THE DEGREE OF SHAPE PRESERVING WEIGHTED POLYNOMIAL APPROXIMATION

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ABSTRACT. We analyze the degree of shape preserving weighted polynomial approximation for exponential weights on the whole real line. In particular, we establish a Jackson type estimate.

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1. INTRODUCTION

Shape preserving polynomial approximation has been an active research topic for decades. There are many interesting features, and a great many complex examples, and exceptional cases. Perhaps the oldest modern result is due to O. Shisha [14]. For continuous $f : [-1, 1] \to \mathbb{R}$, let

$$E_n [f] = \inf_{\deg(P) \le n} \|f - P\|_{L_\infty[-1,1]} .$$

In addition, let

$$E_n^{(1)} [f] = \inf_{\deg(P) \le n} \left\{ \|f - P\|_{L_\infty[-1,1]} : P \text{ monotone in } [-1, 1] \right\} .$$

Shisha [14] essentially proved that when f' is non-negative and continuous, for $n \ge 1$,

$$E_n^{(1)} [f] \le 2E_{n-1} \left[f'\right] . \tag{1.1}$$

This simple estimate is disappointing, in that one loses a factor of $\frac{1}{n}$, when compared to Jackson-Favard estimates. However, it is best possible in the class of functions to which it applies [13].

Similar results hold for convex functions, and more generally, k-monotone functions. Recall that a function f is called k-monotone, if for any distinct $x_0, x_1, ..., x_k$ in the interval of definition,

$$[x_0, x_1, ..., x_k, f] = \sum_{i=0}^{k} \frac{f (x_i)}{\omega' (x_i)} \ge 0,$$

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where

$$\omega(x) = \prod_{j=0}^{k} (x - x_j).$$

The case $k = 1$ corresponds to monotone functions, and $k = 2$ to convex functions. The natural generalisation of (1.1) to k-monotone functions is

$$E_n^{(k)}[f] \le 2E_{n-k}\left[f^{(k)}\right],$$

for $n \ge k$. Again, this is a disappointing estimate, as one loses a factor of n^{-k} when compared with unconstrained approximation. However, it turns out that this estimate may not, in general, be improved, see [4]. See also [3], [5].

A recent interesting paper of O. Maizlish [10] seems to be the first extending shape preserving approximation to weighted polynomial approximation on the whole real line. Recall that for $\alpha > 0$,

$$W_\alpha(x) = \exp(-|x|^\alpha),\ x \in \mathbb{R},$$

is an exponential weight, often called a Freud weight. The polynomials are dense in the weighted space of continuous functions generated by W_α iff $\alpha \ge 1$. Thus, if $\alpha \ge 1$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, with

$$\lim_{|x|\to\infty} (fW_\alpha)(x) = 0,$$

while

$$E_n[f]_{W_\alpha} = \inf_{\deg(P)\le n} \|(f-P)W_\alpha\|_{L_\infty(\mathbb{R})},$$

we have

$$\lim_{n\to\infty} E_n[f]_{W_\alpha} = 0.$$

This is a special case of the classical solution of Bernstein's weighted polynomial approximation problem, involving more general weights W, by Achieser, Mergelyan, and Pollard [2], [7], [9].

For $W_\alpha, \alpha > 1$, the Jackson theorem takes the form

$$E_n[f]_{W_\alpha} \le Cn^{-1+1/\alpha} \left\|f'W_\alpha\right\|_{L_\infty(\mathbb{R})},$$

provided f' is continuous in $\mathbb{R}$. Here C is independent of f and n. Interestingly enough, there is no estimate of this type for W_1, even though the polynomials are dense. There are Jackson theorems involving weighted moduli of continuity, see [1], [8], [9].

