

ORTHOGONAL POLYNOMIALS FOR EXPONENTIAL WEIGHTS $x^{2\rho}e^{-2Q(x)}$ ON $[0, d)$

ELI LEVIN¹ AND DORON LUBINSKY²

ABSTRACT. Let $I = [0, d)$, where d is finite or infinite. Let $W_\rho(x) = x^\rho \exp(-Q(x))$, where $\rho > -\frac{1}{2}$ and Q is continuous and increasing on I , with limit ∞ at d . We study the orthonormal polynomials associated with the weight W_ρ^2 , obtaining bounds on the orthonormal polynomials, zeros, and Christoffel functions. In addition, we obtain restricted range inequalities.

1. INTRODUCTION AND RESULTS

Let

$$(1.1) \quad I = [0, d),$$

where $0 < d \leq \infty$. Let $Q : I \rightarrow [0, \infty)$, and

$$(1.2) \quad W = \exp(-Q).$$

We call W an exponential weight on I . Typical examples would be

$$W(x) := \exp(-x^\alpha), x \in [0, \infty),$$

where $\alpha > \frac{1}{2}$ or

$$W(x) = \exp(-(1-x)^{-\alpha}), x \in [0, 1),$$

where $\alpha > 0$. For $\rho > -\frac{1}{2}$, we set

$$W_\rho(x) := x^\rho W(x), x \in I.$$

The orthonormal polynomial of degree n for W^2 is denoted by $p_n(W^2, x)$ or just $p_n(x)$. That for W_ρ^2 is denoted by $p_n(W_\rho^2, x)$ or just $p_{n,\rho}(x)$.

Thus

$$(1.3) \quad \int_I p_{n,\rho}(x) p_{m,\rho}(x) x^{2\rho} W^2(x) dx = \delta_{mn}$$

and

$$p_{n,\rho}(x) = \gamma_{n,\rho} x^n + \dots,$$

where $\gamma_{n,\rho} = \gamma_n(W_\rho^2) > 0$.

Date: 18 February 2004.

There is a very substantial body of research dealing with exponential weights on a subset of the real line, especially as regards the associated potential theory, weighted approximation, and orthonormal polynomials. For some recent references on orthogonal polynomials for exponential weights, and especially their asymptotics and quantitative estimates, the reader may consult [2], [3], [5], [6], [7], [9], [20], [21], [23].

In our recent monograph [7], we dealt with exponential weights on a real interval (c, d) containing 0 in its interior. A typical example would be the weight

$$W(x) = \begin{cases} \exp(-|x|^\alpha), & x \in (-\infty, 0) \\ \exp(-|x|^\beta), & x \in [0, \infty) \end{cases},$$

where $\alpha, \beta > 1$. In all cases, the exponent Q grows to ∞ at both endpoints of the interval.

In this paper, we look at the ‘‘one-sided’’ case where Q increases from 0 at 0 to ∞ at d . This may be thought of as a limiting case of the two-sided case, in which the exponent to the left of 0 grows to ∞ . However, the results of [7] cannot be applied through such a limit, as the constants in the estimates there are not known to be uniform in the weight. Moreover, there are significant differences in even the formulation of the results - just as there are for the Laguerre and Hermite weights. Nevertheless, we can use the results from [7] by defining an even weight corresponding to the one-sided weight.

Given I and W as in (1.1) and (1.2), we define

$$(1.4) \quad I^* := (-\sqrt{d}, \sqrt{d})$$

and for $x \in I^*$,

$$(1.5) \quad Q^*(x) := Q(x^2);$$

$$(1.6) \quad W^*(x) := \exp(-Q^*(x)).$$

In the special case

$$I = [0, \infty) \text{ and } Q(x) = x,$$

this substitution gives the Hermite polynomials from Laguerre polynomials. In our case, if $p_{2n}(W^{*2}, x)$ denotes the orthonormal polynomial of degree $2n$ for W^{*2} , this substitution yields the identity

$$(1.7) \quad p_{n, -\frac{1}{4}}(x^2) = p_n\left(W_{-\frac{1}{4}}^2, x^2\right) = p_{2n}(W^{*2}, x).$$

Our main focus is bounds on $p_{n,\rho}(x)$ and associated quantities. These include the zeros of $p_{n,\rho}$, which we denote by

$$x_{nn,\rho} < x_{n,n-1,\rho} < \dots < x_{2n,\rho} < x_{1n,\rho},$$

and the Christoffel functions

$$\lambda_n(W_\rho^2, x) = \inf_{\deg(P) \leq n-1} \frac{\int_I (PW_\rho)^2}{P^2(x)}.$$

Before stating some of our results, we need more notation. We say that $f : I \rightarrow (0, \infty)$ is *quasi-increasing* if there exists $C > 0$ such that

$$f(x) \leq C f(y), 0 < x < y < d.$$

Of course, any increasing function is quasi-increasing. The notation

$$f(x) \sim g(x)$$

means that there are positive constants C_1, C_2 such that for the relevant range of x ,

$$C_1 \leq f(x)/g(x) \leq C_2.$$

Similar notation is used for sequences and sequences of functions.

Throughout, C, C_1, C_2, \dots denote positive constants independent of n, x, t and polynomials P of degree at most n . We write $C = C(\lambda)$, $C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ . The same symbol does not necessarily denote the same constant in different occurrences.

Following is our class of weights:

Definition 1.1

Let $W = e^{-Q}$ where $Q : I \rightarrow [0, \infty)$ satisfies the following properties:

- (a) $\sqrt{x}Q'(x)$ is continuous in I , with limit 0 at 0 and $Q(0) = 0$;
- (b) Q'' exists in $(0, d)$, while Q^{*n} is positive in $(0, \sqrt{d})$;
- (c)

$$(1.8) \quad \lim_{x \rightarrow d^-} Q(x) = \infty;$$

(d) The function

$$(1.9) \quad T(x) := \frac{xQ'(x)}{Q(x)}, x \in (0, d)$$

is quasi-increasing in $(0, d)$, with

$$(1.10) \quad T(x) \geq \Lambda > \frac{1}{2}, x \in (0, d).$$

(e) There exists $C_1 > 0$ such that

$$(1.11) \quad \frac{|Q''(x)|}{Q'(x)} \leq C_1 \frac{Q'(x)}{Q(x)}, \text{ a.e. } x \in (0, d).$$

Then we write $W \in \mathcal{L}(C^2)$. If also there exists a compact subinterval J of I^* , and $C_2 > 0$ such that

$$(1.12) \quad \frac{Q^{*''}(x)}{|Q^{*'}(x)|} \geq C_2 \frac{|Q^{*'}(x)|}{Q^*(x)}, \text{ a.e. } x \in I^* \setminus J,$$

then we write $W \in \mathcal{L}(C^2+)$.

Remarks

(a) The simplest case of the above definition is when $I = [0, \infty)$ and

$$C \geq T \geq \Lambda > \frac{1}{2} \text{ in } (0, \infty).$$

Thus,

$$T \sim 1 \text{ in } (0, \infty).$$

This is the one-sided version of the Freud case, for $T = O(1)$ forces Q to be of at most polynomial growth. Moreover, T is then automatically quasi-increasing in $(0, d)$. Typical examples then would be

$$Q(x) = Q_\alpha(x) = x^\alpha, \quad x \in [0, \infty)$$

where $\alpha > \frac{1}{2}$. For this choice, we see that

$$T(x) = \alpha, \quad x \in (0, \infty).$$

Note that for the case $\alpha = \frac{1}{2}$, which forms the boundary in the one-sided case between determinate and indeterminate weights, there are added complications in the behaviour of the orthonormal polynomials and related quantities. For this phenomenon in the case of even Freud weights, see, [4], [17] for example. This explains our restriction (1.6), namely $T \geq \Lambda > \frac{1}{2}$, which forces Q to grow at least as fast as $x^\Lambda \gg x^{1/2}$ if I is unbounded. For such Q , most of our results for $p_{n,\rho}$ follow from results of Kasuga and Sakai [5]. They considered generalized Freud weights $|x|^{2\rho} \exp(-2Q^*(x))$ on \mathbb{R} .

(b) A more general example satisfying the above conditions is

$$Q(x) = Q_{k,\alpha}(x) = \exp_k(x^\alpha) - \exp_k(0), \quad x \in [0, \infty)$$

where $\alpha > \frac{1}{2}$ and $k \geq 0$. Here we set

$$\exp_0(x) := x$$

and for $k \geq 1$,

$$\exp_k(x) = \underbrace{\exp(\exp(\exp \dots \exp(x)))}_{k \text{ times}}$$

is the k th iterated exponential. In particular,

$$\exp_k(x) = \exp(\exp_{k-1}(x)).$$

(c) An example on the finite interval $I = [0, 1)$ is

$$Q(x) = Q^{(k,\alpha)}(x) = \exp_k((1-x)^{-\alpha}) - \exp_k(1), x \in [0, 1),$$

where $\alpha > 0$ and $k \geq 0$.

(d) The class $\mathcal{L}(C^2+)$ was formulated in such a way that $W^* \in \mathcal{F}(C^2+)$, the most explicit class of weights from [7]. We use the letter \mathcal{L} to indicate that, analogous to the Laguerre weights, we are working on (a subset of) the positive real axis.

Potential theory plays a key role in analysis of exponential weights, and one of the important quantities there is the Mhaskar-Rakhmanov-Saff number a_t , [13], [19], [11], [20] defined for $t > 0$ as the positive root of the equation

$$(1.13) \quad t = \frac{1}{\pi} \int_0^1 \frac{a_t u Q'(a_t u)}{\sqrt{u(1-u)}} du.$$

If $xQ'(x)$ is strictly increasing and continuous, with limits 0 and ∞ at 0 and d respectively, a_t is uniquely defined. Moreover, a_t is an increasing function of $t \in (0, \infty)$, with

$$\lim_{t \rightarrow \infty} a_t = d.$$

The interval

$$(1.14) \quad \Delta_t = [0, a_t], t > 0,$$

plays a key role in analysis of weighted polynomials. For example, [12], [13], the Mhaskar-Saff identity asserts that if P is a polynomial of degree $\leq n$, then

$$(1.15) \quad \|Pe^{-Q}\|_{L_\infty(I)} = \|Pe^{-Q}\|_{L_\infty(0, a_n)} = \|Pe^{-Q}\|_{L_\infty(\Delta_n)}$$

and a_n is, as $n \rightarrow \infty$, the "smallest" number for which this holds.

One of our main results is:

Theorem 1.2

Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}(C^2)$. Let $p_{n,\rho}(x)$ be the n th orthonormal polynomial for the weight W_ρ^2 . Then uniformly for $n \geq 1$,

$$(1.16) \quad \sup_{x \in I} |p_{n,\rho}(x)| W(x) \left(x + \frac{a_n}{n^2}\right)^\rho |x + a_n n^{-2}|^{1/4} (a_n - x)^{1/4} \sim 1.$$

We shall prove this in Section 8. Let

$$(1.17) \quad \eta_t = (tT(a_t))^{-2/3}, t > 0,$$

and

$$(1.18) \quad \varphi_t(x) := \begin{cases} \frac{\sqrt{x+a_t t^{-2}}(a_{2t}-x)}{t\sqrt{a_t-x+a_t\eta_t}}, & x \in [0, a_t]; \\ \varphi_t(a_t), & x > a_t; \\ \varphi_t(0), & x < 0. \end{cases}$$

For the Christoffel functions, we shall prove:

Theorem 1.3

Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}(C^2)$.

(a) Let $L > 0$. Then uniformly for $n \geq n_0$ and $x \in [a_n n^{-2}/L, a_n(1 + L\eta_n)]$, we have

$$(1.19) \quad \lambda_n(W_\rho^2, x) \sim \varphi_n(x)W^2(x) \left(x + \frac{a_n}{n^2}\right)^{2\rho}.$$

(b) Moreover, there exist $C, n_0 > 0$ such that uniformly for $n \geq n_0$ and $x \in I$,

$$(1.20) \quad \lambda_n(W_\rho^2, x) \geq C\varphi_n(x)W^2(x) \left(x + \frac{a_n}{n^2}\right)^{2\rho}.$$

We shall prove this in Section 6 for generalized L_p Christoffel functions involving exponentials of potentials. For the zeros, we prove:

Theorem 1.4

Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}(C^2)$.

(a) There exists $C > 0$ such that for $n \geq 1$ and $1 \leq j \leq n-1$,

$$(1.21) \quad x_{j+1, n, \rho} - x_{jn, \rho} \leq C\varphi_n(x_{jn}).$$

(b) For each fixed j and n , $x_{jn, \rho}$ is a non-decreasing function of ρ .

(c)

$$(1.22) \quad x_{nn, \rho} \sim a_n n^{-2},$$

and

$$(1.23) \quad a_n(1 - C\eta_n) \leq x_{1n, \rho} < a_{n+\rho+\frac{1}{4}}.$$

If in addition $W \in \mathcal{L}(C^2+)$, then for large enough n ,

$$(1.24) \quad 1 - \frac{x_{1n, \rho}}{a_n} \sim \eta_n.$$

We shall prove this in section 7. Finally, we note a restricted range inequality, which will be proved in Section 5. In the sequel, we let \mathcal{P}_n

denote the polynomials of degree $\leq n$.

Theorem 1.5

Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}(C^2)$. Let $0 < p \leq \infty$ and $L, \lambda \geq 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$.

(a) There exist C_1, n_0 such that for $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$(1.25) \quad \| (PW)(x) x^\beta \|_{L_p(I)} \leq C_1 \| (PW)(x) x^\beta \|_{L_p[La_n n^{-2}, a_n(1-\lambda\eta_n)]}.$$

(b) Given $r > 1$, there exist C_2, n_0, α such that for $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$(1.26) \quad \| (PW)(x) x^\beta \|_{L_p(a_{rn}, d)} \leq \exp(-C_2 n^\alpha) \| (PW)(x) x^\beta \|_{L_p(\Delta_n)}.$$

We note that all the above results are valid under weaker conditions on W . All we need is that W^* satisfies the conditions for the corresponding result in [7]. However, for simplicity, we use just one class of weights in this paper.

This paper is organised as follows. In the next section, we relate $\mathcal{L}(C^2)$ to a class of weights from [7]. In Section 3, we state some technical estimates, most following from results in [7]. In Section 4 we formulate some potential theoretic estimates. In Section 5, we shall state and prove restricted range inequalities. In Section 6, we state and prove estimates for Christoffel functions. In Section 7, we state and prove estimates for zeros of orthogonal polynomials. Finally in Section 8, we state and prove our bounds for orthogonal polynomials.

2. CLASSES OF WEIGHTS W AND W^*

The class $\mathcal{L}(C^2)$ was defined in such a way that W^* becomes part of the corresponding class in [7, p. 7], namely the class $\mathcal{F}(C^2)$: In its formulation, there are some simplifications due to the fact that W^* is even.

Definition 2.1

Let $W^* = e^{-Q^*}$ where $Q^* : I^* \rightarrow [0, \infty)$ satisfies the following properties:

- (a) $Q^{*'} is continuous in I^* and $Q^*(0) = 0$;$
- (b) $Q^{*''}$ exists and is positive in $I^* \setminus \{0\}$;
- (c)

$$\lim_{x \rightarrow \sqrt{d}^-} Q^*(t) = \infty;$$

(d) *The function*

$$(2.1) \quad T^*(x) := \frac{xQ^{*'}(x)}{Q^*(x)},$$

is quasi-increasing in $(0, \sqrt{d})$, with

$$(2.2) \quad T^*(x) \geq \Lambda^* > 1, x \in I^* \setminus \{0\}.$$

(e) *There exists $C_1 > 0$ such that*

$$(2.3) \quad \frac{Q^{*''}(x)}{|Q^{*'}(x)|} \leq C_1 \frac{|Q^{*'}(x)|}{Q^*(x)}, \text{ a.e. } x \in I^* \setminus \{0\}.$$

Then we write $W^* \in \mathcal{F}(C^2)$. If also there exists a compact subinterval J of the open interval I^* , and $C_2 > 0$ such that

$$(2.4) \quad \frac{Q^{*''}(x)}{|Q^{*'}(x)|} \geq C_2 \frac{|Q^{*'}(x)|}{Q^*(x)}, \text{ a.e. } x \in I^* \setminus J,$$

then we write $W^* \in \mathcal{F}(C^2+)$.

Lemma 2.2

(I)

$$W \in \mathcal{L}(C^2) \Leftrightarrow W^* \in \mathcal{F}(C^2).$$

(II)

$$W \in \mathcal{L}(C^2+) \Leftrightarrow W^* \in \mathcal{F}(C^2+).$$

Proof

(I) We first show that

$$W \in \mathcal{L}(C^2) \Rightarrow W^* \in \mathcal{F}(C^2).$$

Now $Q^{*'}(x) = 2Q'(x^2)x$ is continuous in $I^* \setminus \{0\}$ and by hypothesis (a) in Definition 1.1 has limit 0 at 0, so is continuous in I^* . So (a) in Definition 2.1 is satisfied. We see that (b), (c), (d) in Definition 2.1 follow directly from those in Definition 1.1, if we set $\Lambda^* = 2\Lambda$ and observe that

$$(2.5) \quad T^*(x) = 2T(x^2) \geq 2\Lambda =: \Lambda^*, x \in I^* \setminus \{0\}.$$

Finally, for $x \in (0, \sqrt{d})$, (1.10) and (1.11) give

$$\begin{aligned} 0 &< \frac{Q^{*''}(x)}{Q^{*'}(x)} = \frac{1}{x} + 2\frac{Q''(x^2)}{Q'(x^2)}x \\ &\leq \frac{T(x)}{\Lambda x} + 2C_1\frac{Q'(x^2)}{Q(x^2)}x \\ &= \frac{Q^{*'}(x)}{Q^*(x)} \left[\frac{1}{2\Lambda} + C_1 \right], \end{aligned}$$

so (2.3) in Definition 2.1 is satisfied. Thus $W^* \in \mathcal{F}(C^2)$.

