}{a_n} \sim \eta_n. \tag{12}$$

We also obtained estimates on spacing between successive zeros, equilibrium densities, Christoffel functions, and Markov-Bernstein inequalities [2]. We shall outline our proof of the bounds of the orthogonal polynomials in Section 2. We close this section with an explicit class of weights, for which all three results above, hold. See [2] for larger classes of weights to which these results apply.

Definition 4
Let $W = e^{-Q}$, where $Q : I \to [0, \infty)$ satisifes the following properties:
(a) $\sqrt{x}Q'(x)$ is continuous in $(0, d)$, with limit 0 at 0;
(b) Q'' exists in $(0, d)$ while $Q^*(x) := Q(x^2)$ is convex in $\left(0, \sqrt{d}\right)$;
(c) $Q(0) = 0$ and $\lim_{t \to d} Q(t) = \infty$;
(d) The function T above is "quasi-increasing" in the sense that there exists $A > 0$ such that

$$0 \leq x < y < d \Rightarrow T(x) \leq AT(y); \tag{13}$$

and

$$T(x) \geq \Lambda > \frac{1}{2} \text{ in } [0, d); \tag{14}$$

(e) There exists $C_1 > 0$ such that

$$\left|\frac{Q''(x)}{Q'(x)}\right| \leq C_1 \left|\frac{Q'(x)}{Q(x)}\right|, \text{ a.e. } x \in (0, d) \tag{15}$$

Then we write $W \in \mathcal{L}(C^2)$. If also for some $0 < c < \sqrt{d}$, and some $C_2 > 0$,

$$\left|\frac{Q^{*\prime\prime}(x)}{Q^{*\prime}(x)}\right| \geq C_2 \left|\frac{Q^{*\prime}(x)}{Q^{*}(x)}\right|, \text{ a.e. } x \in \left(c, \sqrt{d}\right),$$

then we write $W \in \mathcal{L}(C^2+)$

We note that (13) and (15) are regularity conditions, while (14) is a lower growth condition, forcing $Q(x)$ to grow faster than x^Λ at ∞ if I is unbounded.

2 Ideas behind the Proof of Theorem 1

Step 1: The Christoffel-Darboux Formula
Let us start with the Christoffel-Darboux formula

$$\begin{aligned} K_n(x,t) &= \sum_{j=0}^{n-1} p_j(x)\, p_j(t) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(t) - p_{n-1}(x)\, p_n(t)}{x-t}. \end{aligned} \tag{16}$$

Here γ_n is the leading coefficient of p_n. Now we set $t = x_{jn}$, a zero of p_n and solve for $p_n(x)$, giving

$$p_n(x) = K_n(x, x_{jn}) \frac{x - x_{jn}}{\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn})}. \tag{17}$$

Next introduce the notation for the Christoffel function

$$\lambda_n(x) = 1 / \sum_{j=0}^{n-1} p_j^2(x).$$

Applying Cauchy-Schwarz in (17) gives

$$|p_n(x)| \leq \lambda_n^{-1}(x)\, \lambda_n^{-1}(x_{jn}) \left|\frac{x - x_{jn}}{\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn})}\right|. \tag{18}$$

This on its own would not be of much use. But now we derive
Step 2: An identity for $\lambda_n^{-1}(x_{jn})$
First, let $x = x_{jn}$ in (16), then let $t \to x_{jn}$ and apply l'Hospital's rule, giving

$$\lambda_n^{-1}(x_{jn}) = K_n(x_{jn}, x_{jn}) = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_{jn})\, p_{n-1}(x_{jn}). \tag{19}$$

Now we need an identity for $p_n'(x_{jn})$. Since $xp_n'(x)$ is a polynomial of degree $n-1$, we can represent it as an orthogonal expansion in terms of $\{p_j\}_{j=0}^n$ and use the reproducing kernel formula at $x = x_{jn}$:

$$\begin{aligned} x_{jn}p_n'(x_{jn}) &= \int_I K_{n+1}(t, x_{jn})\, tp_n'(t)\, W_\rho^2(t)\, dt \\ &= \int_I K_n(t, x_{jn})\, tp_n'(t)\, W_\rho^2(t)\, dt, \end{aligned}$$