Let $k \ge 1$, and let

(1.2)
$$E_n^{(k)}[f]_{W_\alpha} = \inf_{\deg(P)\le n} \left\{\|(f-P)W_\alpha\|_{L_\infty(\mathbb{R})} : P \text{ is } k-\text{monotone in } \mathbb{R}\right\}.$$

Maizlish proved that if f is k times continuously differentiable on $\mathbb{R}$ and $f^{(k)}$ is non-negative, then

$$\lim_{n\to\infty} E_n^{(k)}[f]_{W_\alpha} = 0.$$

Somewhat more is true: let

$$\mu(x) = \sqrt{f^{(k)}\left(2^{1/\alpha}x\right)},\ x \in \mathbb{R},$$

and

$$r_n = 4\left(\frac{2n}{\alpha}\right)^{1/\alpha}, n \geq 1.$$

Maizlish also proved that then there exists a polynomial P_n of degree at most $2n+k$ that is $k-$monotone, and such that

$$\|(f-P_n)W_\alpha\|_{L_\infty[-r_n,r_n]} \leq M_1 E_n[\mu]_{W_\alpha} \|\mu W_\alpha\|_{L_\infty(\mathbb{R})}$$

and

$$\|(f-P_n)W_\alpha\|_{L_\infty(\mathbb{R}\setminus[-r_n,r_n])} \leq M_1 n^{-1+1/\alpha} E_n[\mu]_{W_\alpha} \|\mu W_\alpha\|_{L_\infty(\mathbb{R})}.$$

Here M_1 is independent of f and n. Note that μ can be somewhat less smooth than $f^{(k)}$.

In this paper, we prove results of this type that are closer in spirit to the unweighted Shisha type theorems. Throughout, $[x]$ denotes the greatest integer $\leq x$.

Theorem 1.1
Let $\alpha > 1$, and $k \geq 1$. Let $A > 1$. There exist $B, C > 0$ with the following property: for every $f : \mathbb{R} \to \mathbb{R}$ that is k times continuously differentiable and $k-$monotone, satisfying

$$\lim_{|x|\to\infty} \left(f^{(k)} W_\alpha\right)(x) = 0, \tag{1.3}$$

we have for $n \geq 1$,

$$E^{(k)}_{[An]+k}[f]_{W_\alpha} \leq C\left[E_n\left[f^{(k)}\right]_{W_\alpha} + \left\|f^{(k)}W_\alpha\right\|_{L_\infty(\mathbb{R})} e^{-Bn}\right]. \tag{1.4}$$

Conversely, given any $B > 0$, there exists sufficiently large A for which this last inequality holds for all $n \geq 1$.

We may replace the geometric factors e^{-Bn} by factors that decay more slowly, and then allow $[An]$ to be replaced by something smaller. We may also consider more general Freud weights, or even exponential weights on a finite interval. For simplicity, we shall consider only even weights $W = e^{-Q}$, defined on a symmetric interval $I = (-d, d)$, where $0 < d \leq \infty$. Accordingly, we define

$$E_n[f]_W = \inf_{\deg(P)\leq n} \|(f-P)W\|_{L_\infty(I)}, \tag{1.5}$$

and

$$E^{(k)}_n[f]_W = \inf_{\deg(P)\leq n} \left\{\|(f-P)W\|_{L_\infty(I)} : P \text{ is } k-\text{monotone in } I\right\}. \tag{1.6}$$

We start with a generalization of Theorem 1.1 for Freud weights:

Theorem 1.2
Let $W = e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, and Q' is continuous in $\mathbb{R}$, while Q'' exists in $(0, \infty)$. Assume in addition, that
(i) $Q' > 0$ in $(0, \infty)$ and $Q(0) = 0$;
(ii) $Q'' > 0$ in $(0, \infty)$;
(iii) For some $\Gamma, \Lambda > 1$,

$$\Gamma \geq \frac{tQ'(t)}{Q(t)} \geq \Lambda,\ t \in (0, \infty); \tag{1.7}$$

(iv)

$$\frac{Q''(t)}{Q'(t)} \leq C_1 \frac{Q'(t)}{Q(t)},\ \ t \in (0, \infty). \tag{1.8}$$

Let $A > 1$ and $2 \leq \ell_n \leq An + 1$, $n \geq 1$. Let $k \geq 1$. There exist $B, C > 0$ with the following property: for every $f : \mathbb{R} \to \mathbb{R}$ that is k times continuously differentiable and $k-$monotone, satisfying

$$\lim_{|x| \to \infty} \left(f^{(k)} W \right)(x) = 0,$$

we have for $n \geq 1$,

$$E^{(k)}_{n+\ell_n+k}[f]_W \leq C \left[E_n \left[f^{(k)} \right]_W + \left\| f^{(k)} W \right\|_{L_\infty(\mathbb{R})} e^{-Bn^{-1/2}\ell_n^{3/2}} \right]. \tag{1.9}$$