Conversely, suppose that $W^* \in \mathcal{F}(C^2)$. We shall check that (e) of Definition 1.1 holds for W . The remaining properties follow directly. Using (2.2) and (2.3) of Definition 2.1,

$$\begin{aligned} 2 \left| x^2 \frac{Q''(x^2)}{Q'(x^2)} \right| &= \left| \frac{xQ^{*''}(x)}{Q^{*'}(x)} - 1 \right| \\ &\leq Cx \frac{Q^{*'}(x)}{Q^*(x)} + \frac{T^*(x)}{\Lambda^*} \\ &= 2 \left(C + \frac{1}{\Lambda^*} \right) \frac{x^2 Q'(x^2)}{Q(x^2)}. \end{aligned}$$

Then (1.11) of Definition 1.1 follows.

(II) This follows from (I) as (1.12) in Definition 1.1 is the same as (2.4) in Definition 2.1. ■

In the sequel, we shall denote the positive Mhaskar-Rakhmanov-Saff for the weight W^* by a_t^* , $t > 0$. Thus a_t^* is defined by

$$t = \frac{1}{\pi} \int_{-a_t^*}^{a_t^*} \frac{xQ^{*'}(x)}{\sqrt{a_t^{*2} - x^2}} dx = \frac{2}{\pi} \int_0^1 \frac{a_t^* u Q^{*'}(a_t^* u)}{\sqrt{1 - u^2}} du.$$

In terms of Q , we see that this becomes

$$\frac{t}{2} = \frac{1}{\pi} \int_0^1 \frac{a_t^{*2} v Q'(a_t^{*2} v)}{\sqrt{v(1-v)}} dv,$$

so that uniqueness of the definition of a_t gives

$$(2.6) \quad a_{t/2} = a_t^{*2}.$$

We shall also use the quantity

$$(2.7) \quad \eta_t = (tT(a_t))^{-2/3},$$

and its analogue for Q^*

$$(2.8) \quad \eta_t^* = \{tT^*(a_t^*)\}^{-2/3}.$$

We see that

$$(2.9) \quad \eta_{2t}^* = \{4tT(a_t)\}^{-2/3} = 4^{-2/3}\eta_t.$$

3. TECHNICAL ESTIMATES

In this section, we record a number of technical estimates for Q and a_t . Throughout we assume that $W \in \mathcal{L}(C^2)$.

Lemma 3.1

(a) *Uniformly for $t > 0$, we have*

$$(3.1) \quad Q'(a_t) \sim \frac{t}{a_t} \sqrt{T(a_t)};$$

$$(3.2) \quad Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}.$$

(b) *For $t \geq r > 0$,*

$$(3.3) \quad 1 \leq \frac{a_t}{a_r} \leq C\left(\frac{t}{r}\right)^{1/\Lambda}.$$

In particular for fixed $L > 1$ and uniformly for $t > 0$,

$$(3.4) \quad a_{Lt} \sim a_t.$$

(c) *Fix $L > 0$. Then uniformly for $t > 0$,*

$$(3.5) \quad Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), j = 0, 1.$$

Moreover,

$$(3.6) \quad T(a_{Lt}) \sim T(a_t) \text{ and } \eta_{Lt} \sim \eta_t.$$

(d) *For some $\varepsilon > 0$, and for large enough t ,*

$$(3.7) \quad T(a_t) \leq Ct^{2-\varepsilon}$$

and

$$(3.8) \quad \eta_t T(a_t) \leq Ct^{-\varepsilon} = o(1).$$

Proof

(a) Recall that Q^* is even, and that $a_t = (a_{2t}^*)^2$. Lemma 3.4 in [7, p. 69] gives

$$Q^{*'}(a_{2t}^*) \sim \frac{t}{a_{2t}^*} \sqrt{T^*(a_{2t}^*)}.$$

(Note that in the notation of [7], $\delta_t^* = a_t^*$ because Q^* is even). Then the relationship between Q and Q^* and T and T^* give (3.1). The relation (3.2) now follows from the identity

$$Q(x) = xQ'(x)/T(x).$$

(b) From Lemma 3.5(c) in [7, p. 72], we have as $\delta_{2t}^* = a_{2t}^*$ in the even case,

$$1 \leq \frac{a_{2t}^*}{a_{2r}^*} \leq C \left(\frac{t}{r} \right)^{1/\Lambda^*}$$

for $t > r > 0$. As $\Lambda^* = 2\Lambda$, the result follows.

(c) This follows similarly from Lemma 3.5(b) in [7, p. 72] and the relations between Q, Q^*, T, T^* .

(d) These follow similarly from Lemma 3.7 in [7, p. 76] and from (2.9). ■

Some further estimates involving a_t :

Lemma 3.2

(a) We have for $t > 0$,

$$(3.9) \quad \left| 1 - \frac{a_t}{a_s} \right| \sim \frac{1}{T(a_t)} \left| 1 - \frac{t}{s} \right|, \frac{1}{2} \leq \frac{s}{t} \leq 2.$$

(b) Given fixed $L > 1$, we have uniformly for $t > 0$,

$$(3.10) \quad \left| 1 - \frac{a_{Lt}}{a_t} \right| \sim \frac{1}{T(a_t)}.$$

Proof

These follow from Lemma 3.11 in [7, pp. 81-82] and the identities relating T, T^*, a_t, a_t^* . ■

Lemma 3.3

(a) For $x \in [0, a_t)$,

$$(3.11) \quad Q'(x)\sqrt{x} \leq \frac{Ct}{\sqrt{a_t - x}}.$$

(b) Fix $L > 0$. Then for $t > 0$ and $x \in [La_t t^{-2}, a_t]$,

$$(3.12) \quad \frac{a_t}{t^2} Q'(x) \left(1 - \frac{x}{a_t} \right) \leq C/\sqrt{T(a_t)} \leq C_1.$$

Proof

(a) From Lemma 3.8(a) in [7, p. 77],

$$Q^{*'}(y) \leq \frac{Ct}{\sqrt{a_{2t}^*(a_{2t}^* - y)}},$$

for $y \in [0, a_{2t}^*) = [0, \sqrt{a_t})$. Setting $y = \sqrt{x}$ gives

$$\begin{aligned} Q'(x) \sqrt{x} &\leq \frac{Ct}{\sqrt{\sqrt{a_t}(\sqrt{a_t} - \sqrt{x})}} \\ &= \frac{Ct \left(\sqrt{\sqrt{a_t} + \sqrt{x}} \right)}{\sqrt{\sqrt{a_t}(a_t - x)}} \\ &\leq \frac{Ct}{\sqrt{a_t - x}}. \end{aligned}$$

(b) By Lemma 3.8(b) in [7, p. 77],

$$\frac{a_{2t}^*}{t} Q^{*'}(y) \left(1 - \frac{y}{a_{2t}^*} \right) \leq C / \sqrt{T^*(a_{2t}^*)}.$$

Setting $y = \sqrt{x}$ gives

$$\frac{\sqrt{a_t}}{t} \sqrt{x} Q'(x) \left(1 - \sqrt{\frac{x}{a_t}} \right) \leq C / \sqrt{T(a_t)}.$$

Multiplying by $\frac{\sqrt{a_t}}{t\sqrt{x}} \left(1 + \sqrt{\frac{x}{a_t}} \right)$ gives

$$\frac{a_t}{t^2} Q'(x) \left(1 - \frac{x}{a_t} \right) \leq \frac{C}{t} \sqrt{\frac{a_t}{xT(a_t)}} \leq \frac{C}{\sqrt{T(a_t)}},$$

provided $x \geq La_t t^{-2}$, some fixed $L > 0$. ■

4. POTENTIAL THEORY

Let us assume that the function $\sqrt{x}Q'(x)$ is increasing in I , with limit 0 at 0 and limit ∞ at d . This is essentially equivalent to Q^* being convex on I^* . We recall [7], [20] that, given $t > 0$, there is a unique positive measure μ_t of total mass t , and a unique constant c_t , such that

$$(4.1) \quad V^{\mu_t}(x) + Q(x) \begin{cases} = c_t, & x \in S(\mu_t) \\ > c_t, & x \in I \setminus S(\mu_t) \end{cases},$$

where $S(\mu_t)$ denotes the support of the measure μ_t , and

$$V^{\mu_t}(x) = \int \log \frac{1}{|x-s|} d\mu_t(s)$$

is the corresponding potential. This measure μ_t is the equilibrium measure for the external field Q . In this section, we relate μ_t to the corresponding measure μ_t^* for Q^* , and hence establish some basic results about μ_t .

Given $t > 0$, we let μ_t^* denote the equilibrium measure for Q^* so that

$$(4.2) \quad V^{\mu_t^*}(x) + Q^*(x) \begin{cases} = c_t^*, & x \in S(\mu_t^*) \\ > c_t^*, & x \in I^* \setminus S(\mu_t^*) \end{cases} .$$

We let σ_t and σ_t^* denote the densities for μ_t and μ_t^* respectively. Under mild conditions on Q or Q^* , there is a simple relationship between the supports $S(\mu_t^*)$, $S(\mu_t)$, the densities σ_t^* , σ_t , and the associated potentials:

Theorem 4.1

Let $\sqrt{x}Q'(x)$ be increasing in I , with limit 0 at 0 and limit ∞ at d . Assume moreover, that

$$(4.3) \quad 0 = Q(0) < Q(x), x \in (0, d) .$$

Let $t > 0$.

(a) μ_t is absolutely continuous with respect to Lebesgue measure and its density σ_t is given by

$$(4.4) \quad \sigma_t(x) = \frac{1}{2\sqrt{x}} \sigma_{2t}^*(\sqrt{x}), x \in (0, (a_{2t}^*)^2),$$

where σ_{2t}^* is the density of the equilibrium measure μ_{2t}^* for Q^* .

(b) Moreover,

$$(4.5) \quad V^{\mu_t}(z^2) = V^{\mu_{2t}^*}(z), z \in \mathbb{C};$$

$$(4.6) \quad a_t = (a_{2t}^*)^2;$$

$$(4.7) \quad c_t = c_{2t}^* = \int_0^t \log \frac{4}{a_s} ds.$$

Proof

Let ν denote the measure with density given by (4.4). We shall show that ν has mass t and satisfies (4.1) with some constant c_t . Uniqueness of the equilibrium measure then gives the result. First recall that Q^* is even, so that its equilibrium density is also even. Moreover the hypotheses above on Q imply that Q^* satisfies the hypotheses of Theorem

2.4 in [7, pp. 40-41]. Now

$$\begin{aligned} \int_0^{a_{2t}^*} \frac{1}{2\sqrt{x}} \sigma_{2t}^*(\sqrt{x}) dx &= \int_0^{a_{2t}^*} \sigma_{2t}^*(s) ds \\ &= \frac{1}{2} \int_{-a_{2t}^*}^{a_{2t}^*} \sigma_{2t}^*(s) ds = t. \end{aligned}$$

Next,

$$V^{\mu_{2t}^*}(z) = \int_{-a_{2t}^*}^{a_{2t}^*} \log \frac{1}{|z-s|} \sigma_{2t}^*(s) ds = \int_{-a_{2t}^*}^{a_{2t}^*} \log \frac{1}{|z+s|} \sigma_{2t}^*(s) ds$$

by evenness of σ_{2t}^* . Therefore,

$$\begin{aligned} V^{\mu_{2t}^*}(z) &= \frac{1}{2} \int_{-a_{2t}^*}^{a_{2t}^*} \log \frac{1}{|z^2-s^2|} \sigma_{2t}^*(s) ds \\ &= \int_0^{a_{2t}^*} \log \frac{1}{|z^2-y|} \sigma_{2t}^*(\sqrt{y}) \frac{dy}{2\sqrt{y}} \\ &= V^\nu(z^2), \end{aligned}$$

where ν denotes the measure on $[0, (a_{2t}^*)^2]$, with density given by the right-hand side of (4.4). Next, let $x \in [0, (a_{2t}^*)^2]$ and write $x = y^2$, where $y \in [0, a_{2t}^*]$. Then

$$\begin{aligned} V^\nu(x) + Q(x) &= V^\nu(y^2) + Q(y^2) \\ &= V^{\mu_{2t}^*}(y) + Q^*(y) \\ &= c_{2t}^*, \end{aligned}$$

by the equilibrium relation (4.2) for Q^* . Uniqueness of the equilibrium measure shows that

$$\nu = \mu_t$$

and that (4.1) holds. We proved (4.6) at the end of Section 2. Finally, from uniqueness of c_t followed by (2.34) in [7, p. 46],

$$\begin{aligned} c_t &= c_{2t}^* \\ &= \int_0^{2t} \log \frac{2}{a_\tau^*} d\tau \\ &= \int_0^{2t} \log \frac{2}{\sqrt{a_\tau/2}} d\tau \\ &= \int_0^t \log \frac{4}{a_s} ds. \end{aligned}$$

■

Next, we state a formula for, and an estimate of, the density $\sigma_t(x)$:

Theorem 4.2

Let $W \in \mathcal{L}(C^2)$.

(a) For $x \in [0, a_t]$,

$$(4.8) \quad \sigma_t(x) = \frac{1}{\pi^2} \sqrt{\frac{a_t - x}{x}} \int_0^{a_t} \frac{uQ'(u) - xQ'(x)}{u - x} \frac{du}{\sqrt{u(a_t - u)}}.$$

(b) Uniformly for $t > 0$,

$$(4.9) \quad \sigma_t(x) \sim \frac{t\sqrt{a_t - x}}{\sqrt{x}(a_{2t} - x)}, x \in (0, a_t).$$

Proof

(a) From (5.23) in [7, p. 116],

$$\sigma_{2t}^*(y) = \frac{\sqrt{a_{2t}^{*2} - y^2}}{\pi^2} \int_{-a_{2t}^*}^{a_{2t}^*} \frac{Q^{*'}(s) - Q^{*'}(y)}{s - y} \frac{ds}{\sqrt{a_{2t}^{*2} - s^2}}.$$

Using (4.4), $Q^{*'}(s) = 2sQ'(s^2)$ and some elementary manipulations, we obtain (4.11).

(b) Recall from Lemma 2.2 that

$$W \in \mathcal{L}(C^2) \Leftrightarrow W^* \in \mathcal{F}(C^2).$$

Then we may apply Theorem 5.3 in [7, p.111]: uniformly in t and y ,

$$\sigma_{2t}^*(y) \sim \frac{t\sqrt{a_{2t}^{*2} - y^2}}{a_{4t}^{*2} - y^2}, y \in \Delta_t.$$

Then (4.4) gives the result. ■

Recall that we defined φ_t at (1.18). Theorem 4.2 shows that φ_t is basically the reciprocal of σ_t . More precisely, if $\beta, \varepsilon > 0$ are fixed then for $t > 0$,

$$(4.10) \quad \varphi_t(x) \sim \sigma_t^{-1}(x), x \in [\beta a_t t^{-2}, a_t(1 - \varepsilon \eta_t)],$$

The following lemma involving φ_t will be useful:

Lemma 4.3

Let $W \in \mathcal{L}(C^2)$. Given $A, B \in \mathbb{R}$ with $A < B$, there exists $M > 0, t_0 > 0$ such that

$$(4.11) \quad \sigma_t(x + \lambda \sigma_t^{-1}(x)) \sim \sigma_t(x), x \in [M a_t t^{-2}, a_t(1 - M \eta_t)],$$

and

$$(4.12) \quad \varphi_t(x + \lambda \varphi_t(x)) \sim \varphi_t(x), x \in I,$$

uniformly for $\lambda \in [A, B]$, $t \geq t_0$, and for x in the above intervals. Conversely, given $M > 0$, there exist $t_0, \varepsilon > 0$ such that (4.11) and (4.12) hold provided $|\lambda| \leq \varepsilon$ and $t \geq t_0$.

Proof

(I) We prove the second statement (4.12). Then (4.11) follows from Theorem 4.2. In view of the definition (1.18) of φ_t , we need to show that for the given A, B and $\lambda \in [A, B]$, there exists $M > 0$ such that for $x \in [Ma_t t^{-2}, a_t(1 - M\eta_t)]$,

$$(4.13) \quad x + a_t t^{-2} \sim (x + \lambda \varphi_t(x)) + a_t t^{-2};$$

$$(4.14) \quad a_{2t} - x \sim a_{2t} - (x + \lambda \varphi_t(x));$$

$$(4.15) \quad a_t - x + a_t \eta_t \sim a_t - (x + \lambda \varphi_t(x)) + a_t \eta_t.$$

We do the first and third of these; the second is easier than the third, because a_{2t} is larger than $a_t + a_t \eta_t$ for large t . Let

$$D = \max\{|A|, |B|\}.$$

Proof of (4.13)

If first $x \in [Ma_t t^{-2}, a_{t/2}]$, then from (1.18),

$$(4.16) \quad \begin{aligned} \frac{|\lambda| \varphi_t(x)}{x + a_t t^{-2}} &\leq \frac{D(a_{2t} - x)}{t \sqrt{x + a_t t^{-2}} \sqrt{a_t - x + a_t \eta_t}} \\ &\leq \frac{C(a_{2t} - a_t) + (a_t - x)}{t \sqrt{x} \sqrt{a_t - x}} \\ &\leq \frac{C}{t} \frac{1}{\sqrt{Ma_t t^{-2}}} \left[\frac{a_{2t} - a_t}{\sqrt{a_t - a_{t/2}}} + \sqrt{a_t - a_{t/2}} \right]. \end{aligned}$$

We continue this using (3.10), (3.4) and (3.6) as

$$\leq \frac{C}{\sqrt{a_t M}} \left[\sqrt{\frac{a_t}{T(a_t)}} + \sqrt{\frac{a_t}{T(a_t)}} \right] \leq \frac{C}{\sqrt{M}}.$$

Next, if $x \in [a_{t/2}, a_t(1 - M\eta_t)]$, (4.16) becomes

$$\begin{aligned} \frac{|\lambda| \varphi_t(x)}{x + a_t t^{-2}} &\leq \frac{D(a_{2t} - a_{t/2})}{t \sqrt{a_{t/2}} \sqrt{a_t \eta_t}} \\ &\leq \frac{C}{t T(a_t) \sqrt{\eta_t}} = C \eta_t = O(t^{-2/3}), \end{aligned}$$

by (3.10) again. Together the above estimates show that if t is large enough and M is large enough, we have

$$\frac{|\lambda| \varphi_t(x)}{x + a_t t^{-2}} \leq \frac{1}{2}$$

for the specified range of x, t, λ . So we have (4.13).