as $p_n(x_{jn}) = 0$. We integrate by parts in this, using the fact that the integrand vanishes at 0 and d, obtaining, because of orthogonality

$$\begin{aligned} x_{jn}p_n'(x_{jn}) &= -\int_I p_n(t)\, \frac{d}{dt}\left[K_n(t, x_{jn})\, tW_\rho^2(t)\right] dt \\ &= -\int_I p_n(t)\, K_n(t, x_{jn})\, t\left\{-2Q'(t) + \frac{2\rho}{t}\right\} W_\rho^2(t)\, dt \\ &= 2\int_I p_n^2(t)\, K_n(t, x_{jn})\, tQ'(t)\, W_\rho^2(t)\, dt. \end{aligned}$$

The Christoffel-Darboux formula allows us to continue this as

$$x_{jn}p_n'(x_{jn}) = 2\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn}) \int_I \frac{tQ'(t)}{t - x_{jn}} p_n^2(t)\, W_\rho^2(t)\, dt.$$

Using orthogonality once more, we can rewrite this as

$$p_n'(x_{jn}) = \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn})\, A_n^{\#}(x_{jn}),$$

where

$$A_n^{\#}(x) = \frac{2}{x}\int_I \frac{tQ'(t) - xQ'(x)}{t - x} p_n^2(t)\, W_\rho^2(t)\, dt. \tag{20}$$

Together with (19), this gives

$$\lambda_n^{-1}(x_{jn}) = \left(\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn})\right)^2 A_n^{\#}(x_{jn}). \tag{21}$$

Step 3 The Main Bound for $p_n(x)$

Solving for $\left|\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn})\right|$ in this last identity and substituting in (18) gives the bound

$$|p_n(x)| \le \lambda_n^{-1}(x)\, |x - x_{jn}| \left(\lambda_n^{-1}(x_{jn})\, A_n^{\#}(x_{jn})\right)^{1/2}. \tag{22}$$

This is valid for any x and any j. The reason this is useful, is that we have matching upper and lower bounds for Christoffel functions, derived using its extremal property,

$$\lambda_n(x) = \inf\left\{\frac{\int_I P^2 W_\rho^2}{P^2(x)} : \deg(P) \leq n-1\right\}.$$

For $x \in [x_{nn}, x_{1n}]$, we choose x_{jn} to be the closest zero of p_n to x. Then for some C independent of n, j, x,

$$|x - x_{jn}| \leq C\lambda_n(x).$$

(This estimate is proved using our estimates on λ_n and what are essentially Posse-Markov-Stieltjes type inequalities). If we let

$$h_n(x) = |x(a_n - x)|^{1/2},$$

we obtain from (22),

$$\left(p_n^2 W_\rho^2 h_n\right)(x) \leq C_0 \left(h_n \lambda_n^{-1} W_\rho^2 A_n^{\#}\right)(x_{jn}), x \in [x_{nn}, x_{1n}]. \tag{23}$$

(Replacing $h_n(x)$ by $h_n(x_{jn})$ can be justified). On the right-hand side, we have $h_n\lambda_n^{-1}$, for which good bounds are available, but $A_n^{\#}$ is unknown. It is here that the difficult part of the proof arises. We split the integral (20) defining $A_n^{\#}$ into a number of integrals. We show that for each $\varepsilon > 0$, there exists C_ε such that

$$\left(h_n \lambda_n^{-1} W_\rho^2 A_n^{\#}\right)(x) \leq \varepsilon \sup_{[x_{nn}, x_{1n}]} \left(p_n^2 W_\rho^2 h_n\right) + C_\varepsilon,$$

with C_ε independent of n, x. Then we plug this estimate in (23) giving

$$\sup_{[x_{nn}, x_{1n}]} \left(p_n^2 W_\rho^2 h_n\right) \leq C_0 \varepsilon \sup_{[x_{nn}, x_{1n}]} \left(p_n^2 W_\rho^2 h_n\right) + C_0 C_\varepsilon.$$

Since C_0 does not depend on ε, we choose $\varepsilon = \frac{1}{2C_0}$ and obtain

$$\sup_{[x_{nn}, x_{1n}]} \left(p_n^2 W_\rho^2 h_n\right) \leq C_1,$$

for all $n \geq 1$. Restricted range inequalities such as (9) then give the rest. ■

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