Observe that Theorem 1.1 is the special case in which $Q(x) = |x|^\alpha$ and $\ell_n = [(A-1)n]$. Given a positive integer j, if we choose

$$\ell_n = \left[rn^{1/3} (\log n)^{2/3} \right],$$

with large enough r, we obtain

$$E^{(k)}_{n+\left[rn^{1/3}(\log n)^{2/3}\right]+k}[f]_W \leq C \left[E_n \left[f^{(k)} \right]_W + \left\| f^{(k)} W \right\|_{L_\infty(\mathbb{R})} n^{-j} \right]. \tag{1.10}$$

Finally, we turn to general even exponential weights. For these, we need the concept of the nth Mhaskar-Rakhmanov-Saff number a_n, associated with $W = e^{-Q}$. This is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}}. \tag{1.11}$$

It is uniquely defined if $tQ'(t)$ is positive and strictly increasing in $(0, d)$ with limits 0 and ∞ at 0 and d respectively. One of its features is the Mhaskar-Saff identity [6], [12]

$$\|PW\|_{L_\infty(I)} = \|PW\|_{L_\infty[-a_n, a_n]}, \tag{1.12}$$

for all polynomials P of degree $\leq n$. Moreover, a_n is essentially the smallest number for which this holds. We shall also need the function

$$T(x) = \frac{xQ'(x)}{Q(x)}, x \in (0,d). \tag{1.13}$$

We shall say that T is *quasi-increasing* in $(0,d)$ if there exists $C>0$ such that

$$T(x) \leq CT(y) \text{ for all } 0<x<y<d.$$

Our most general theorem is:

Theorem 1.3
Let $I=(-d,d)$, where $0<d\leq\infty$. Let $W=e^{-Q}$, where $Q: I\to\mathbb{R}$ is even, and Q' is continuous in I, while Q'' exists in $(0,d)$. Assume in addition, that
(i) $Q(0)=0$ and $\lim_{t\to d-} Q(t)=\infty$;
(ii) $Q'>0$ in $(0,d)$;
(iii) $Q''>0$ in $(0,d)$;
(iv) For some $\Lambda>1$,

$$T(t) \geq \Lambda,\ t\in(0,d), \tag{1.14}$$

while T is quasi-increasing there.
(v)

$$\frac{Q''(t)}{Q'(t)} \leq C_1 \frac{Q'(t)}{Q(t)},\ t\in(0,d). \tag{1.15}$$

Let $A>1$ and $2\leq \ell_n \leq An+1$, $n\geq 1$. Let $k\geq 1$. There exist $B,C>0$ with the following property: for every $f: I\to\mathbb{R}$ that is k times continuously differentiable and $k-$monotone, and for which

$$\lim_{|x|\to d-} \left(f^{(k)}W\right)(x)=0,$$

we have for $n\geq 1$,

$$E^{(k)}_{n+\ell_n+k}[f]_W \leq C\left(E_n\left[f^{(k)}\right]_W + \left\|f^{(k)}W\right\|_{L_\infty(I)} e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right).$$

(1.16)

Here a_n is the nth Mhaskar-Rakhmanov-Saff number for Q.

Examples of such weights on the interval $(-1,1)$ include

$$W(x) = \exp\left(1-\left(1-x^2\right)^{-\alpha}\right) \tag{1.17}$$

or

$$W(x) = \exp\left(\exp_k(1) - \exp_k\left(\left(1-x^2\right)^{-\alpha}\right)\right), \tag{1.18}$$

where $\alpha > 1$, and

$$\exp_k = \exp\underbrace{(\exp(\ldots\exp()))}_{k\text{ times}}$$

is the kth iterated exponential. On the whole real line, in addition to the Freud weights, one may choose

$$W(x) = \exp\left(\exp_k(0) - \exp_k(|x|^\alpha)\right), \tag{1.19}$$

where $k \geq 1$ and $\alpha > 1$. For W of (1.17), [6, p. 31, Example 3]

$$T(a_n) \sim n^{\frac{1}{\alpha + \frac{1}{2}}}.$$

This means that the ratio of the two sides is bounded above and below by positive constants independent of n. For W of (1.18), [6, p. 33, Example 4]

$$T(a_n) \sim (\log_k n)^{1+\frac{1}{\alpha}} \prod_{j=1}^{k-1} \log_j n,$$

where

$$\log_k = \log\underbrace{(\log(\ldots\log()))}_{k\text{ times}}$$

is the kth iterated logarithm. For W of (1.19), [6, p. 30, Example 2]

$$T(a_n) \sim \prod_{j=1}^{k} \log_j n.$$

We note that all our weights lie in the class $\mathcal{F}\left(C^2\right)$ considered in [6, p. 7]. We may actually consider the non-even weights there, as well as the more general class $\mathcal{F}\left(Lip\frac{1}{2}\right)$, but avoid this for notational simplicity.