Proof of (4.15)

Now for $x \in [Ma_t t^{-2}, a_t(1 - M\eta_t)]$,

$$\begin{aligned} & \left| 1 - \frac{a_t - (x + \lambda\varphi_t(x)) + a_t\eta_t}{a_t - x + a_t\eta_t} \right| \\ &= \frac{|\lambda|\varphi_t(x)}{a_t - x + a_t\eta_t} \\ &\leq \frac{D\sqrt{x + a_t t^{-2}}(a_{2t} - x)}{t(a_t - x + a_t\eta_t)^{3/2}} \\ &\leq C \frac{\sqrt{a_t} a_{2t} - a_t + a_t - x}{t(a_t - x)^{3/2}} \\ &\leq C \frac{\sqrt{a_t}}{t} \left(\frac{a_t}{T(a_t)[Ma_t\eta_t]^{3/2}} + \frac{1}{\sqrt{Ma_t\eta_t}} \right), \end{aligned}$$

by (3.10) and as $x \geq a_t(1 - M\eta_t)$. We continue this, using the definition of η_t , as

$$\leq C \left(\frac{1}{M^{3/2}} + \frac{1}{M^{1/2}} \left(\frac{T(a_t)}{t^2} \right)^{1/3} \right) \leq \frac{C}{M^{1/2}},$$

by (3.7). Since C is independent of M , we obtain, if M is large enough,

$$\left| 1 - \frac{a_t - (x + \lambda\varphi_t(x)) + a_t\eta_t}{a_t - x + a_t\eta_t} \right| \leq \frac{1}{2}$$

for the specified range of x, t, λ . So we have (4.15).

The converse part of the lemma follows similarly. ■

Lemma 4.4

Let $M > 0$. There exists t_0 such that for $t \geq t_0$ and $x \in I$,

$$(4.17) \quad \varphi_{t+M}(x) \sim \varphi_t(x).$$

Proof

This follows easily from (3.9) and the definition of φ_t . ■

5. RESTRICTED RANGE INEQUALITIES

For $t \geq 0$, we denote by \mathbb{P}_t the set of all functions of the form

$$(5.1) \quad P(z) = c \exp\left(\int \log |z - \xi| d\nu(\xi)\right),$$

where $\nu \geq 0$, $\|\nu\| \leq t$, $c \geq 0$, and the support of ν is compact. These are the exponentials of potentials of mass $\leq t$. In particular if $t \geq n$, then $\mathcal{P}_n \subset \mathbb{P}_t$. Note too that for $P \in \mathbb{P}_t$, we have $P(z^2) \in \mathbb{P}_{2t}$. Recall also the notation

$$\Delta_t = [0, a_t].$$

In this section, we present L_p analogues of the Mhaskar-Saff inequality for the class \mathbb{P}_t .

Theorem 5.1

Let $W := e^{-Q}$ where $Q : I \rightarrow [0, \infty)$ is such that $Q^*(x) = Q(x^2)$ is convex in I^* . Assume moreover that $Q(d-) = \infty$ and $Q(x) > 0 = Q(0)$, $x \in I \setminus \{0\}$. Let $0 < p < \infty$ and $\beta > -\frac{1}{p}$. Let $P \in \mathbb{P}_{t-\beta-\frac{3}{2p}} \setminus \{0\}$.

Then

$$(5.2) \quad \|(PW)(x)x^\beta\|_{L_p(I \setminus \Delta_t)} < \|(PW)(x)x^\beta\|_{L_p(\Delta_t)};$$

and

$$(5.3) \quad \|(PW)(x)x^\beta\|_{L_p(I)} < 2^{1/p} \|(PW)(x)x^\beta\|_{L_p(\Delta_t)}.$$

In particular this holds for not-identically vanishing polynomials P of degree $\leq t - \beta - \frac{3}{2p}$. For $p = \infty$, (5.2) and (5.3) remain valid with $<$ replaced by \leq , provided $\beta \geq 0$.

Under additional assumptions, we can improve the above result, and “go back” into the interval Δ_t , giving a Schur type inequality. Recall the numbers

$$\eta_t = \{tT(a_t)\}^{-2/3}, t > 0,$$

which are small for large t .

Theorem 5.2

Let $W \in \mathcal{L}(C^2)$. Let $0 < p \leq \infty$ and $L, \lambda \geq 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$.

(a) There exist C_1, t_0 such that for $t \geq t_0$ and $P \in \mathbb{P}_t$,

$$(5.4) \quad \|(PW)(x)x^\beta\|_{L_p(I)} \leq C_1 \|(PW)(x)x^\beta\|_{L_p[L a_t t^{-2}, a_t(1-\lambda\eta_t)]}.$$

(b) For $t, \kappa > 0$, define

$$(5.5) \quad H(\kappa, t) := \frac{\min\{\kappa, T(a_t)^{-1}\}}{\eta_t}.$$

There exist C_2, C_3 independent of t, κ, P with the following properties: for $t > 0$ and $P \in \mathbb{P}_t$,

(5.6)

$$\| (PW)(x) x^\beta \|_{L_p(a_t(1+\kappa), d)} \leq C_2 \exp(-C_3 H(\kappa, t)^{3/2}) \| (PW)(x) x^\beta \|_{L_p(\Delta_t)}.$$

Furthermore, given $r > 1$, we have for some $t_0, \alpha > 0$ and $t \geq t_0$,

$$(5.7) \quad \| (PW)(x) x^\beta \|_{L_p(a_{rt}, d)} \leq \exp(-Ct^\alpha) \| (PW)(x) x^\beta \|_{L_p(\Delta_t)}.$$

We note that the conditions on W may be relaxed; all we need is that W^* satisfy the hypotheses of Theorem 4.2 in [7, p. 96]. We begin with a Lemma which is similar to Lemma 4.4 in [7, p. 99ff.]. Recall that the Green's function for $\mathbb{C} \setminus [a, b]$ with pole at ∞ is

$$g_{[a,b]}(z) = \log \left| \frac{2}{b-a} \left(z - \frac{a+b}{2} \right) + \frac{2}{b-a} \sqrt{(z-a)(z-b)} \right|.$$

It is harmonic in $\mathbb{C} \setminus [a, b]$, equal to 0 on $[a, b]$, and behaves like $\log |z| + O(1)$ as $z \rightarrow \infty$.

Lemma 5.3

Let $\Delta = [a, b] \ni 0$ and $0 < p \leq \infty$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$. Let $\Omega \geq 0, t > 0, c \in \mathbb{C}$, and ν be a non-negative Borel measure with compact support and total mass $\leq \Omega$. Let

$$P(z) := c \exp\left(\int \log |z - y| d\nu(y) \right).$$

Let $\alpha \in \mathbb{R}$ and U be a function harmonic in $\mathbb{C} \setminus \Delta$ with

$$(5.8) \quad U(z) = \alpha \log |z| + o(1), \quad z \rightarrow \infty.$$

Assume moreover, that on Δ , U has boundary values U_\pm from the upper and lower half plane that satisfy

$$U_+ = U = U_-$$

where $U \in L_p(\Delta)$. Let g_Δ denote the Green's function for $\mathbb{C} \setminus \Delta$. Then

(5.9)

$$\| P(x) e^{U(x) - (\Omega + \alpha + \frac{2}{p} + \max\{0, \beta\}) g_\Delta(x)} |x|^\beta \|_{L_p(\mathbb{R} \setminus \Delta)} \leq C \| (Pe^U)(x) |x|^\beta \|_{L_p(\Delta)}.$$

Here $C = C(\beta)$ only. If $\beta \geq 0$, we can take $C = 1$.

Proof

We assume $p < \infty$. (The case $p = \infty$ follows by letting $p \rightarrow \infty$). The proof is similar to Lemma 4.4 in [7, p. 99 ff.]. We note that it suffices to prove this with ν having total mass Ω . For, $g_\Delta \geq 0$, so the left-hand

side of (5.9) decreases as we increase Ω . Hence if we have (5.9) with Ω replaced by $\|\nu\|$, then it holds as stated. Thus we assume ν has total mass Ω . We may also clearly assume $c = 1$.

Let $g_\Delta(z, x)$ denote the Green's function for the exterior of an interval Δ with pole at x . In the special case $x = \infty$, we have already used the notation $g_\Delta(x) = g_\Delta(x, \infty)$. In the case $x \in \Delta$, we just set $g_\Delta(z, x) \equiv 0$. Now assume $x \notin \Delta$. The Green's function $g_\Delta(z, x)$ has the following properties:

- (i) $g_\Delta(z, x) + \log |z - x|$ is harmonic (as a function of z) in $\mathbb{C} \setminus \Delta$;
- (ii) $g_\Delta(z, x) = 0, z \in \Delta$ and $g_\Delta(z, x) \geq 0$ on \mathbb{C} .

Define the function

$$\begin{aligned} g(z) &:= \frac{1}{t} \int \{\log |z - x| + g_\Delta(z, x)\} d\nu(x) \\ &\quad + \frac{1}{t} U(z) - \frac{\Omega + \alpha}{t} g_\Delta(z) + \frac{\beta}{t} (\log |z| + g_\Delta(z, 0) - g_\Delta(z)) \\ &=: g_1(z) + \frac{1}{t} U(z) - \frac{\Omega + \alpha}{t} g_\Delta(z) + \frac{\beta}{t} (\log |z| + g_\Delta(z, 0) - g_\Delta(z)). \end{aligned}$$

Now (as in [7, pp. 99-100]) g_1 is harmonic in $\mathbb{C} \setminus \Delta$ and

$$g_1(z) = \frac{\Omega}{t} \log |z| + \frac{1}{t} \int g_\Delta(\infty, x) d\nu(x) + o(1), z \rightarrow \infty.$$

Next, $\frac{1}{t} U - \frac{\Omega + \alpha}{t} g_\Delta$ is harmonic in $\mathbb{C} \setminus \Delta$, and behaves like

$$\frac{1}{t} (\alpha - \Omega - \alpha) \log |z| + \text{Constant} + o(1) = -\frac{\Omega}{t} \log |z| + \text{Constant} + o(1), z \rightarrow \infty.$$

Finally, $\beta (\log |z| + g_\Delta(z, 0) - g_\Delta(z))$ is harmonic in $\mathbb{C} \setminus \Delta$ and has a finite limit at ∞ . It follows that g is harmonic in $\overline{\mathbb{C}} \setminus \Delta$, for it has a finite limit at ∞ . Hence it has a single-valued harmonic conjugate $\tilde{g}(z)$ there. Then the function

$$f(z) := \exp(g(z) + i\tilde{g}(z))$$

is analytic and single-valued in $\overline{\mathbb{C}} \setminus \Delta$ and has no zeros there, so we may define a single-valued branch of $f^{tp/2}(z)$ in $\overline{\mathbb{C}} \setminus \Delta$. Let $\tilde{g}_\Delta(z)$ denote the harmonic conjugate of $g_\Delta(z)$ in $\overline{\mathbb{C}} \setminus \Delta$ so that

$$A(z) := \exp(g_\Delta(z) + i\tilde{g}_\Delta(z))$$

is analytic there except for a simple pole at ∞ .

Now let us look at the boundary values $(f^t)_\pm$ of f^t . In Δ , we have

$$(5.10) \quad \left| (f^t)_\pm(x) \right| = \exp(tg_\pm(x)) = |P|(x) e^{U(x)} |x|^\beta.$$

Moreover in $\mathbb{R} \setminus \Delta$,

$$(5.11) \quad |f^t(x)| = |P|(x) e^{U(x)} |x|^\beta e^{h(x)},$$

where

$$(5.12) \quad h(x) = \int g_\Delta(x, y) d\nu(y) - (\Omega + \alpha) g_\Delta(x) + \beta \{g_\Delta(x, 0) - g_\Delta(x)\}.$$

Now we consider two subcases.

Case I: $\beta \geq 0$

Since $\beta \geq 0$ and $g_\Delta \geq 0$, we see that

$$(5.13) \quad h(x) \geq -(\Omega + \alpha + \beta) g_\Delta(x).$$

Next, we apply Lemma 4.3 in [7, p. 98] (with $p = 2$) to the analytic function $f^{tp/2}/A$, obtaining

$$(5.14) \quad \|f^{tp/2}/A\|_{L_2(\mathbb{R} \setminus \Delta)} \leq \frac{1}{2} \left\{ \|f_+^{tp/2}/A_+\|_{L_2(\Delta)} + \|f_-^{tp/2}/A_-\|_{L_2(\Delta)} \right\}.$$

Then (5.10-5.13) and the fact that $|A_\pm| = 1$ in Δ while $|A| = \exp(g_\Delta)$ in the rest of the real line give (5.9) with $C = 1$.

Case II: $-\frac{1}{p} < \beta < 0$

We use g above, but with $\beta = 0$, so that in Δ ,

$$(5.15) \quad \left| (f^t)_\pm(x) \right| = \exp(tg_\pm(x)) = |P|(x) e^{U(x)}.$$

Moreover in $\mathbb{R} \setminus \Delta$, (5.11) holds with $\beta = 0$ and with

$$(5.16) \quad h(x) = \int g_\Delta(z, x) d\nu(x) - (\Omega + \alpha) g_\Delta(x).$$

As above, we may choose a single-valued branch of $f^{tp/2}/A$ in $\mathbb{C} \setminus \Delta$. Since this function vanishes at ∞ , Cauchy's integral formula gives

$$(f^{tp/2}/A)(z) = \frac{1}{2\pi i} \int_a^b \frac{(f^{tp/2}/A)_+(x) - (f^{tp/2}/A)_-(x)}{t - z} dt,$$

$z \notin \Delta$. We may rewrite this as

$$(f^{tp/2}/A)(z) = \frac{1}{2} \left(H \left[(f^{tp/2}/A)_+ \right] (z) - H \left[(f^{tp/2}/A)_- \right] (z) \right),$$

where H denotes the Hilbert transform. Then we may apply the weighted inequality for the Hilbert transform [14, p. 440],

$$\| H[F](x) |x|^\gamma \|_{L_2(\mathbb{R})} \leq C \| F(x) |x|^\gamma \|_{L_2(\mathbb{R})},$$

valid if $\gamma \in (-\frac{1}{2}, \frac{1}{2})$ and provided the right-hand side is finite. Choosing $F = (f^{tp/2}/A)_\pm$ and $\gamma = \frac{\beta p}{2} \in (-\frac{1}{2}, 0)$ gives

$$\begin{aligned} & \int_{\mathbb{R} \setminus \Delta} |f^{tp/2}/A|^2(x) |x|^{\beta p} dx \\ & \leq C \left[\int_{\Delta} \left| (f^{tp/2}/A)_+ \right|^2(x) |x|^{\beta p} dx + \int_{\Delta} \left| (f^{tp/2}/A)_- \right|^2(x) |x|^{\beta p} dx \right] \\ & \leq 2C \int_{\Delta} |Pe^U|^p(x) |x|^{\beta p} dx, \end{aligned}$$

by (5.15). Finally (5.11), (5.16) and the fact that in this case

$$h(x) \geq -(\Omega + \alpha) g_{\Delta}(x), \quad x \notin \Delta,$$

give the result. ■

Proof of Theorem 5.1

We do this in 2 steps.