The main new idea in this paper over that of Maizlish is the use of non-negative polynomials, obtained from discretizing potentials, and that were constructed in [6, Theorem 7.4, p. 171]. We shall use many of Maizlish's ideas, as well as devices from the unweighted theory of shape preserving approximation. The proofs are contained in the next section.

2. Proof of Theorem 1.3

We begin with some background on potential theory with external fields [12]. Let us assume the hypotheses of Theorem 1.3. The Mhaskar-Rakhmanov-Saff number a_t may be defined by (1.11), for any $t > 0$, not just for integer n: thus for $t > 0$,

$$t = \frac{2}{\pi}\int_0^1 a_t u Q'(a_t u) \frac{du}{\sqrt{1-u^2}}.$$

The function $t \to a_t$ is a continuous strictly increasing function of t, so has an inverse function b, defined by

$$b(a_t) = t,\ t > 0.$$

For each $t > 0$, there is an equilibrium density σ_t, that satisfies

$$\int_{-a_t}^{a_t} \sigma_t = t.$$

The equilibrium potential

$$V^{\sigma_t}(z) = \int_{-a_t}^{a_t} \log \frac{1}{|z-u|} \sigma_t(u)\, du$$

satisfies

$$V^{\sigma_t} + Q = c_t \text{ in } [-a_t, a_t],$$

where c_t is a characteristic constant. We shall need mostly the function

$$U_t(x) = -\left(V^{\sigma_t}(x) + Q(x) - c_t\right), x \in I.$$

It satisfies

$$U_t(x) = 0,\ x \in [-a_t, a_t];$$

$$U_t(x) < 0,\ x \in I \backslash [-a_t, a_t].$$

We shall need an alternative representation for U_t. For an interval $[a, b]$, the Green's function for $\mathbb{C} \backslash [a, b]$ with pole at ∞, is

$$g_{[a,b]}(z) = \log \left| \frac{2}{b-a} \left(z - \frac{a+b}{2} + \sqrt{(z-a)(z-b)} \right) \right|.$$

It vanishes on $[a, b]$, is non-negative in the plane, and behaves like $\log |z| + O(1)$, as $z \to \infty$. There is the representation [6, Corollary 2.9, p. 50]

$$U_t(x) = -\int_t^{b_x} g_{[-a_\tau, a_\tau]}(x)\, d\tau,\ x \in [0, d). \tag{2.1}$$

It is really this that we shall need, not so much the other quantities above.

Lemma 2.1
(a) For $n \geq 1$, and polynomials P_n of degree $\leq n$,

$$|P_n W|(x) \leq e^{U_n(x)} \|P_n W\|_{L_\infty(\mathbb{R})}, |x| > a_n. \tag{2.2}$$

(b) Let $D > 1$. For $n \leq m \leq Dn$, and $x \geq a_m$,

$$(U_n - U_m)(x) \leq -C \frac{n}{T(a_n)^{1/2}} \left(1 - \frac{n}{m}\right)^{3/2}. \tag{2.3}$$

Here C is independent of m, n, x.
Proof
(a) This is a classical inequality of Mhaskar and Saff that can be found, for example, in [6, Lemma 4.4, p. 99] or [12, p. 153, Thm. 2.1].
(b) From (2.1), for $x > a_m$,

$$U_n(x) - U_m(x) = -\int_n^m g_{[-a_\tau, a_\tau]}(x)\, d\tau,\ x \in [0, d).$$

Here for each $\tau \in [n, m]$, $g_{[-a_\tau, a_\tau]}(x)$ is an increasing function of $x \geq a_m$, as the Green's function $g_{[a,b]}$ increases as we move to the right of $[a, b]$. It follows that for $x \geq a_m$,

$$\begin{aligned} & U_n(x) - U_m(x) \\ \leq \quad & U_n(a_m) - U_m(a_m) = -\int_n^m g_{[-a_\tau, a_\tau]}(a_m)\, d\tau. \end{aligned} \tag{2.4}$$

Next, by Lemma 4.5(a) in [6, p. 101], followed by (3.51) of Lemma 3.11(a) in [6, p. 81], for $\tau \in [n, m]$,

$$g_{[-a_\tau, a_\tau]}(a_m) \geq C \left(\frac{a_m}{a_\tau} - 1 \right)^{1/2} \geq \frac{C}{T(a_n)^{1/2}} \left(\frac{m}{\tau} - 1 \right)^{1/2}$$

(Note that in the even case, in [6], $\delta_n = a_n$, and $a_{2n} \leq C a_n$). Then (2.3) follows easily from (2.4). ■

We also need polynomials constructed by discretizing the potential V^{σ_t}. The method is due to Totik, but the form we need was proved in [6, Theorem 7.4, p. 171]:

Lemma 2.2
There exists $C_0 > 1$ with the following property: for even $n \geq 2$, there exists a polynomial R_n of degree $\leq n$ such that

$$1 \leq R_n W \leq C_0 \text{ in } [-a_n, a_n]; \tag{2.5}$$

and moreover,

$$R_n W \geq e^{U_n} \text{ in } I. \tag{2.6}$$

Now we can use this to generate non-negative weighted polynomial approximations to non-negative functions:

Lemma 2.3
Let $g : I \to \mathbb{R}$ be a continuous non-negative function such that

$$\|gW\|_{L_\infty(I)} = 1, \tag{2.7}$$

and

$$\lim_{|x| \to d} (gW)(x) = 0.$$

Assume that $D > 0$ and $\{\ell_n\}$ is a sequence of positive integers with $2 \leq \ell_n \leq Dn + 1$. Then there exist $B, C > 0$, and for $n \geq 1$, a polynomial $P_n^{\#}$ of degree $\leq n + \ell_n$ such that

$$P_n^{\#} \geq 0 \text{ in } I \tag{2.8}$$

and

$$\left\| \left(g - P_n^{\#} \right) W \right\|_{L_\infty(I)} \leq C \left(E_n[g]_W + e^{-B(nT(a_n))^{-1/2} \ell_n^{3/2}} \right). \tag{2.9}$$

Here $C \neq C(n,g)$.

Proof

Choose a polynomial P_n such that

$$\|(g-P_n)W\|_{L_\infty(I)} = E_n[g]_W.$$

As $g \geq 0$, we have

$$P_nW \geq -E_n[g]_W \text{ in } [-a_n, a_n]. \tag{2.10}$$

Let $m = m(n) = 2\left[\frac{n+\ell_n}{2}\right]$, an even integer. Note that $m \geq n+1$. Let R_m be the polynomial of Lemma 2.2. Let

$$S_n(x) = P_n(x) + \left(E_n[g]_W + e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right) R_m(x),$$

a polynomial of degree $\leq m$. From (2.5) and (2.10), we have in $[-a_m, a_m]$,

$$(S_nW)(x) \geq 0.$$

From Lemma 2.1(a), for $|x| \in (a_n, d)$,

$$\begin{aligned} |P_nW|(x) &\leq \|P_nW\|_{L_\infty(I)} e^{U_n(x)} \\ &\leq \left(\|gW\|_{L_\infty(I)} + E_n[g]_W\right) e^{U_n(x)} \\ &\leq 2e^{U_n(x)}, \end{aligned}$$

recall our normalization (2.7). Then from Lemma 2.2, for $x \in (a_m, d)$,

$$(S_nW)(x) \geq -2e^{U_n(x)} + \left(E_n[g]_W + e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right) e^{U_m(x)}.$$

This will be non-negative if

$$(U_n - U_m)(x) \leq \log\left(\frac{E_n[g]_W + e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}}{2}\right).$$