Step 1: Apply Lemma 5.3 to the weight W^*

We apply Lemma 5.3 with $\beta = 0$ there, with $\Delta = \Delta_{2t}^* = [-a_{2t}^*, a_{2t}^*]$, and with

$$U(z) = V^{\mu_{2t}^*}(z) + 2\beta \log |z|.$$

Then

$$U(z) = (2\beta - 2t) \log |z| + o(1), \quad z \rightarrow \infty,$$

so (5.8) holds with $\alpha = 2\beta - 2t$. Also, by (4.2),

$$\begin{aligned} U(x) &= -Q^*(x) + c_{2t}^* + 2\beta \log |x|, \quad x \in \Delta_{2t}^*; \\ U(x) &> -Q^*(x) + c_{2t}^* + 2\beta \log |x|, \quad x \in I^* \setminus \Delta_{2t}^*. \end{aligned}$$

Then (5.9) implies (recall that $C = 1$ as we use Lemma 5.3 with $\beta = 0$),

$$\| (RW^*)(x) |x|^{2\beta} e^{-(\Omega + 2\beta - 2t + \frac{2}{p})g_{\Delta_{2t}^*}(x)} \|_{L_p(I \setminus \Delta_{2t}^*)} \leq \| (RW^*)(x) |x|^{2\beta} \|_{L_p(\Delta_{2t}^*)},$$

provided $R \in \mathbb{P}_\Omega$. In particular, as $g_{\Delta_{2t}^*} > 0$ outside Δ_{2t}^* , we obtain

$$(5.17) \quad \| (RW^*)(x) |x|^{2\beta} \|_{L_p(I \setminus \Delta_{2t}^*)} < \| (RW^*)(x) |x|^{2\beta} \|_{L_p(\Delta_{2t}^*)},$$

provided

$$\Omega \leq 2t - 2\beta - \frac{2}{p}.$$

Step 2: Transfer estimates to W

Let $P \in \mathbb{P}_{t-\beta-\frac{3}{2p}} \setminus \{0\}$, and

$$R(y) = P(y^2) |y|^{1/p} \in \mathbb{P}_{2t-2\beta-\frac{2}{p}}.$$

Since RW^* is even, (5.17) gives

$$2 \int_{a_{2t}^*}^{\sqrt{d}} (RW^*)^p(y) y^{2p\beta} dy < 2 \int_0^{a_{2t}^*} (RW^*)^p(y) y^{2p\beta} dy.$$

The substitution $x = y^2$ and the fact that $a_{2t}^* = \sqrt{a_t}$ gives (5.2). ■

We begin the proof of Theorem 5.2 with

Lemma 5.4

Let $W \in \mathcal{L}(C^2)$. Let $0 < p \leq \infty$ and $\lambda \geq 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$. There exist C_1, t_0 such that for $P \in \mathbb{P}_t$,

$$(5.18) \quad \|(PW)(x) x^\beta\|_{L_p(I)} \leq C_1 \|(PW)(x) x^\beta\|_{L_p[0, a_t(1-\lambda\eta_t)]}.$$

Proof

Case I: $\beta \geq 0$

Let

$$\tau = t + \beta + \frac{1}{2p},$$

and

$$R(y) = P(y^2) y^{2\beta+\frac{1}{p}} \in \mathbb{P}_{2\tau},$$

so we can apply Theorem 4.2(a) in [7, p. 96] to deduce that

$$\|RW^*\|_{L_p(I^*)} \leq C \|RW^*\|_{L_p(-a_{2\tau}^*(1-\lambda\eta_{2\tau}^*), a_{2\tau}^*(1-\lambda\eta_{2\tau}^*))}.$$

Here we need t large enough and C is independent of R, t, τ . On making the substitutions $x = y^2$ in the integrals in the norms, and using $a_{2\tau}^* = \sqrt{a_\tau}$, we obtain

$$\|(PW)(x) x^\beta\|_{L_p(I)} \leq C \|(PW)(x) x^\beta\|_{L_p[0, a_\tau(1-\lambda\eta_{2\tau}^*)^2]}.$$

Here in view of (2.9),

$$(1 - \lambda\eta_{2\tau}^*)^2 = 1 - 2^{-1/3}\lambda\eta_\tau + o(\eta_\tau).$$

Moreover, by (3.9),

$$a_\tau/a_t = 1 + O\left(\frac{1}{t\Gamma(a_t)}\right) = 1 + o(\eta_t),$$

while by (3.6), $\eta_\tau \sim \eta_t$. Then (5.18) follows for large enough t , if we change λ a little. ■

Lemma 5.5

Let $W \in \mathcal{L}(C^2)$. Let $0 < p \leq \infty$ and $\lambda \geq 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$. There exist C_1, t_0 such that for $P \in \mathbb{P}_t$,

(5.19)

$$\| (PW)(x) x^\beta \|_{L_p[0, La_t t^{-2}]} \leq C_1 \| (PW)(x) x^\beta \|_{L_p[La_t t^{-2}, a_t(1-\lambda\eta_t)]}.$$

Proof

Let us write for large enough t ,

$$a_t(1 - \lambda\eta_t) = a_\tau \text{ and } J = [La_t t^{-2}, a_\tau].$$

In view of (3.9), we see that

$$\lambda\eta_t = 1 - \frac{a_\tau}{a_t} \sim \frac{1}{T(a_t)} \left(1 - \frac{\tau}{t}\right)$$

whence

$$(5.20) \quad t - \tau \sim \eta_t t T(a_t) = (tT(a_t))^{1/3} = o(t).$$

Let ℓ denote the linear map of J onto $\Delta_\tau = [0, a_\tau]$ so that

$$\ell(z) = (z - La_t t^{-2}) \frac{1 - \lambda\eta_t}{1 - \lambda\eta_t - Lt^{-2}}.$$

Let

$$v(z) := V^{\mu_\tau}(\ell(z)), z \in \mathbb{C}.$$

Then the equilibrium condition (4.1) for V^{μ_τ} yields

$$(5.21) \quad v(x) + Q(\ell(x)) = c_\tau, x \in J.$$

We claim that

$$(5.22) \quad 0 \leq Q(x) - Q(\ell(x)) \leq C, x \in J.$$

Indeed the left inequality follows as Q is increasing, and as $\ell(x) \leq x$. We proceed to prove the right-hand one. For $x \in J$, we have for some ξ between x and $\ell(x)$,

$$\begin{aligned} Q(x) - Q(\ell(x)) &= Q'(\xi)(x - \ell(x)) \\ &= Q'(\xi) Lt^{-2} \frac{a_t(1 - \lambda\eta_t) - x}{1 - \lambda\eta_t - Lt^{-2}}. \end{aligned}$$

Here $x \geq \xi \geq \ell(x)$, so we can continue this as

$$Q(x) - Q(\ell(x)) \leq \frac{Q'(\xi)(a_t - \xi)Lt^{-2}}{1 - \lambda\eta_t - Lt^{-2}} \leq C,$$

by (3.12). Here we need t large enough, and C is independent of x, t . So we have (5.22). Then we may recast (5.21) as

$$(5.23) \quad |v(x) + Q(x) - c_\tau| \leq C, x \in J.$$

Next, v is harmonic outside J , and

$$v(z) = -\tau \log |z| + \text{Constant} + o(1), z \rightarrow \infty.$$

We apply Lemma 5.3 to $U = v - \text{Constant}$, $\Omega = t$, $\alpha = -\tau$, $\Delta = J$. We obtain

$$\begin{aligned} & \| P(x) \exp \left\{ v(x) - c_\tau - \left(t - \tau + \frac{2}{p} + \max\{0, \beta\} \right) g_J(x) \right\} x^\beta \|_{L_p[0, La_t t^{-2}]} \\ & \leq C \| (P \exp(v - c_\tau))(x) x^\beta \|_{L_p(J)} \leq C_1 \| (PW)(x) x^\beta \|_{L_p(J)}, \end{aligned}$$

by (5.23). Then we obtain (5.19) provided

$$v(x) - c_\tau - \left(t - \tau + \frac{2}{p} + \max\{0, \beta\} \right) g_J(x) \geq -Q - C \text{ on } [0, La_t t^{-2}].$$

Since Q is bounded on $[0, La_t t^{-2}]$, we can establish the right-hand side without Q . Now for any $[a, b]$, $g_{[a, b]}$ is positive and decreasing on $(-\infty, a]$. Moreover, v is increasing on $(-\infty, La_t t^{-2}]$. Therefore it suffices to show that

$$(5.24) \quad v(0) - c_\tau \geq -C;$$

$$(5.25) \quad \left(t - \tau + \frac{2}{p} + \max\{0, \beta\} \right) g_J(0) \leq C.$$

To prove (5.24), we observe that as $Q(0) = 0$, (4.1) gives

$$\begin{aligned} v(0) - c_\tau &= V^{\mu_\tau}(\ell(0)) - V^{\mu_\tau}(0) \\ &= \int_0^{a_\tau} \log \left| \frac{s}{s - \ell(0)} \right| d\mu_\tau(s). \end{aligned}$$

Since for $s \geq |\ell(0)|$,

$$\log \left| \frac{s}{s - \ell(0)} \right| \sim -\frac{|\ell(0)|}{s}$$

and since $a_\tau \sim a_{2\tau} \sim a_t$, we can use the estimate for μ'_τ in (4.9) to obtain

$$\begin{aligned} & v(0) - c_\tau \\ & \geq C \left[\int_0^{|\ell(0)|} \log \left| \frac{s}{s + |\ell(0)|} \right| \frac{\tau}{\sqrt{sa_\tau}} ds - \int_{|\ell(0)|}^{\frac{1}{2}a_\tau} \frac{|\ell(0)|}{s} \frac{\tau}{\sqrt{sa_\tau}} ds - \int_{\frac{1}{2}a_\tau}^{a_\tau} \frac{|\ell(0)|}{s} \frac{\tau}{\sqrt{a_\tau}} \frac{ds}{\sqrt{a_\tau - s}} \right] \\ & \geq C \frac{\tau}{\sqrt{a_\tau}} \left[\sqrt{|\ell(0)|} \int_0^1 \log \left| \frac{y}{y+1} \right| \frac{dy}{\sqrt{y}} - \sqrt{|\ell(0)|} \int_1^\infty \frac{dy}{y^{3/2}} - \frac{|\ell(0)|}{\sqrt{a_\tau}} \right] \geq -C, \end{aligned}$$

since $|\ell(0)| \sim a_t t^{-2}$; $a_\tau \sim a_{2\tau} \sim a_t$; and $\tau \sim t$. So we have (5.24). Also

$$\begin{aligned} g_J(0) &= \log \left| -\frac{a_\tau + La_t t^{-2}}{a_\tau - La_t t^{-2}} + \frac{2}{a_\tau - La_t t^{-2}} \sqrt{La_t t^{-2} a_\tau} \right| \\ &= \log \left| -1 + O(t^{-1}) \right| = O(t^{-1}), \end{aligned}$$

so from (5.20),

$$\begin{aligned} & \left(t - \tau + \frac{2}{p} + \max\{0, \beta\} \right) g_J(0) \\ & \leq C (tT(a_t))^{1/3} t^{-1} = o(1), \end{aligned}$$

recall (3.7). ■

Proof of Theorem 5.2(a)

This follows directly from Lemmas 5.4 and 5.5. ■

Proof of Theorem 5.2(b) for $\beta \geq 0$

Let $P \in \mathbb{P}_t$. We derive this from Theorem 4.2(b) in [7, p. 96], applied to W^* and P^* , defined by

$$P^*(y) = P(y^2) |y|^{2\beta+1/p} \in \mathbb{P}_{2t+2\beta+\frac{1}{p}}.$$

Since P^*, W^* are even, Theorem 4.2(b) there gives

$$\| P^* W^* \|_{L_p(a_{2t+2\beta+1/p}^*(1+\kappa_1), \sqrt{d})} \leq C_2 \exp(-C_3 H^*(\kappa_1, t)^{3/2}) \| P^* W^* \|_{L_p(\Delta_{2t+2\beta+1/p}^*)},$$

where

$$\begin{aligned} H^*(\kappa_1, t) &= \min \left\{ \kappa_1, T^*(a_{2t+2\beta+1/p})^{-1} \right\} / \eta_{2t+2\beta+1/p}^* \\ &\sim \min \left\{ \kappa_1, T(a_t)^{-1} \right\} / \eta_t, \end{aligned}$$

in view of (2.6), (2.9) and (3.6). On making the substitution $x = y^2$ in the norms and using (5.4), we obtain

$$\| (PW)(x) x^\beta \|_{L_p(a_{t+\beta+1/(2p)}(1+\kappa_1)^2, d)} \leq C_3 \exp(-C_4 H(\kappa_1, t)^{3/2}) \| (PW)(x) x^\beta \|_{L_p(\Delta_t)}.$$

Now, given $\kappa > 0$, let us write

$$a_t (1 + \kappa) = a_{t+\beta+1/(2p)} (1 + \kappa_1)^2.$$

Then by (3.9),

$$\frac{(1 + \kappa_1)^2}{1 + \kappa} = \frac{a_t}{a_{t+\beta+1/(2p)}} = 1 + O\left(\frac{1}{tT(a_t)}\right) = 1 + o(\eta_t),$$

so

$$2\kappa_1 - \kappa = o(\eta_t),$$

and hence if $\kappa_1 \geq \eta_t$,

$$H(\kappa_1, t) \sim H(\kappa, t).$$

Then (5.6) follows. If instead $\kappa_1 < \eta_t$, then both $H(\kappa_1, t)$ and $H(\kappa, t)$ are bounded, and Theorem 5.2(a) gives the result.

We turn to the proof of (5.7). Let $r > 1$, and write

$$a_{rt} = a_t (1 + \kappa),$$

so that

$$\kappa = \frac{a_{rt}}{a_t} - 1 \sim \frac{1}{T(a_t)}$$

and hence

$$H(\kappa, t) \sim \frac{1}{T(a_t) \eta_t} \geq Ct^\alpha,$$

some $\alpha > 0$, by (3.8). Then (5.7) follows from (5.6). ■

Proof of Theorem 5.2(b) for $\beta < 0$

This follows from the decreasing property of x^β in $(0, d)$:

$$\begin{aligned} & \| (PW)(x) x^\beta \|_{L_p(a_t(1+\kappa), d)} \leq C a_t^\beta \| PW \|_{L_p(a_t(1+\kappa), d)} \\ & \leq C a_t^\beta \exp(-C_3 H(\kappa, t)^{3/2}) \| PW \|_{L_p(\Delta_t)} \\ & \leq C \exp(-C_3 H(\kappa, t)^{3/2}) \| (PW)(x) x^\beta \|_{L_p(\Delta_t)}. \end{aligned}$$

Here we have used the case $\beta = 0$ of Theorem 5.2(b). ■

6. CHRISTOFFEL FUNCTIONS

Christoffel functions are crucially important in analysis of orthogonal polynomials and weighted approximation theory [16]. In this section we shall estimate generalized and classical L_p Christoffel functions for $0 < p \leq \infty$. As in the previous section, we denote the exponentials of potentials with mass $\leq t$ by \mathbb{P}_t , so

(6.1)

$$\mathbb{P}_t = \left\{ c \exp\left(\int \log |z - \xi| d\nu(\xi)\right) : c \geq 0, \nu \geq 0, \|\nu\| \leq t, S(\nu) \text{ is compact} \right\}.$$

Our L_p Christoffel functions are defined as follows: for $0 < p < \infty$,

$$(6.2) \quad \Lambda_{t,p}(W, z) := \inf_{P \in \mathbb{P}_t} (\|PW\|_{L_p(I)} / |P(z)|)^p, z \in \mathbb{C}.$$

The polynomial analogues of $\Lambda_{t,p}$ are for $n \geq 1$,

$$(6.3) \quad \lambda_{n,p}(W, z) := \inf_{P \in \mathcal{P}_n} (\|PW\|_{L_p(I)} / |P(z)|)^p, z \in \mathbb{C}.$$

It is clear that

$$(6.4) \quad \Lambda_{n,p}(W, z) \leq \lambda_{n,p}(W, z).$$

The $\lambda_{n,p}(W, \cdot)$ are weighted analogues of the L_p Christoffel functions introduced by P. Nevai [15]. However, the classical Christoffel function is

$$(6.5) \quad \lambda_n(W^2, x) := \inf_{P \in \mathcal{P}_{n-1}} \left(\int_I (PW)^2 \right) / P^2(x).$$

We see that

$$(6.6) \quad \lambda_n(W^2, x) = \lambda_{n-1,2}(W, x).$$

In describing our result, we shall need the auxiliary function φ_t introduced in (1.18).

Theorem 6.1

Let $0 < p < \infty$; $\rho > -\frac{1}{p}$; $L > 0$ and let $W \in \mathcal{L}(C^2)$.

(a) Then uniformly for $t \geq t_0$ and $x \in J_t = [0, a_t(1 + L\eta_t)]$, we have

$$(6.7) \quad \Lambda_{t,p}(W_\rho, x) \sim \varphi_t(x) W^p(x) \left(x + \frac{a_t}{t^2}\right)^{\rho p}.$$

(b) Moreover, there exist $C, t_0 > 0$ such that uniformly for $t \geq t_0$ and $x \in I$,

$$(6.8) \quad \Lambda_{t,p}(W_\rho, x) \geq C \varphi_t(x) W^p(x) \left(x + \frac{a_t}{t^2}\right)^{\rho p}.$$

For the polynomial analogues $\lambda_{n,p}$ of $\Lambda_{n,p}$, we prove:

Theorem 6.2

Let $0 < p < \infty$; $\rho > -\frac{1}{p}$; $L > 0$ and let $W \in \mathcal{L}(C^2)$.

(a) Then uniformly for $n \geq n_0$ and $x \in J_n = [0, a_n(1 + L\eta_n)]$, we have

$$(6.9) \quad \lambda_{n,p}(W_\rho, x) \sim \varphi_n(x)W^p(x) \left(x + \frac{a_n}{n^2}\right)^{\rho p}.$$

(b) Moreover, there exist $C, n_0 > 0$ such that uniformly for $n \geq n_0$ and $x \in I$,

$$(6.10) \quad \lambda_{n,p}(W_\rho, x) \geq C\varphi_n(x)W^p(x) \left(x + \frac{a_n}{n^2}\right)^{\rho p}.$$

Note that Theorem 1.3 follows directly from Theorem 6.2 and (6.6). We begin with a lemma:

Lemma 6.3

Let $\rho \in \mathbb{R}$ and $L \in (0, 1)$. For $n \geq 1$, there exist polynomials R_n of degree $\leq n$ such that,

$$(6.11) \quad R_n(x) \sim \left(x + \frac{a_n}{n^2}\right)^\rho, x \in [0, a_{2n}];$$

$$(6.12) \quad |R'_n(x)| \leq Cx^{\rho-1}, x \in [La_n n^{-2}, a_{2n}].$$

Proof

Suppose first that $|\rho| < \frac{1}{2}$. Consider the Jacobi weight

$$w(x) = (1-x)^{-\rho} (1-x^2)^{-1/2}, x \in (-1, 1).$$

It is known [18, p. 36] that their Christoffel functions satisfy

$$n^{-1}\lambda_n^{-1}(w, x) \sim (1-x+n^{-2})^\rho,$$

uniformly for $n \geq 1$ and $x \in (-1, 1)$. Moreover, for any fixed $\varepsilon > 0$, in $[-1 + \varepsilon n^{-2}, 1 - \varepsilon n^{-2}]$,

$$n|\lambda'_n(w, x)| \leq C(1-x)^{-\rho} (1-x^2)^{-1}.$$

Let k be a positive integer and $\lfloor \frac{n}{k} \rfloor$ denote the largest integer $\leq \frac{n}{k}$. We set

$$R_n(x) = n^{-1}\lambda_{\lfloor \frac{n}{k} \rfloor}^{-1} \left(w, 1 - \frac{x}{a_{2n}}\right) a_{2n}^\rho.$$

It is straightforward to check that (6.11) and (6.12) follow. The degree of R_n is at most $2n/k \leq n$, if $k \geq 2$. For general ρ , we choose a positive integer ℓ such that $|\rho/\ell| < \frac{1}{2}$ and form the polynomial R_n for ρ/ℓ , and

then raise it to the power ℓ . If $k > 2\ell$, the resulting polynomial will have degree at most n . ■

The Proof of the Lower Bounds for the Christoffel Functions in Theorem 6.1

Let us set $\tau = -\frac{1}{2p}$. We do this in three steps:

Step 1: The case $\rho = \tau$

Recall that we define

$$W^*(x) = \exp(-Q^*(x)) = \exp(-Q(x^2)), x \in I^* = (-\sqrt{d}, \sqrt{d})$$

and that then $W^* \in \mathcal{F}(C^2)$. From [7, Theorem 1.13, p. 20], we have for $\sqrt{x} \in [0, \sqrt{d}]$,

$$\begin{aligned} \inf_{P \in \mathbb{P}_{2t}} \frac{\int_{I^*} |PW^*|^p(u) du}{|PW^*|^p(\sqrt{x})} &= \Lambda_{2t,p}(W^*, \sqrt{x}) / W^{*p}(\sqrt{x}) \\ &\geq C\varphi_{2t}^*(\sqrt{x}), \end{aligned}$$

where in $[-a_{2t}^*, a_{2t}^*]$,

$$\varphi_{2t}^*(u) = \frac{|u^2 - a_{4t}^{*2}|}{t\sqrt{(|u + a_{2t}^*| + a_{2t}^*\eta_{2t}^*)(|u - a_{2t}^*| + a_{2t}^*\eta_{2t}^*)}}$$

and φ_{2t}^* is defined to be constant in $(-\infty, a_{2t}^*]$ and $[a_{2t}^*, \infty)$. We see that in $[0, a_t]$,

$$(6.13) \quad \varphi_{2t}^*(\sqrt{x}) \sim \frac{a_{2t} - x}{t\sqrt{a_t - x + a_t\eta_t}} \sim \varphi_t(x) / \sqrt{x + a_t t^{-2}}.$$

In (a_t, d) , we obtain instead $\varphi_{2t}^*(\sqrt{x}) \sim \varphi_t(a_t)$. We make the substitutions $u = \sqrt{v}$ and $P(\sqrt{v}) = P_0(v)$ in the integral, and note that if $P_0(v) \in \mathbb{P}_t$, then $P(u) = P_0(v^2) \in \mathbb{P}_{2t}$. We deduce that

$$\inf_{P_0 \in \mathbb{P}_t} \frac{\int_I |P_0 W|^p(v) \frac{1}{\sqrt{v}} dv}{|P_0 W|^p(x)} \geq C\varphi_t(x) / \sqrt{x + a_t t^{-2}}$$

and hence

$$\Lambda_{t,p}(W_\tau, x) / [W(x)(x + a_t t^{-2})^\tau]^p \geq C\varphi_t(x),$$

provided $\sqrt{x} \in [0, \sqrt{d}]$, which is equivalent to $x \in [0, d]$.

Step 2: The case $\rho > \tau$

Assume that $x \in [0, d]$. Note that if $P(v) \in \mathbb{P}_t$, then $P(v)(v + a_t t^{-2})^{\rho-\tau} \in$

$\mathbb{P}_{t+\rho-\tau}$. Then

$$\begin{aligned}
 & \Lambda_{t,p}(W_\rho, x) / [W(x) (x + a_t t^{-2})^\rho]^p \\
 & \geq \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_t} (|PW_\tau|(v) v^{\rho-\tau})^p dv}{(|PW|(x) (x + a_t t^{-2})^{\rho-\tau} (x + a_t t^{-2})^\tau)^p} \\
 & \geq C \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_t} (|PW_\tau|(v) (v + a_t t^{-2})^{\rho-\tau})^p dv}{(|PW|(x) (x + a_t t^{-2})^{\rho-\tau} (x + a_t t^{-2})^\tau)^p} \\
 & \geq C \inf_{P \in \mathbb{P}_{t+\rho-\tau}} \frac{\int_I |PW_\tau|(v)^p dv}{(|PW|(x) (x + a_t t^{-2})^\tau)^p},
 \end{aligned}$$

by our restricted range inequalities. Using the result from Step 1, we continue this as

$$\begin{aligned}
 & = C \Lambda_{t+\rho-\tau,p}(W_\tau, x) / [W(x) (x + a_t t^{-2})^\tau]^p \\
 & \geq C \varphi_{t+\rho-\tau}(x) \sim \varphi_t(x),
 \end{aligned}$$

by Lemma 4.4.

Step 3: The case $\rho < \tau$

We consider two ranges of x .

Range A: $x \in [0, a_t/4]$

Let $n = [t] + 1$. We use the polynomials R_n from Lemma 6.3 that satisfy

$$R_n(v) \sim (v + a_t t^{-2})^{\rho-\tau}, v \in [0, a_{2n}].$$

Then as above

$$\begin{aligned}
 & \Lambda_{t,p}(W_\rho, x) / [W(x) (x + a_t t^{-2})^\rho]^p \\
 & \geq C \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_{2n}} (|PW_\tau|(v) (v + a_t t^{-2})^{\rho-\tau})^p dv}{(|PW|(x) (x + a_t t^{-2})^{\rho-\tau} (x + a_t t^{-2})^\tau)^p} \\
 & \geq C \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_{2n}} |PR_n W_\tau|(v)^p dv}{(|PR_n W|(x) (x + a_t t^{-2})^\tau)^p} \\
 & \geq C \Lambda_{t+n,p}(W_\tau, x) / [W(x) (x + a_t t^{-2})^\tau]^p,
 \end{aligned}$$

by our restricted range inequalities. Using Step 1 above, we continue this as

$$\geq C \varphi_{t+n}(x) \sim \varphi_t(x),$$

as

$$x \in [0, a_t/4] \Rightarrow a_{2t} - x \sim a_t - x,$$

so

$$\varphi_{t+n}(x) \sim \varphi_{2t}(x) \sim \varphi_t(x).$$

Range B: $x \in [a_t/4, d)$

Here as $\rho < \tau$,

$$\begin{aligned} & \Lambda_{t,p}(W_\rho, x) / [W(x)(x + a_t t^{-2})^\rho]^p \\ & \geq C \inf_{P \in \mathbb{P}_t} \frac{\int_0^{a_t} (|PW_\tau|(v) v^{\rho-\tau})^p dv}{(|PW|(x)(x + a_t t^{-2})^{\rho-\tau}(x + a_t t^{-2})^\tau)^p} \\ & \geq C \left(\frac{a_t}{x + a_t t^{-2}} \right)^{(\rho-\tau)p} \inf_{P \in \mathbb{P}_t} \frac{\int_0^{a_t} |PW_\tau|(v)^p dv}{(|PW|(x)(x + a_t t^{-2})^\tau)^p} \\ & \geq C \Lambda_{t,p}(W, x) / [W(x)(x + a_t t^{-2})^\tau]^p \geq C \varphi_t(x). \end{aligned}$$

■

The Proof of the Upper Bounds for the Christoffel Functions implicit in Theorem 6.2(a)

Let us set $\tau = -\frac{1}{2p}$. We do this in three steps:

Step 1: The case $\rho = \tau$

Let

$$W^\#(x) = W^*(x)^{1/2} = \exp\left(-\frac{1}{2}Q^*(x)\right), \quad x \in I^* = (-\sqrt{d}, \sqrt{d}).$$

Then $W^\# \in \mathcal{F}(C^2)$. Let $L > 0$. Denote by $a_n^\#, \varphi_{2n}^\#$ and so on, the analogues of a_n, φ_n for $W^\#$. From [7, Theorem 9.3(c), p. 257], we have for $\sqrt{x} \in [0, a_n^\#(1 + L\eta_n^\#)]$,

$$\begin{aligned} \inf_{P \in \mathcal{P}_n} \frac{\int_{I^*} |PW^\#|^{2p}(u) du}{|PW^\#|^{2p}(\sqrt{x})} &= \lambda_{n,2p}(W^\#, \sqrt{x}) / (W^\#(\sqrt{x}))^{2p} \\ &\leq C \varphi_n^\#(\sqrt{x}) = C \frac{a_n^\#}{n} \frac{\left|1 - \left(\frac{\sqrt{x}}{a_{2n}^\#}\right)^2\right|}{\sqrt{\left|1 - \left(\frac{\sqrt{x}}{a_n^\#}\right)^2\right| + \eta_n^\#}}. \end{aligned}$$

Let $P \in \mathcal{P}_n$ denote a minimizing polynomial, achieving the inf in the left-hand side (a compactness argument shows that it exists). Since $a_n^\# = a_{2n}^* = \sqrt{a_n}$ and $\eta_n^\# \sim \eta_{2n}^* \sim \eta_n^* \sim \eta_n$, we can reformulate the

above as

$$\frac{\int_{I^*} |P^2 W^*|^p(u) du}{|P^2 W^*|^p(\sqrt{x})} \leq C \frac{\sqrt{a_n}}{n} \frac{\left|1 - \frac{x}{a_{2n}}\right|}{\sqrt{\left|1 - \frac{x}{a_n}\right| + \eta_n}}.$$

Now let us define a polynomial S_n of degree $\leq n$ by

$$S_n(u^2) = P(u)^2 + P(-u)^2.$$

Then S_n is a nonnegative polynomial with

$$S_n(x) \geq P^2(\sqrt{x}).$$

As W^* is even, we deduce that for $x \in [0, a_n(1 + L\eta_n)]$,

$$\frac{\int_{I^*} |S_n(u^2) W^*(u)|^p du}{|S_n(x) W^*(\sqrt{x})|^p} \leq C \frac{\sqrt{a_n}}{n} \frac{\left|1 - \frac{x}{a_{2n}}\right|}{\sqrt{\left|1 - \frac{x}{a_n}\right| + \eta_n}}.$$

A substitution $u = \sqrt{v}$ gives

$$\lambda_{n,p}(W_\tau, x) / \left(W(x) \left(x + \frac{a_n}{n^2}\right)^\tau\right)^p \leq \frac{\int_{I^*} |(S_n W)(v)|^p \frac{1}{\sqrt{v}} dv}{(S_n W)^p(x) \left(x + \frac{a_n}{n^2}\right)^{-1/2}} \leq C \varphi_n(x),$$

provided $x \in [0, a_n(1 + L\eta_n)]$.

Step 3: The case $\rho > \tau$

We consider two ranges of x .

Range A: $x \in [0, a_{n/4}]$

We use the polynomials $R_{\lfloor n/2 \rfloor}$ from Lemma 6.3 of degree $\leq n/2$ that satisfy

$$R_{\lfloor n/2 \rfloor}(v) \sim (v + a_n n^{-2})^{\tau - \rho}, v \in [0, a_n].$$

Then as above, our restricted range inequality gives

$$\begin{aligned}
& \lambda_{n,p}(W_\rho, x) / [W(x) (x + a_n n^{-2})^\rho]^p \\
& \leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} (|PW_\tau|(v) (v + a_n n^{-2})^{\rho-\tau})^p dv}{(|PW|(x) (x + a_n n^{-2})^{\rho-\tau} (x + a_n n^{-2})^\tau)^p} \\
& \leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} |PW_\tau/R_{[n/2]}|(v)^p dv}{(|PW/R_{[n/2]}|(x) (x + a_n n^{-2})^\tau)^p} \\
& \leq C \inf_{P_1 \in \mathcal{P}_{[n/2]}} \frac{\int_I |P_1 W_\tau|(v)^p dv}{(|P_1 W|(x) (x + a_n n^{-2})^\tau)^p} \\
& = C \lambda_{[n/2],p}(W_\tau, x) / (W(x) (x + a_n n^{-2})^\tau)^p \\
& \leq C \varphi_{[n/2]}(x) \sim \varphi_n(x),
\end{aligned}$$

by the result of Step 1 above, and as

$$x \in [0, a_{n/4}] \Rightarrow a_{[n/2]} - x \sim a_n - x \sim a_{2n} - x,$$

so

$$\varphi_{[n/2]}(x) \sim \varphi_n(x).$$

Range B: $x \in [a_{n/4}, d)$

We use our restricted range inequalities and $\rho > \tau$ to deduce that

$$\begin{aligned}
& \lambda_{n,p}(W_\rho, x) / [W(x) (x + a_n n^{-2})^\rho]^p \\
& \leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} (|PW_\tau|(v) (v + a_n n^{-2})^{\rho-\tau})^p dv}{(|PW|(x) (x + a_n n^{-2})^{\rho-\tau} (x + a_n n^{-2})^\tau)^p} \\
& \leq C \left(\frac{a_n^{\rho-\tau}}{(x + a_n n^{-2})^{\rho-\tau}} \right)^p \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} |PW_\tau|(v)^p dv}{(|PW|(x) (x + a_n n^{-2})^\tau)^p} \\
& \leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} |PW_\tau|(v)^p dv}{(|PW|(x) (x + a_n n^{-2})^\tau)^p} \leq C \varphi_n(x),
\end{aligned}$$

by the Results of Step 1.

Step 3: The case $\rho < \tau$

We let ℓ be a fixed integer $> \tau - \rho$. We use the fact that if $P_1 \in \mathcal{P}_{n-\ell}$,

then $P(u) = P_1(u)(u + a_n n^{-2})^\ell \in \mathcal{P}_n$. Then

$$\begin{aligned}
 & \lambda_{n,p}(W_\rho, x) / [W(x)(x + a_n n^{-2})^\rho]^p \\
 & \leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} |PW_\rho|(v)^p dv}{(|PW|(x)(x + a_n n^{-2})^\rho)^p} \\
 & \leq C \inf_{P_1 \in \mathcal{P}_{n-\ell}} \frac{\int_{a_n/n^2}^{a_n} (|P_1 W_\rho|(v)(v + a_n n^{-2})^\ell)^p dv}{(|P_1 W|(x)(x + a_n n^{-2})^{\rho+\ell})^p} \\
 & \leq C \inf_{P_1 \in \mathcal{P}_{n-\ell}} \frac{\int_{a_n/n^2}^{a_n} |P_1 W_{\rho+\ell}|(v)^p dv}{(|P_1 W|(x)(x + a_n n^{-2})^{\rho+\ell})^p} \\
 & \leq C \lambda_{n-\ell}(W_{\rho+\ell}, x) / (W(x)(x + a_n n^{-2})^{\rho+\ell})^p \\
 & \leq C \varphi_{n-\ell}(x) \sim \varphi_n(x),
 \end{aligned}$$

by the results of Step 2, since $\ell + \rho > \tau$, and by Lemma 4.4. ■

Proof of the Rest of Theorems 6.1 and 6.2

If we combine the lower bounds for $\Lambda_{t,p}$ and the upper bounds for $\lambda_{n,p}$, we obtain, for the relevant range of x ,

$$\begin{aligned}
 C_1 \varphi_t(x) & \leq \Lambda_{t,p}(W_\rho, x) / (W(x)(x + a_t t^{-2})^\rho)^p \\
 & \leq \lambda_{[t],p}(W, x) / (W(x)(x + a_t t^{-2})^\rho)^p \\
 & \leq C_2 \varphi_{[t]}(x) \sim \varphi_t(x).
 \end{aligned}$$

With $n = [t]$, this then gives the \sim relations in both Theorems 6.1(a) and 6.2(a). ■

7. ZEROS OF ORTHOGONAL POLYNOMIALS

The n th orthonormal polynomial $p_{n,\rho}(x)$ has zeros $\{x_{jn,\rho}\}_{j=1}^n$, where

$$0 < x_{nn,\rho} < x_{n-1,n,\rho} < \dots < x_{2n,\rho} < x_{1n,\rho} < d.$$

In our estimation of $p_{n,\rho}(x)$, we shall need bounds on the zeros and on the spacing between the zeros. In this section, we establish these, thereby also obtaining Theorem 1.4.

We begin by showing that all the zeros of $p_n(W_\rho^2, x)$ lie in $\Delta_{n+\rho+\frac{1}{4}}$, as a simple consequence of our restricted range inequality Theorem 5.1.

Theorem 7.1

Let $W := e^{-Q}$ where $Q : I \rightarrow [0, \infty)$ is such that $Q^*(x) = Q(x^2)$ is

convex in I^* . Assume moreover, that $Q(d-) = \infty$ and $Q(x) > 0 = Q(0)$, $x \in I \setminus \{0\}$. Let $\rho > -\frac{1}{2}$. Then for $n \geq 1$,

$$(7.1) \quad x_{1n,\rho} < a_{n+\rho+\frac{1}{4}}.$$

It is interesting that for $\rho = 0$ and for weights on the whole real line, $a_{n+\frac{1}{4}}$ has to be replaced by $a_{n+\frac{1}{2}}$ [7]. The reason for the better estimate here comes from the slightly different restricted range inequalities we obtain for subintervals of $(0, \infty)$. We note that it is possible to prove a generalisation of Theorem 7.1 for L_p extremal polynomials, as in [7].

There are a number of simple monotonicity and interlacing properties for the zeros of the orthogonal polynomials:

Theorem 7.2

Let W be a continuous function on I such that W^2 has all finite power moments. Let $\rho > -\frac{1}{2}$ and let ℓ be a positive integer.

(a) For each $n \geq j \geq 1$, $x_{jn,\rho}$ is a non-decreasing function of ρ .

(b)

$$x_{1n,\rho} \leq x_{1n,\rho+\ell} \leq x_{1,n+\ell,\rho}.$$

(c) For each $j \in \{1, 2, 3, \dots, n - 2\ell\}$, $p_{n,\ell+\rho}$ has at least one zero in $[x_{j+2\ell,n,\rho}, x_{jn,\rho}]$. Moreover, for each $j \in \{2\ell + 2, 2\ell + 3, \dots, n\}$,

$$(7.2) \quad x_{jn,\rho+\ell} \leq x_{j-2\ell-1,n,\rho} \leq x_{j-2\ell-1,n,\rho+\ell}.$$

We note that in the special case of Laguerre weights $x^\rho e^{-x}$, the monotonicity of the zeros in ρ is classical [22, pp. 122-123]. On the more quantitative side, we prove:

Theorem 7.3

Let $W \in \mathcal{L}(C^2)$ and $\rho > -\frac{1}{2}$.