From Lemma 2.1(b), it suffices in turn that for some large enough C,

$$C\frac{n}{T(a_n)^{1/2}}\left(1-\frac{n}{m}\right)^{3/2} \geq \left|\log\left(\frac{e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}}{2}\right)\right|,$$

or

$$\frac{C}{(nT(a_n))^{1/2}}\ell_n^{3/2} \geq 2B(nT(a_n))^{-1/2}\ell_n^{3/2}.$$

So we can choose $B = C/2$, and ensure non-negativity of S_n in $[0,d)$. The interval $(-d,0)$ may be handled similarly. Finally,

$$\begin{aligned} &\|(g-S_n)W\|_{L_\infty(I)} \\ \leq\ &\|(g-P_n)W\|_{L_\infty(I)} + \left(E_n[g]_W + e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right)\|R_mW\|_{L_\infty(I)} \\ \leq\ &E_n[g]_W + C_0\left(E_n[g]_W + e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right), \end{aligned}$$

where C_0 is as in Lemma 2.2. ∎

Proof of Theorem 1.3
By the last lemma, we can choose a polynomial P_n of degree $\leq n+\ell_n$ such that $P_n \geq 0$ in I, and

$$\begin{aligned}
&\left\| \left(f^{(k)} - P_n\right) W \right\|_{L_\infty(I)} \\
\leq\ & C\left(E_n\left[f^{(k)}\right]_W + \left\|f^{(k)}W\right\|_{L_\infty(I)} e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right) =: M_n,
\end{aligned}$$

say. We have taken account of the need to divide $f^{(k)}$ by $\left\|f^{(k)}W\right\|_{L_\infty(I)}$, in order to satisfy the normalization (2.7). Now, let

$$P_n^*(x) = \int_0^x \int_0^{t_{k-1}} ... \int_0^{t_1} P_n(t_0)\, dt_0 dt_1 ... dt_{k-1} + \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j.$$

Then P_n^* is k monotone. For $x > 0$, we have, following Maizlish's ideas,

$$\begin{aligned}
& |(f - P_n^*) W| (x) \\
(2.11) =\ & \left| W(x) \int_0^x \int_0^{t_{k-1}} ... \int_0^{t_1} \left(f^{(k)} - P_n\right)(t_0)\, dt_0 dt_1 ... dt_{k-1} \right| \\
\leq\ & M_n W(x) \int_0^x \int_0^{t_{k-1}} ... \int_0^{t_1} W^{-1}(t_0)\, dt_0 dt_1 ... dt_{k-1} \\
(2.12) =\ & M_n \int_0^x \frac{W(x)}{W(t_{k-1})} \int_0^{t_{k-1}} \frac{W(t_{k-1})}{W(t_{k-2})} ... \int_0^{t_1} \frac{W(t_1)}{W(t_0)} dt_0 dt_1 ... dt_{k-1}.
\end{aligned}$$

Fix $r \in (0, d)$. Here by monotonicity of Q, for $t_1 > 0$,

$$\int_0^{t_1} \frac{W(t_1)}{W(t_0)} dt_0 \leq t_1$$

while by its convexity, for $t_1 \geq r$,

$$\int_r^{t_1} \frac{W(t_1)}{W(t_0)} dt_0 \leq \int_r^{t_1} e^{-Q'(r)(t_1 - t_0)} dt_0 \leq \frac{1}{Q'(r)}.$$

It follows that for all $t \in (0, d)$,

$$\int_0^{t_1} \frac{W(t_1)}{W(t_0)} dt_0 \leq r + \frac{1}{Q'(r)}.$$

Applying this repeatedly to (2.11) gives

$$|(f - P_n^*) W| (x) \leq M_n \left(r + \frac{1}{Q'(r)}\right)^k.$$

The case $x < 0$ is similar, so we obtain

$$E_{n+\ell_n+k}^{(k)}[f]_W \leq \left(r + \frac{1}{Q'(r)}\right)^k C\left(E_n\left[f^{(k)}\right]_W + \left\|f^{(k)}W\right\|_{L_\infty(I)} e^{-B(nT(a_n))^{-1/2}\ell_n^{3/2}}\right).$$

■

Proof of Theorem 1.2

This is a special case of Theorem 1.3, where T is bounded above and below by positive constants. ■

Proof of Theorem 1.1

This is the special case $W = W_\alpha$ of Theorem 1.2. We can choose

$$\ell_n = [(A-1)\, n]$$

when that is at least 2. For the remaining finitely many n, we can set $\ell_n = 2$ and use the elementary inequality

$$E_k^{(k)}[f]_{W_\alpha} \leq C \left\| f^{(k)} W \right\|_{L_\infty(\mathbb{R})}.$$

The fact that we may choose B as large as we please, with correspondingly large A, is easily seen from the proof of Lemma 2.3. ■

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