(a) Uniformly for $n \geq 1$,

$$(7.3) \quad x_{nn,\rho} \sim a_n n^{-2}.$$

(b) For n large enough,

$$1 - \frac{x_{1n,\rho}}{a_n} \sim \eta_n.$$

(c) For some $C > 0$,

$$(7.4) \quad x_{j-1,n,\rho} - x_{jn,\rho} \leq C\varphi_n(x_{jn}), 2 \leq j \leq n.$$

We begin with

The Proof of Theorem 7.1

We use the well known formula

$$(7.5) \quad x_{1n,\rho} = \max_{P \in \mathcal{P}_{n-1}} \frac{\int_I x (PW_\rho)^2(x) dx}{\int_I (PW_\rho)^2(x) dx}.$$

This is an easy consequence of the Gauss quadrature formula for W_ρ^2 , see for example [22, p.188]. In turn this implies that for $r > 0$,

$$(7.6) \quad 1 - \frac{x_{1n,\rho}}{a_r} = \min_{P \in \mathcal{P}_{n-1}} \frac{\int_I (1 - \frac{x}{a_r})(PW_\rho)^2(x) dx}{\int_I (PW_\rho)^2(x) dx}.$$

Now we proceed as in the proof of Theorem 11.1 in [7, p. 315]. Let $t = n + \rho + \frac{1}{4}$, $p = 2$, and $r = t$. We note first that for $P \in \mathcal{P}_{n-1} \setminus \{0\}$

$$\left| 1 - \frac{x}{a_t} \right|^{1/2} |P(x)| \in \mathbb{P}_{n-\frac{1}{2}} = \mathbb{P}_{t-\rho-\frac{3}{2p}}.$$

Then Theorem 5.1 with the above choices of t, p and with $\beta = \rho$ gives

$$\int_{I \setminus \Delta_t} \left| 1 - \frac{x}{a_t} \right| (PW_\rho)^2(x) dx < \int_{\Delta_t} \left| 1 - \frac{x}{a_t} \right| (PW_\rho)^2(x) dx.$$

Since $1 - \frac{x}{a_t} > 0$ in the right-hand integral except when $x = a_t$, we deduce that

$$\int_I \left(1 - \frac{x}{a_t}\right) (PW_\rho)^2(x) dx > 0.$$

Then (7.6) gives

$$\begin{aligned} 1 - \frac{x_{1n}}{a_t} &> 0 \\ \Rightarrow x_{1n} &< a_t = a_{n+\rho+\frac{1}{4}}. \end{aligned}$$

■

Proof of Theorem 7.2

(a) If w_1 and w_2 are positive continuous weights on a compact interval $[a, b]$ and w_2/w_1 is a strictly increasing function in $[a, b]$, then a classical result [22, Thm. 6.12.2, p. 116] asserts that

$$x_{jn}(w_1) < x_{jn}(w_2),$$

where $x_{jn}(w_k)$ denotes the j th zero of $p_n(w_k)$. In our situation, if $\Delta > \rho$, W_Δ/W_ρ is a strictly increasing function in I . However, the classical result cannot be applied directly to W_Δ and W_ρ , since I is not compact. (However Szego applies the result to Laguerre weights without further explanation). We can replace I by $I_\varepsilon = [\varepsilon, \inf \{d - \varepsilon, \frac{1}{\varepsilon}\}]$, where $\varepsilon > 0$

is small, and apply the result to the weights W_ρ and W_Δ restricted to I_ε . If we fix n , and let $\varepsilon \rightarrow 0+$, and use continuity in ε , of the orthogonal polynomial of degree n with respect to the weight W_ρ^2 restricted to I_ε , we then obtain the result.

(b) By (a),

$$x_{1n,\rho} \leq x_{1n,\rho+\ell}.$$

Moreover, the extremal formula (7.5) gives

$$\begin{aligned} x_{1n,\rho+\ell} &= \sup_{\deg(P) \leq n-1} \frac{\int_I x P^2(x) x^{2\ell} W_\rho^2(x) dx}{\int_I P^2(x) x^{2\ell} W_\rho^2(x) dx} \\ &\leq \sup_{\deg(P) \leq n+\ell-1} \frac{\int_I x P^2(x) W_\rho^2(x) dx}{\int_I P^2(x) W_\rho^2(x) dx} = x_{1,n+\ell,\rho}. \end{aligned}$$

(c) By the Gauss quadrature formula, if P is a polynomial of degree $\leq n - 2\ell - 1$, we have

$$\begin{aligned} &\sum_{j=1}^n \lambda_n(W_\rho^2, x_{jn,\rho}) x_{jn,\rho}^{2\ell} p_{n,\rho+\ell}(x_{jn,\rho}) P(x_{jn,\rho}) \\ &= \int_I x^{2\ell} p_{n,\rho+\ell}(x) P(x) W_\rho^2(x) dx \\ &= \int_I p_{n,\rho+\ell}(x) P(x) W_{\rho+\ell}^2(x) dx = 0. \end{aligned}$$

That is, setting

$$w_{jn} = \lambda_n(W_\rho^2, x_{jn,\rho}) x_{jn,\rho}^{2\ell} |p_{n,\rho+\ell}(x_{jn,\rho})| \geq 0,$$

we have for $P \in \mathcal{P}_{n-2\ell-1}$,

$$\sum_{j=1}^n w_{jn} \text{sign}(p_{n,\rho+\ell}(x_{jn,\rho})) P(x_{jn,\rho}) = 0.$$

It then follows that for $k = 1, 2, \dots, n-2\ell$, the sequence $\{p_{n,\rho+\ell}(x_{jn,\rho})\}_{j=k}^{k+2\ell}$ contains a sign change or zero. Indeed if there is a k for which this is not true, on setting

$$P(x) = \sigma \prod_{j=1, j \notin \{k, k+1, \dots, k+2\ell\}}^n (x - x_{jn,\rho})$$

where

$$\sigma = \text{sign}(p_{n,\rho+\ell}(x_{kn,\rho})),$$

we obtain

$$\begin{aligned} & \sum_{j=k}^{k+2\ell} w_{jn} |P(x_{jn,\rho})| \\ &= \sum_{j=1}^n w_{jn} \text{sign}(p_{n,\rho+\ell}(x_{jn,\rho})) P(x_{jn,\rho}) = 0. \end{aligned}$$

This is not possible as $P(x_{jn,\rho}) \neq 0$, $k \leq j \leq k + 2\ell$ and all of w_{jn} , $k \leq j \leq k + 2\ell$ are of one sign. So $\{p_{n,\rho+\ell}(x_{jn,\rho})\}_{j=k}^{k+2\ell}$ has a zero or sign change for each k , and then the first part follows. Similarly, $\{p_{n,\rho+\ell}(x_{jn,\rho})\}_{j=1}^n$ has at least $n - 2\ell$ sign changes or zeros in the sequence $\{x_{jn,\rho}\}_{j=1}^n$. (If not, we choose P to be a polynomial of degree $m \leq n - 2\ell - 1$ with sign changes to match the m sign changes in $\{p_{n,\rho+\ell}(x_{jn,\rho})\}_{j=1}^n$, but not vanishing at any of $x_{jn,\rho}$, leading to a contradiction as above).

Next, the above shows that for the smallest zeros of $p_{n,\rho}$ and $p_{n,\rho+\ell}$, we have

$$x_{nn,\rho} \leq x_{nn,\rho+\ell} \leq x_{n-2\ell,n,\rho}.$$

Let us now generalize this. If for some k ,

$$x_{kn,\rho+\ell} > x_{k-2\ell-1,n,\rho},$$

then in the interval $[x_{nn,\rho}, x_{k-2\ell-1,n,\rho}]$, $p_{n,\rho+\ell}$ has at most $n - k$ zeros, and hence in $\{x_{jn,\rho}\}_{j=k-2\ell-1}^n$, $p_{n,\rho+\ell}$ has at most $n - k + 1$ sign changes. In the remaining $\{x_{jn,\rho}\}_{j=1}^{k-2\ell-1}$, $p_{n,\rho+\ell}$ has at most $k - 2\ell - 2$ sign changes, giving a total of at most $n - 2\ell - 1$ sign changes in $\{x_{jn,\rho}\}_{j=1}^n$, a contradiction. So

$$x_{kn,\rho+\ell} \leq x_{k-2\ell-1,n,\rho}.$$

The right-hand inequality in (7.2) follows from (a). ■

Next we record the desired inequalities for the zeros of $p_{n,-1/4}$, which follow from results in [7].

Lemma 7.4

Let $W \in \mathcal{L}(C^2)$ and $\tau = -\frac{1}{4}$.

(a) Uniformly for $n \geq 1$,

$$(7.7) \quad x_{nn,\tau} \sim a_n n^{-2}.$$

(b) For n large enough

$$(7.8) \quad 1 - \frac{x_{1n,\tau}}{a_n} \sim \eta_n.$$

(c) For some $C > 0$,

$$(7.9) \quad x_{j,n,\tau} - x_{j+1,n,\tau} \sim \varphi_n(x_{jn}), 1 \leq j \leq n-1.$$

Proof

(b) Recall from (1.7) that

$$p_n(W_\tau^2, t^2) = p_{2n}(W^{*2}, t)$$

so

$$(7.10) \quad x_{jn,\tau} = (x_{j,2n}^*)^2.$$

By Theorem 1.19(f) in [7, p. 23],

$$1 - \frac{x_{1,2n}^*}{a_{2n}^*} \sim \eta_{2n}^*,$$

so

$$1 - \frac{x_{1n,\tau}}{a_n} = 1 - \left(\frac{x_{1,2n}^*}{a_{2n}^*} \right)^2 \sim \eta_{2n}^* \sim \eta_n.$$

(a) Next, as W^* is even, the spacing in [7, Theorem 1.19(e), p. 23] gives

$$\begin{aligned} 2x_{n,2n}^* &= x_{n,2n}^* - x_{n+1,2n}^* \sim \varphi_{2n}^*(x_{n,2n}^*) \\ &\sim \frac{a_{2n}^*}{n} \frac{\left| 1 - \frac{x_{n+1,2n}^*}{a_{2n}^*} \right|}{\sqrt{\left| 1 - \frac{x_{n+1,2n}^*}{a_{2n}^*} \right| + \eta_{2n}^*}} \sim \frac{\sqrt{a_n}}{n}, \end{aligned}$$

whence

$$x_{nn,\tau} = (x_{n,2n}^*)^2 \sim \frac{a_n}{n^2}.$$

(c) By (7.10),

$$\begin{aligned} x_{jn,\tau} - x_{j+1,n,\tau} &= (x_{j,2n}^* + x_{j+1,2n}^*) (x_{j,2n}^* - x_{j+1,2n}^*) \\ &\sim x_{j,2n}^* \varphi_{2n}^*(x_{j,2n}^*) \sim \varphi_n(x_{jn,\tau}), \end{aligned}$$

by (6.13). ■

Proof of Theorem 7.3(a), (b)

(a) By the classical extremal property for smallest zeros, and our restricted range inequalities,

$$\begin{aligned} x_{nn,\rho} &= \inf_{\deg(P) \leq n-1} \frac{\int_I x P^2(x) W_\rho^2(x) dx}{\int_I P^2(x) W_\rho^2(x) dx} \\ &\geq \frac{a_n}{n^2} \inf_{\deg(P) \leq n-1} \frac{\int_{a_n/n^2}^{a_n} P^2(x) W_\rho^2(x) dx}{\int_I P^2(x) W_\rho^2(x) dx} \\ &\geq C \frac{a_n}{n^2}. \end{aligned}$$

Next, choose a positive integer ℓ such that $\ell + \tau > \rho$. By Theorem 7.2(a),

$$x_{nn,\rho} \leq x_{nn,\ell+\tau}$$

and by Theorem 7.2(b),

$$x_{nn,\ell+\tau} \leq x_{n,n+\ell,\tau}.$$

Using Lemma 7.4(b) and the spacing in Lemma 7.4(c), we obtain (as ℓ is fixed),

$$x_{n,n+\ell,\tau} \leq C \frac{a_n}{n^2}$$

and hence

$$x_{nn,\rho} \leq C \frac{a_n}{n^2}.$$

(b) **Case I:** $\rho > \tau$.

Choose a positive integer ℓ such that $\ell + \tau > \rho$. By Theorem 7.2(a)

$$x_{1n,\tau} \leq x_{1n,\rho} \leq x_{1n,\ell+\tau}$$

and by Theorem 7.2(b),

$$x_{1n,\ell+\tau} \leq x_{1,n+\ell,\tau}.$$

Then

$$\begin{aligned} 1 - \frac{x_{1n,\rho}}{a_n} &\geq 1 - \frac{x_{1,n+\ell,\tau}}{a_n} \\ &= 1 - \frac{x_{1,n+\ell,\tau}}{a_{n+\ell}} + \frac{x_{1,n+\ell,\tau}}{a_n} \left(\frac{a_n}{a_{n+\ell}} - 1 \right). \end{aligned}$$

Here from (3.9),

$$\frac{a_n}{a_{n+\ell}} - 1 = O\left(\frac{1}{nT(a_n)}\right) = o(\eta_n),$$

while from Lemma 7.4(b),

$$1 - \frac{x_{1,n+l,\tau}}{a_{n+l}} \sim \eta_{n+l} \sim \eta_n.$$

So at least for large enough n ,

$$1 - \frac{x_{1n,\rho}}{a_n} \geq C\eta_n.$$

In the other direction, Lemma 7.4(b) gives

$$1 - \frac{x_{1n,\rho}}{a_n} \leq 1 - \frac{x_{1n,\tau}}{a_n} \leq C\eta_n.$$

Case II: $\rho < \tau$

Choose a positive integer ℓ such that $\ell + \rho > \tau$. Here Theorem 7.2 (a), (b) give

$$x_{1n,\rho} \leq x_{1n,\tau} \leq x_{1n,\ell+\rho} \leq x_{1,n+l,\rho}.$$

Then

$$C\eta_n \geq 1 - \frac{x_{1n,\tau}}{a_n} \geq 1 - \frac{x_{1,n+l,\rho}}{a_n}.$$

Much as above this yields, for large enough n ,

$$1 - \frac{x_{1,n+l,\rho}}{a_{n+l}} \leq C\eta_{n+l},$$

and hence for large enough n ,

$$1 - \frac{x_{1n,\rho}}{a_n} \leq C\eta_n.$$

In the other direction,

$$1 - \frac{x_{1n,\rho}}{a_n} \geq 1 - \frac{x_{1n,\tau}}{a_n} \geq C\eta_n.$$

■

Our proof of Theorem 7.3 (c) is based on an extension of a classical inequality of Erdős and Turan for sums of successive fundamental polynomials. One such extension was presented in [8], and reproduced in [7, p. 320 ff.]. That required Q to be convex, which is not always true for the weights in this work. So we present another extension, which allows $xQ'(x)$ to be increasing, but holds only on subintervals of $(0, \infty)$. Yet another extension was given in [24].

We note that it is possible to give another proof of Theorem 7.3(c) based on the estimates in Lemma 7.4, and the inequalities in Theorem 7.2. But we feel the following lemma is of independent interest.

Lemma 7.5

Let

$$0 \leq a \leq y_1 < y_2 < \dots < y_m \leq b$$

and $\{\ell_j(x)\}_{j=1}^m \subseteq \mathcal{P}_{m-1}$ denote the corresponding fundamental polynomials of Lagrange interpolation, so that

$$\ell_j(y_k) = \delta_{jk}, 1 \leq j, k \leq m.$$

Let $w : (a, b) \rightarrow (0, \infty)$ and assume that $q := \log \frac{1}{w}$ is such that q' exists and such that $xq'(x)$ is non-decreasing in $[y_1, y_m]$. Then for $1 \leq j \leq m-1$,

$$(7.11) \quad \ell_j(x)w^{-1}(y_j)w(x) + \ell_{j+1}(x)w^{-1}(y_{j+1})w(x) \geq 1, x \in [y_j, y_{j+1}].$$

We first need a zero counting lemma:

Lemma 7.6

Under the hypotheses of Lemma 7.5, if $P \in \mathcal{P}_m$ has only real zeros, all lying in $[s, t] \subset (0, \infty)$, and s, t are zeros, then $(Pw)'$ has at most $m-1$ distinct zeros lying in $[s, t] \cap (a, b)$.

Proof

Let

$$0 < s = x_1 < x_2 < \dots < x_k = t$$

denote the distinct zeros of P , with multiplicities n_1, n_2, \dots, n_k respectively. Since

$$(Pw)' = 0 \Rightarrow P' - q'P = 0,$$

we see that zeros of $(Pw)'$ occur where P has a multiple zero or where

$$g(x) := \frac{P'(x)}{P(x)} = \sum_{j=1}^k \frac{n_j}{x - x_j}$$

has $g(x) = q'(x)$. Now we count the zeros of $g - q'$. Since we are working on a subinterval of $(0, \infty)$, this is the same as counting the zeros of the function $xg(x) - xq'(x)$. Here

$$\frac{d}{dx}(xg(x)) = - \sum_{j=1}^k \frac{x_j n_j}{(x - x_j)^2} < 0,$$

so $xg(x) - xq'(x)$ is strictly decreasing in $(x_j, x_{j+1}) \cap (a, b)$, so has at most one zero there. (There will be exactly one zero if $(x_j, x_{j+1}) \subset$

(a, b)). Thus $(Pw)'$ has at most one zero in $(x_j, x_{j+1}) \cap (a, b)$, $1 \leq j < k$, and zeros at x_j iff $n_j \geq 2$. Then in $[s, t] \cap (a, b)$, $(Pw)'$ has at most

$$k - 1 + \sum_{j=1}^k \max\{0, n_j - 1\} \leq -1 + \sum_{j=1}^k n_j = m - 1$$

distinct zeros. ■

We turn to the

Proof of Lemma 7.5

Now that we have Lemma 7.6, this is identical to that of Lemma 11.8 in [7, p. 322], but we include the details for the reader's convenience. Fix j and let

$$P(x) := \ell_j(x)/w(y_j) + \ell_{j+1}(x)/w(y_{j+1}).$$

Then $P \in \mathcal{P}_{m-1}$ has $m - 2$ zeros at $\{y_1, y_2, \dots, y_{j-1}, y_{j+2}, \dots, y_m\}$ and

$$(Pw)(y_j) = 1 = (Pw)(y_{j+1}).$$

Its remaining zero must also be real. By Rolle's theorem, $(Pw)'$ has a zero in (y_k, y_{k+1}) for

$$k \in \{1, 2, \dots, m - 1\} \setminus \{j - 1, j + 1\}$$

a total of $m - 3$ distinct zeros. From the lemma, it can have at most $m - 2$ distinct zeros in $[y_1, y_m]$. We claim that

$$(7.12) \quad (Pw)'(y_j) \geq 0 \geq (Pw)'(y_{j+1}).$$

Once we have proved this, it follows that $(Pw)'$ has exactly one zero in (y_j, y_{j+1}) at its local maximum in this interval (otherwise it would have to have ≥ 3 distinct zeros in this interval, giving $\geq m - 1$ zeros in all, which is impossible: a sketch of the situation will assist the reader). Then Pw increases from 1 at y_j to its maximum and then decreases again to 1 at y_{j+1} , giving (7.11).

We proceed to prove (7.12). Suppose first that $2 \leq j \leq m - 2$ and suppose for example $(Pw)'(y_{j+1}) > 0$. Then we see that $(Pw)'$ must have at least one zero in (y_{j+1}, y_{j+2}) (recall that $(Pw)(y_{j+1}) = 1$; $(Pw)(y_{j+2}) = 0$, again a sketch will help). Then we already have counted $m - 2$ distinct zeros of $(Pw)'$, so there are no more. But then $(Pw)'(y_j) < 0$ (for else, (Pw) has at least one local maximum and minimum in $[y_j, y_{j+1}]$ so $(Pw)'$ has 2 zeros there, and this is impossible: consider separately the cases $(Pw)'(y_j) = 0$ or > 0). Since $(Pw)(y_j) = 1 > 0 = (Pw)(y_{j-1})$, $(Pw)'$ has one more zero in (y_{j-1}, y_j) giving $\geq m - 1$ zeros, which is impossible. So in this case we have the right-hand side of (7.12) and the other half of (7.12) is similar (or

can be deduced by considering $(Pw)(-x)$ with points $-y_j, 1 \leq j \leq m$. For $j = 1, 2$ or $m-1, m$, this argument requires minor modifications. ■

Finally, we turn to:

Proof of Theorem 7.3(c)

Let $\{\ell_{jn}\}_{j=1}^n$ denote the fundamental polynomials of Lagrange interpolation at the zeros $\{x_{jn,\rho}\}_{j=1}^n$ of the orthogonal polynomials $p_{n,\rho}(x)$, so that

$$\ell_{jn}(x_{kn}) = \delta_{jk}, 1 \leq j, k \leq n.$$

A classical formula for the weights in the Gauss quadrature formula is

$$\lambda_{jn} := \lambda_n(W_\rho^2, x_{jn,\rho}) = \int_I \ell_{jn}^2 W_\rho^2.$$

Then applying Lemma 7.5 with $w = W^2$,

$$\begin{aligned} \lambda_{jn} W^{-2}(x_{jn,\rho}) + \lambda_{j-1,n} W^{-2}(x_{j-1,n,\rho}) &= \int_I (\ell_{jn}^2 W^{-2}(x_{jn,\rho}) + \ell_{j-1,n}^2 W^{-2}(x_{j-1,n,\rho})) W_\rho^2 \\ &\geq \int_{x_{jn,\rho}}^{x_{j-1,n,\rho}} (\ell_{jn}^2 W^{-2}(x_{jn,\rho}) + \ell_{j-1,n}^2 W^{-2}(x_{j-1,n,\rho})) W_\rho^2 \\ &\geq \frac{1}{2} \int_{x_{jn,\rho}}^{x_{j-1,n,\rho}} (\ell_{jn} W^{-1}(x_{jn,\rho}) + \ell_{j-1,n} W^{-1}(x_{j-1,n,\rho}))^2 W_\rho^2 \\ (7.13) \quad &\geq \frac{1}{2} \int_{x_{jn,\rho}}^{x_{j-1,n,\rho}} x^{2\rho} dx \geq C(x_{j-1,n,\rho}^{2\rho+1} - x_{jn,\rho}^{2\rho+1}). \end{aligned}$$

(We used the inequality $s^2 + t^2 \geq \frac{1}{2}(s+t)^2$ in the second last line). The inequality

$$y^{2\rho+1} - x^{2\rho+1} \geq C_0(y-x) \max\{y^{2\rho}, x^{2\rho}\}, y > x > 0,$$

where C_0 is independent of x and y , enables us to reformulate the above as

$$\lambda_{jn} W^{-2}(x_{jn,\rho}) + \lambda_{j-1,n} W^{-2}(x_{j-1,n,\rho}) \geq C(x_{j-1,n,\rho} - x_{jn,\rho}) \max\{x_{j-1,n,\rho}^{2\rho}, x_{jn,\rho}^{2\rho}\}.$$

Using our estimates for Christoffel functions in Theorem 6.2 (recall also (6.6)), we obtain for some $C \neq C(j, n)$

$$\begin{aligned} & (x_{j-1, n, \rho} - x_{jn, \rho}) \max \{x_{j-1, n, \rho}^{2\rho}, x_{jn, \rho}^{2\rho}\} \\ & \leq C(\varphi_n(x_{jn, \rho})x_{jn, \rho}^{2\rho} + \varphi_n(x_{j-1, n, \rho})x_{j-1, n, \rho}^{2\rho}) \\ & \leq C(\varphi_n(x_{jn, \rho}) + \varphi_n(x_{j-1, n, \rho})) \max \{x_{j-1, n, \rho}^{2\rho}, x_{jn, \rho}^{2\rho}\}, \end{aligned}$$

so

$$x_{j-1, n, \rho} - x_{jn, \rho} \leq C(\varphi_n(x_{jn, \rho}) + \varphi_n(x_{j-1, n, \rho})).$$

But if, for example, $\varphi_n(x_{jn, \rho}) < \varphi_n(x_{j-1, n, \rho})$ this gives

$$x_{j-1, n, \rho} - x_{jn, \rho} \leq C\varphi_n(x_{j-1, n, \rho})$$

and then Lemma 4.3 shows that

$$(7.14) \quad \varphi_n(x_{jn, \rho}) \sim \varphi_n(x_{j-1, n, \rho})$$

So the desired inequality follows. The case $\varphi_n(x_{jn, \rho}) \geq \varphi_n(x_{j-1, n, \rho})$ is similar. ■

8. BOUNDS ON ORTHOGONAL POLYNOMIALS

We prove Theorem 1.2, which we reformulate as:

Theorem 8.1

Let $W \in \mathcal{L}(C^2)$, $\rho > -\frac{1}{2}$ and let $p_{n, \rho}(x)$ be the n th orthonormal polynomial for the weight W^2 . Then uniformly for $n \geq 1$,

$$(8.1) \quad \sup_{x \in I} |p_{n, \rho}(x)| W(x) \left(x + \frac{a_n}{n^2}\right)^\rho \left|x + \frac{a_n}{n^2}\right| (|a_n - x| + a_n \eta_n)^{1/4} \sim 1.$$

The proof of Theorem 8.1 is similar in spirit - and easier - than its analogue for weights on two-sided intervals, Theorem 12.1 in [7, p. 326]. The broad outlines of the method were introduced by Bonan [1] and extended by Mhaskar [10], and the authors. The method has also recently been used by Kasuga and Sakai in [5].

We shall first prove the upper bound for $x \in [\varepsilon a_n, a_n]$, any $0 < \varepsilon < 1$, and then treat the rest of the range of x . Before proceeding to the first step, let us recall some notation: the zeros of $p_{n, \rho}(x) = p_n(W_\rho^2, x)$ are denoted by

$$0 < x_{nn, \rho} < x_{n-1, n, \rho} < \dots < x_{2n, \rho} < x_{1n, \rho} < d$$

and $\gamma_{n,\rho}$ denotes the (positive) leading coefficient of $p_{n,\rho}(x)$. The n th reproducing kernel function is

$$(8.2) \quad K_{n,\rho}(x, t) := K_n(W_\rho^2, x, t) := \sum_{j=0}^{n-1} p_{j,\rho}(x)p_{j,\rho}(t).$$

The Christoffel-Darboux formula provides an alternative representation for K_n :

$$(8.3) \quad K_{n,\rho}(x, t) = \frac{\gamma_{n-1,\rho} p_{n,\rho}(x)p_{n-1,\rho}(t) - p_{n,\rho}(t)p_{n-1,\rho}(x)}{\gamma_{n,\rho} (x - t)}.$$

Letting $t \rightarrow x$ gives

$$(8.4) \quad \lambda_{n,\rho}^{-1}(x) := \lambda_n^{-1}(W_\rho^2, x) = K_{n,\rho}(x, x) = \frac{\gamma_{n-1,\rho} [p'_{n,\rho}(x)p_{n-1,\rho}(x) - p'_{n-1,\rho}(x)p_{n,\rho}(x)]}{\gamma_{n,\rho}}$$

and in particular for $x = x_{jn,\rho}$ we obtain

$$(8.5) \quad \lambda_{n,\rho}^{-1}(x_{jn,\rho}) = \frac{\gamma_{n-1,\rho} p'_{n,\rho}(x_{jn,\rho})p_{n-1,\rho}(x_{jn,\rho})}{\gamma_{n,\rho}}.$$

Lemma 8.2

Let $\rho > -\frac{1}{4}$ and $0 < \varepsilon < 1$. Let $W \in \mathcal{L}(C^2)$. Then uniformly for $n \geq 1$,

$$(8.6) \quad \sup_{x \in [\varepsilon a_n, a_{2n}]} |p_{n,\rho}(x) | W(x)x^\rho | x(a_n - x) |^{1/4} \leq C.$$

Proof

Let $\tau = -\frac{1}{4}$. First recall that $a_{2n}^{*2} = a_n$ and a substitution in the integral defining orthogonality of $p_{2n}(W^{*2}, t)$ gives

$$p_{n,\tau}(t^2) = p_{n,-1/4}(t^2) = p_{2n}(W^{*2}, t).$$

Then the bounds for the latter polynomials in [7, Theorem 1.17, p. 22] give for $t \in I^*$

$$|p_{n,\tau}(t^2)| W(t^2) = |p_{2n}(W^{*2}, t) W^*(t)| \leq C |a_{2n}^{*2} - t^2|^{-1/4}.$$

Then for $n \geq 1$ and any fixed j , our restricted range inequalities and (3.9) give

$$(8.7) \quad \sup_{x \in I} |p_{n+j,\tau} W|(x) |a_n - x|^{1/4} \leq C.$$

Now choose non-negative integers k, ℓ such that

$$(8.8) \quad k + \rho > -\frac{1}{2} \text{ and } \ell - \rho > 0.$$

Also let

$$(8.9) \quad \beta := 2\rho - \ell + k + \frac{1}{2}.$$

For a fixed $x \in [\varepsilon a_n, a_{2n}]$, let

$$(8.10) \quad S(t) := t^\ell x^\beta.$$

We may write

$$(8.11) \quad \begin{aligned} p_{n,\rho} S(x) &= \int_I K_{n+\ell+1,\tau}(x,t) (p_{n,\rho} S)(t) W_\tau^2(t) dt \\ &= \int_I K_{n+\ell+1,\tau}(x,t) p_{n,\rho}(t) [x^\beta - t^\beta] t^\ell W_\tau^2(t) dt \\ &\quad + \int_I K_{n+\ell+1,\tau}(x,t) p_{n,\rho}(t) t^{\beta+\ell} W_\tau^2(t) dt \\ &=: I_1 + I_2. \end{aligned}$$

Estimation of I_2

By choice of β , orthogonality, and then Cauchy-Schwarz,

$$\begin{aligned} |I_2| &= \left| \int_I \left(t^k \sum_{j=n-k}^{n+\ell} p_{j,\tau}(x) p_{j,\tau}(t) \right) p_{n,\rho}(t) W_\rho^2(t) dt \right| \\ &\leq \left[\int_I \left(t^k \sum_{j=n-k}^{n+\ell} p_{j,\tau}(x) p_{j,\tau}(t) \right)^2 W_\rho^2(t) dt \right]^{1/2}. \end{aligned}$$

Now we use our restricted range inequality Theorem 5.2(c), and then (8.7) to obtain, for $x \leq a_{2n}$,

$$\begin{aligned} &|I_2| W(x) |a_n - x|^{1/4} \\ &\leq C \left[\int_0^{a_{2n}} \frac{t^{2k+2\rho}}{|a_n - t|^{1/2}} dt + O(e^{-nC}) \right]^{1/2} \\ &\leq C a_n^{\rho+k+1/4} \left[\int_0^{C_0} \frac{s^{2\rho+2k}}{|1-s|^{1/2}} ds + O(e^{-nC}) \right]^{1/2}, \end{aligned}$$

provided C_0 is so large that $a_{2n}/a_n \leq C_0$. Here the integral converges as $2\rho + 2k > -1$. Since $x \in [\varepsilon a_n, a_{2n}]$, we obtain

$$(8.12) \quad |I_2| W(x) |a_n - x|^{1/4} \leq C x^{\rho+k+1/4}.$$

Estimation of I_1

By the Christoffel-Darboux identity,

$$I_1 = \frac{\gamma_{n+\ell,\tau}}{\gamma_{n+\ell+1,\tau}} \{p_{n+\ell+1,\tau}(x) I_{1,1} - p_{n+\ell,\tau}(x) I_{1,2}\},$$

where

$$\begin{aligned} I_{1,1} &= \int_I p_{n+\ell,\tau}(t) p_{n,\rho}(t) t^\ell \left(\frac{x^\beta - t^\beta}{x-t} \right) W_\tau^2(t) dt; \\ I_{1,2} &= \int_I p_{n+\ell+1,\tau}(t) p_{n,\rho}(t) t^\ell \left(\frac{x^\beta - t^\beta}{x-t} \right) W_\tau^2(t) dt. \end{aligned}$$

Now our restricted range inequality Theorem 5.2(a), applied to W_ρ^2 gives for any m and ρ ,

$$\begin{aligned} \frac{\gamma_{m-1,\rho}}{\gamma_{m,\rho}} &= \int_I x p_{m,\rho}(x) p_{m-1,\rho}(x) W_\rho^2(x) dx \\ (8.13) \quad &\leq C a_m \int_0^{a_m} |p_{m,\rho}(x) p_{m-1,\rho}(x)| W_\rho^2(x) dx \leq C a_m. \end{aligned}$$

Using this, our bound (8.7), Theorem 5.2(c), and Cauchy-Schwarz gives

$$\begin{aligned} &|I_1| W(x) |a_n - x|^{1/4} \\ &\leq C a_n \left(\int_0^{a_{2n}} \frac{t^{2\ell-2\rho-1}}{|a_n - t|^{1/2}} \left(\frac{x^\beta - t^\beta}{x-t} \right)^2 dt + O(e^{-n^C}) \right)^{1/2}. \end{aligned}$$

Let $\chi = x/a_n$. The substitution $t = a_n s$ gives for some C_1 ,

$$\begin{aligned} &|I_1| W(x) |a_n - x|^{1/4} \\ &\leq C a_n^{\ell-\rho+\beta-1/4} \left(\int_0^{C_1} \frac{s^{2\ell-2\rho-1}}{|1-s|^{1/2}} \left(\frac{\chi^\beta - s^\beta}{\chi-s} \right)^2 ds + O(e^{-n^C}) \right)^{1/2}. \end{aligned}$$

We claim that the term $\left(\frac{\chi^\beta - s^\beta}{\chi-s} \right)^2$ is bounded independently of n, x, s . Indeed as $\chi \in [\varepsilon, C]$, we see that for $s \in [0, \varepsilon/2]$,

$$\left(\frac{\chi^\beta - s^\beta}{\chi-s} \right)^2 \leq \left(\frac{\chi^{\beta-1}}{2} \right)^2 \leq C,$$

and for $s \in [\varepsilon/2, C]$, the mean value theorem gives for some ξ between s and χ ,

$$\left(\frac{\chi^\beta - s^\beta}{\chi-s} \right)^2 = (\beta \xi^{\beta-1})^2 \leq C.$$

So, using our choice of β ,

$$\begin{aligned} & |I_1| W(x) |a_n - x|^{1/4} \\ & \leq C a_n^{\rho+k+1/4} \left(\int_0^{C_1} \frac{s^{2\ell-2\rho-1}}{|1-s|^{1/2}} ds + O(e^{-nC}) \right) \leq C a_n^{\rho+k+1/4}, \end{aligned}$$

since $\ell - \rho > 0$. As $x \in [\varepsilon a_n, a_{2n}]$, this leads to the estimate

$$|I_1| W(x) |a_n - x|^{1/4} \leq C x^{\rho+k+1/4}.$$

Finally, combining this last estimate, (8.11) and (8.12), and since

$$S(x) = x^{\beta+\ell} = x^{2\rho+k+1/2},$$

we obtain,

$$|p_{n,\rho}(x) W(x)| x^\rho |x(a_n - x)|^{1/4} \leq C.$$

■

The method for the rest of the range involves the function

$$(8.14) \quad A_{n,\rho}^\#(x) := \frac{2}{x} \int_I (p_{n,\rho} W_\rho)^2(t) \overline{Q(x,t)} dt$$

where

$$\overline{Q(x,t)} := \frac{xQ'(x) - tQ'(t)}{x-t}.$$

The first step involves an identity for $p'_{n,\rho}(x_{jn,\rho})$:

Lemma 8.3

$$(8.15) \quad p'_n(x_{jn,\rho}) = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} A_{n,\rho}^\#(x_{jn,\rho}) p_{n-1,\rho}(x_{jn,\rho}).$$

Proof

Let $K_{n,\rho}(x,t)$ denote the reproducing kernel for the weight W_ρ^2 . Since $p'_{n,\rho}$ has degree $\leq n-1$,

$$\begin{aligned} x_{jn,\rho} p'_{n,\rho}(x_{jn,\rho}) &= \int_I K_{n+1,\rho}(x_{jn,\rho}, t) t p'_{n,\rho}(t) W_\rho^2(t) dt \\ &= \int_I K_{n,\rho}(x_{jn,\rho}, t) t p'_{n,\rho}(t) W_\rho^2(t) dt, \end{aligned}$$

since $p_{n,\rho}(x_{jn,\rho}) = 0$. We integrate this last relation by parts. Using the fact that the integrand vanishes at 0 (recall that $1 + 2\rho > 0$) and d , as well as orthogonality, we obtain

$$x_{jn,\rho} p'_{n,\rho}(x_{jn,\rho}) = \int_I p_{n,\rho}(t) K_{n,\rho}(x_{jn,\rho}, t) 2t Q'(t) W_\rho^2(t) dt.$$

Next, the Christoffel-Darboux formulagives

$$(8.16) \quad x_{jn,\rho} p'_{n,\rho}(x_{jn,\rho}) = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}(x_{jn,\rho}) \left[2 \int_I \frac{p_{n,\rho}^2(t)}{t - x_{jn,\rho}} t Q'(t) W_\rho^2(t) dt \right].$$

Then orthogonality gives

$$\begin{aligned} p'_{n,\rho}(x_{jn,\rho}) &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}(x_{jn,\rho}) \frac{2}{x_{jn,\rho}} \int_I p_{n,\rho}^2(t) \left[\frac{tQ'(t) - x_{jn,\rho}Q'(x_{jn,\rho})}{t - x_{jn,\rho}} \right] W_\rho^2(t) dt \\ &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} A_{n,\rho}^\#(x_{jn,\rho}) p_{n-1,\rho}(x_{jn,\rho}). \end{aligned}$$

■

The next step is to use this identity to bound $p_n(x)$ in terms of $A_n^\#$ and λ_n :

Lemma 8.4

For $1 \leq j \leq n$,

$$(8.17) \quad |p_{n,\rho}(x)| \leq |x - x_{jn,\rho}| [\lambda_{n,\rho}(x)^{-1} A_{n,\rho}^\#(x_{jn,\rho})]^{1/2}.$$

Proof

Applying the Cauchy-Schwartz inequality to $K_{n,\rho}(x, x_{jn,\rho})$ gives

$$|K_{n,\rho}(x, x_{jn,\rho})| \leq \lambda_{n,\rho}^{-1/2}(x) \lambda_{n,\rho}^{-1/2}(x_{jn,\rho})$$

while (8.5) and Lemma 8.3 give

$$\lambda_{n,\rho}^{-1}(x_{jn,\rho}) = \left[\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}(x_{jn,\rho}) \right]^2 A_{n,\rho}^\#(x_{jn,\rho}).$$

Applying this identity and the last inequality to the Christoffel-Darboux formula (8.3) in the form

$$p_{n,\rho}(x) = K_{n,\rho}(x, x_{jn,\rho})(x - x_{jn,\rho}) / \left[\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}(x_{jn,\rho}) \right]$$

gives the result. ■

For a given x , we can choose $x_{jn,\rho}$ to be the closest zero of $p_{n,\rho}$ to x , and use our bounds for $x - x_{jn,\rho}$ from Theorem 7.3 together with our bounds for $\lambda_{n,\rho}$ from Theorem 1.3 to obtain a bound involving $A_{n,\rho}^\#(x_{jn,\rho})$. Choose $M > 1$ such that for large enough n ,

$$(8.18) \quad x_{nn,\rho} > \frac{a_n}{Mn^2}.$$

(This is possible by Theorem 7.3). We fix $\varepsilon \in (0, \frac{1}{2})$ and set

$$(8.19) \quad \mathcal{J}_n := \left[\frac{a_n}{Mn^2}, \varepsilon a_n \right].$$

In the sequel, we also need the notation

$$(8.20) \quad \Psi_n(x) := (p_{n,\rho}W)^2(x) \left(x + \frac{a_n}{n^2} \right)^{2\rho} |x(x - a_n)|^{1/2}$$

and

$$(8.21) \quad \Theta_n(x) := A_{n,\rho}^\#(x)\varphi_n(x) |x(x - a_n)|^{1/2}.$$

The next step is to bound Ψ_n in terms of Θ_n .

Lemma 8.5

Let $x \in \mathcal{J}_n = [\frac{a_n}{Mn^2}, \varepsilon a_n]$ and $x_{jn,\rho}$ denote the closest zero on the left or right to x , restricted to lie in \mathcal{J}_n . Then for some $C_1 \neq C_1(n, \varepsilon, x)$,

$$(8.22) \quad \Psi_n(x) \leq C_1 \Theta_n(x_{jn,\rho}).$$

Proof

From Theorem 7.3

$$(8.23) \quad |x - x_{jn}| \leq C\varphi_n(x_{kn})$$

where k is either $j+1$ or j . As in (7.14), Lemma 4.3 gives

$$\varphi_n(x_{kn,\rho}) \sim \varphi_n(x_{jn,\rho}) \sim \varphi_n(x).$$

Next, from Theorem 1.3,

$$\lambda_{n,\rho}(x)W_\rho^{-2}(x) \left(x + \frac{a_n}{n^2} \right)^{-2\rho} \sim \varphi_n(x) \sim \varphi_n(x_{jn,\rho}).$$

Combining this, (8.17) and (8.23) gives

$$\Psi_n(x) \leq CA_n^\#(x_{jn,\rho})\varphi_n(x_{jn,\rho}) |x(x - a_n)|^{1/2}.$$

It remains to show that

$$|x - a_n| \sim |x_{jn,\rho} - a_n| \quad \text{and} \quad x \sim x_{jn,\rho}.$$

This is easily established:

$$\frac{a_n - x}{a_n - x_{jn,\rho}} = 1 + \frac{x_{jn,\rho} - x}{a_n - x_{jn,\rho}} \leq 1 + \frac{x_{j-1,n,\rho} - x_{j+1,n,\rho}}{a_n - x_{jn,\rho}}$$

$$\leq 1 + \frac{C\varphi_n(x_{jn,\rho})}{a_n - x_{jn,\rho}} \leq 1 + C \frac{\varphi_n(x_{jn,\rho})}{a_n} \leq 1 + C \frac{\sqrt{x_{jn,\rho} + at^{-2}a_{2n}}}{n\sqrt{a_n a_n}} \leq 1 + \frac{C}{n} \leq C$$

by Theorem 7.3(c) and (1.18). Similarly we derive a lower bound. The proof that $x \sim x_{jn,\rho}$ is similar. ■

Now we prove:

Lemma 8.6

Let $\eta > 0$. There exist $\varepsilon \in (0, \frac{1}{2})$, $C(\varepsilon)$, and n_0 such that for $n \geq n_0$,

$$(8.24) \quad \|\Theta_n\|_{L_\infty(\mathcal{I}_n)} \leq C(\varepsilon) + \eta \|\Psi_n\|_{L_\infty(I)}.$$

Proof

We split

$$\begin{aligned} A_{n,\rho}^\#(x) &= \frac{2}{x} \left[\int_0^{\varepsilon a_n} + \int_{\varepsilon a_n}^{a_n} + \int_{a_n}^d \right] (p_{n,\rho} W_\rho)^2(t) \overline{Q(x,t)} dt \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Note that as $x \in \mathcal{I}_n = [\frac{a_n}{Mn^2}, \varepsilon a_n]$, and $\varepsilon < \frac{1}{2}$, (1.18) shows

$$(8.25) \quad \varphi_n(x) |x(x-a_n)|^{1/2} \sim \frac{a_n x}{n}.$$

We shall fix $\eta_1 > 0$ (to be chosen small enough later, depending on η). We can choose ε so small that

$$2\varepsilon a_n \leq a_{\eta_1 n},$$

in view of (3.3). Then

$$\begin{aligned} I_1 &\leq \frac{2}{x} \|\Psi_n\|_{L_\infty(I)} \int_0^{\varepsilon a_n} \frac{\overline{Q(x,t)}}{\sqrt{t(a_n-t)}} dt \\ &\leq \frac{2}{x} \|\Psi_n\|_{L_\infty(I)} \int_0^{a_{\eta_1 n}} \frac{\overline{Q(x,t)}}{\sqrt{t(a_{\eta_1 n}-t)}} dt \sup_{t \in [0, a_{\eta_1 n}]} \sqrt{\frac{a_{\eta_1 n}-t}{a_n-t}} \\ &\leq C \|\Psi_n\|_{L_\infty(I)} \sigma_{\eta_1 n}(x) \frac{1}{\sqrt{x(a_{\eta_1 n}-x)}} \sqrt{\frac{a_{\eta_1 n}}{a_n}} \\ &\leq C \|\Psi_n\|_{L_\infty(I)} \frac{1}{\varphi_{\eta_1 n}(x) \sqrt{x(a_{\eta_1 n}-x)}} \sqrt{\frac{a_{\eta_1 n}}{a_n}}, \end{aligned}$$

by (4.8) and (4.9). Here $x \leq \varepsilon a_n \Rightarrow x \leq \frac{1}{2} a_{\eta_1 n}$. Using (8.25), we deduce that

$$\begin{aligned} I_1 \varphi_n(x) |x(x-a_n)|^{1/2} &\leq C \|\Psi_n\|_{L_\infty(I)} \frac{\eta_1 n}{a_{\eta_1 n} x} \sqrt{\frac{a_{\eta_1 n}}{a_n}} \frac{a_n x}{n} \\ &\leq C \|\Psi_n\|_{L_\infty(I)} \eta_1 \sqrt{\frac{a_n}{a_{\eta_1 n}}} \leq C \Lambda_n \eta_1^{1-\frac{1}{2\lambda}}, \end{aligned}$$

by (3.3). Since $2\Lambda > 1$ and C is independent of $x \in \mathcal{I}_n$ and n and η_1 , we may choose η_1 so small that for all n and $x \in \mathcal{I}_n$,

$$(8.26) \quad I_1 \varphi_n(x) |x(x - a_n)|^{1/2} \leq \eta \|\Psi_n\|_{L^\infty(I)}.$$

Next, by the bounds on p_n that we already have for $x \geq \varepsilon a_n$,

$$\begin{aligned} I_2 &\leq \frac{C}{x} \int_{\varepsilon a_n}^{a_n} \frac{Q(x, t)}{\sqrt{t(a_n - t)}} dt \\ &\leq C \frac{\sigma_n(x)}{\sqrt{x(a_n - x)}} \leq \frac{C}{\varphi_n(x) \sqrt{x(a_n - x)}}, \end{aligned}$$

by (4.8) and (4.9), so

$$(8.27) \quad I_2 \varphi_n(x) \sqrt{x(a_n - x)} \leq C.$$

Finally,

$$\begin{aligned} I_3 &\leq \frac{2}{x} \int_{a_n}^d \frac{tQ'(t)}{t-x} (p_{n,\rho} W_\rho)^2(t) dt \\ &\leq \frac{2}{x(a_n - x)} \int_{a_n}^d tQ'(t) (p_{n,\rho} W_\rho)^2(t) dt. \end{aligned}$$

Here an integration by parts, and orthonormality, give

$$\int_I tQ'(t) (p_{n,\rho} W_\rho)^2(t) dt = n + \rho + \frac{1}{2}.$$

Then

$$I_3 \varphi_n(x) \sqrt{x(a_n - x)} \leq \frac{C}{x a_n} n \frac{a_n x}{n} = C.$$

Putting the estimates together gives

$$\begin{aligned} \Theta_{n,\rho}(x) &= A_{n,\rho}^\#(x) \varphi_n(x) \sqrt{x(a_n - x)} \\ &\leq C + \eta \|\Psi_n\|_{L^\infty(I)}, \end{aligned}$$

uniformly for n large enough and $x \in \mathcal{I}_n$. ■

We turn to the

Proof of Theorem 8.1

Let $0 < \eta < 1$. By the results of Lemma 8.5 and 8.6 we have for some $\varepsilon > 0$ and C_1 independent of n, ε

$$\begin{aligned} \sup_{x \in [a_n/Mn^2, \varepsilon a_n]} |\Psi_n(x)| &\leq C_1 \sup_{x \in [Ma_n/n^2, \varepsilon a_n]} \Theta_n(x) \\ &\leq C_1 (C(\varepsilon) + \eta \|\Psi_n\|_{L^\infty(I)}). \end{aligned}$$

Lemma 8.2 gives

$$\sup_{x \in [\varepsilon a_n, a_{2n}]} |\Psi_n(x)| \leq C_2.$$

Next, our restricted range inequality Theorem 5.2(a) gives for some C_3 independent of n, ε, η

$$\begin{aligned} & \| \Psi_{n,\rho} \|_{L_\infty(I)} \leq C_3 \| \Psi_{n,\rho} \|_{L_\infty(a_n/(Mn^2), a_n)} \\ & \leq C_3 \max \{ C_2, C_1 C(\varepsilon) + C_1 \eta \| \Psi_{n,\rho} \|_{L_\infty(I)} \}. \end{aligned}$$

Since C_1 and C_3 are independent of η , we may choose $\eta = (C_3 C_1)^{-1} / 2$, to obtain

$$\| \Psi_{n,\rho} \|_{L_\infty(I)} \leq C_4.$$

Then it also follows that for $x \in [a_n/Mn^2, a_n(1 - \eta_n)]$,

$$|p_{n,\rho} W|^2(x) \left(x + \frac{a_n}{n^2}\right)^{2\rho} \left| \left(x + \frac{a_n}{n^2}\right) \left((x - a_n)^2 + (a_n \eta_n)^2\right)^{1/2} \right|^{1/2} \leq C.$$

Our restricted range inequality Theorem 5.2(a) shows that this holds for all $x \in I$, and then Theorem 8.1 follows as stated. ■

REFERENCES

- [1] S. Bonan and D.S. Clark, *Estimates of the Hermite and the Freud Polynomials*, J. Approx. Theory, 63(1990), 210-224.
- [2] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes, 1999.
- [3] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides and X. Zhou, *Strong Asymptotics of Orthogonal Polynomials with Respect to Exponential Weights*, Comm. Pure Appl. Math., 52(1999), 1491-1552.
- [4] G. Freud, A. Giroux and Q.I. Rahman, *On Approximation by Polynomials with Weight $\exp(-|x|)$* , Canad J. Math., 30(1978), 358-372.
- [5] T. Kasuga and R. Sakai, *Orthonormal Polynomials with Generalized Freud Type Weights*, J. Approx. Theory, 121(2003), 13-53.
- [6] T. Kriecherbauer and K. T-R McLaughlin, *Strong Asymptotics of Polynomials Orthogonal with Respect to Freud Weights*, International Maths Research Notices, 6(1999), 299-333.
- [7] E. Levin and D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, 2001.
- [8] D.S. Lubinsky, *An Extension of the Erdős-Turan Inequality for Sums of Successive Fundamental Polynomials*, Annals of Numerical Math., 2(1995), 305-309.
- [9] G. Mastroianni, *Polynomial Inequalities, Functional Spaces and Best Approximation on the Real Semiaxis with Laguerre Weights*, ETNA, 14(2003), 142-151.
- [10] H.N. Mhaskar, *Bounds for Certain Freud-type Orthogonal Polynomials*, J. Approx. Theory, 63(1990), 238-254.
- [11] H.N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation*, World Scientific, 1996.

- [12] H.N. Mhaskar and E.B. Saff, *Extremal Problems for Polynomials with Exponential Weights*, Trans. Amer. Math. Soc., 285(1984), 204-234.
- [13] H.N. Mhaskar and E.B. Saff, *Where Does The Sup Norm of a Weighted Polynomial Live?*, Constr. Approx, 1(1985), 71-91.
- [14] B. Muckenhoupt, *Mean Convergence of Hermite and Laguerre Series II*, Trans. Amer. Math. Soc., 147(1970), 433-460.
- [15] P. Nevai, *Orthogonal Polynomials*, Memoirs Amer. Math. Soc., no. 213 (1979).
- [16] P. Nevai, *Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study*, J. Approx. Theory, 48(1986), 3-167.
- [17] P. Nevai and V. Totik, *Weighted Polynomial Inequalities*, Constr. Approx., 2(1986), 113-127.
- [18] P. Nevai and P. Vertesi, *Mean Convergence of Hermite-Fejer Interpolation*, J. Math. Anal. Appl., 105(1985), 26-58.
- [19] E.A. Rakhmanov, *On Asymptotic Properties of Polynomials Orthogonal on the Real Axis*, Math. USSR. Sbornik, 47(1984), 155-193.
- [20] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer, Berlin, 1997.
- [21] H.B. Stahl and V. Totik, *General Orthogonal Polynomials*, Cambridge University Press, Cambridge, 1992.
- [22] G. Szegő, *Orthogonal Polynomials*, American Math. Soc. Colloquium Publications, Vol. 23, Providence, 1975.
- [23] V. Totik, *Weighted Approximation with Varying Weights*, Springer Lecture Notes in Maths., Vol. 1300, Springer, Berlin, 1994.
- [24] P. Vertesi, *On an Interpolatory Inequality of Erdős and Turan and its Application*, Periodica Math. Hungarica, 40(2000), 195-203.

¹MATHEMATICS DEPARTMENT, THE OPEN UNIVERSITY OF ISRAEL, P.O. BOX 39328, RAMAT AVIV, TEL AVIV 61392, ISRAEL, AND, ²THE SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332-0160, USA, LUBINSKY@MATH.GATECH.EDU