Converse Quadrature Sum Inequalities for Freud Weights II

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Abstract

Let $W := \exp(-Q)$, where Q is of smooth polynomial growth at ∞, for example $Q(x) = |x|^{\beta}, \beta > 1$. We call W^2 a *Freud weight.* Let $\{x_{jn}\}_{j=1}^{n}$ and $\{\lambda_{jn}\}_{j=1}^{n}$ denote respectively the zeros of the nth orthonormal polynomial p_n for W^2 and the Christoffel numbers of order n. We establish converse quadrature sum inequalities associated with W, such as

$$\begin{aligned} \| \quad & (PW)(x)(1+|x|)^{r} \|_{L_p(\mathbb{R})} \\ \leq \quad & C \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|^{p}(x_{jn})(1+|x_{jn}|)^{Rp} \end{aligned}$$

with C independent of n and polynomials P of degree $< n$, and suitable restrictions on r, R. We concentrate on the case $p \geq 4$, as the case $p < 4$ was handled earlier. Moreover, we are able to treat a general class of Freud weights, whereas our earlier treatment dealt essentially with $\exp\left(-|x|^{\beta}\right), \beta = 2, 4, 6, \ldots$. Some applications to Lagrange interpolation are presented.

1 Introduction and Results

Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, convex, and is of polynomial growth at ∞. We call W^2 a *Freud weight.* Corresponding to W^2, we can define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \ldots, \gamma_n = \gamma_n(W^2) > 0,$$

such that

$$\int_{-\infty}^{\infty} p_n p_m W^2 = \delta_{mn}.$$

We denote the zeros of p_n by

$$x_{nn} < x_{n-1,n} < ... < x_{1n}.$$

The Lagrange interpolation polynomial $L_n[f]$ to a function $f : \mathbb{R} \to \mathbb{R}$ at $\{x_{jn}\}_{j=1}^n$ is a polynomial of degree at most $n-1$ satisfying

$$L_n[f](x_{jn}) = f(x_{jn}), 1 \leq j \leq n.$$

The convergence as $n \to \infty$ of $L_n[f]$ associated with exponential weights in various settings has been studied by many authors [1], [2], [3], [4], [5], [11], [13], [14], [21], [23]. One of the most successful approaches to establishing mean convergence of L_n is converse quadrature sum, or Marcinkiewicz-Zygmund, inequalities, which have the form

$$\| PW\varphi \|_{L_p(\mathbb{R})} \leq C \left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW\psi|^p (x_{jn}) \right)^{1/p}, \tag{1}$$

where ϕ, ψ are appropriate weighting factors, $1 < p < \infty$, and C is independent of n and polynomials P of degree $< n$. Moreover, $\{\lambda_{jn}\}_{j=1}^n$ are the Christoffel numbers of order n for W^2, so that

$$\int_{-\infty}^{\infty} PW^2 = \sum_{j=1}^{n} \lambda_{jn} P(x_{jn}),$$

for every polynomial P of degree $\leq 2n-1$. Since $x_{j-1,n} - x_{jn}$ has the "same size" as $\lambda_{jn} W^{-2}(x_{jn})$, (with an appropriate definition of x_{0n}), one may replace $\lambda_{jn} W^{-2}(x_{jn})$ in (1) by $x_{j-1,n} - x_{jn}$.

The main feature of this paper is the proof of (1) in L_p spaces with $p \geq 4$, for Freud weights, and under mild conditions on W. In earlier papers [9], we could only establish such inequalities for $p > 4$ under an implicit bound on the orthogonal polynomials associated with the weight W^2. At that time this bound was known only for $W(x) = \exp(-|x|^\alpha), \alpha = 2, 4, 6, ...$. Thus in dispensing with that bound, we remove a major defect of the results of [9].

For those requiring further orientation on Marcinkiewicz inequalities, a survey of these was presented in [10]. Other recent papers involving this technique include [1], [3], [4], [5], [9], [11], [12], [15], [24], [25]. To describe our results, we must define our class of weights:

Definition 1.1
Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, and continuous, Q'' is continuous in $(0, \infty)$ and $Q' > 0$ there, while for some $A, B > 1$,

$$A \leq \frac{(xQ'(x))'}{Q'(x)} \leq B, x \in (0, \infty).$$

Then we write $W \in \mathcal{F}$.

The archetypal example is

$$W(x) = W_\beta(x) := \exp(-|x|^\beta), \beta > 1.$$

An important quantity associated with W^2 is the *Mhaskar-Rakhmanov-Saff number* a_u, the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}} dt, u > 0.$$

One of its features is the Mhaskar-Saff identity [17]

$$\| Pe^{-Q} \|_{L_\infty(\mathbb{R})} = \| Pe^{-Q} \|_{L_\infty[-a_n, a_n]}, P \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ denotes the polynomials of degree $\leq n$. Note that for $Q(x) = |x|^\beta$, $\beta > 0$,

$$a_u = C(\beta) u^{1/\beta}, u > 0,$$

where $C(\beta)$ admits an explicit representation [17]. Our main result is:

Theorem 1.2
Let $W = e^{-Q} \in \mathcal{F}$ and $p \geq 4$. Let $r, R \in \mathbb{R}$ satisfy

$$r < 1 - \frac{1}{p}; r < R; R > -\frac{1}{p} \tag{2}$$

and for some $\delta > 0$,

$$n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} a_n^{r-\min\left\{1-\frac{1}{p}, R\right\}} = O\left(n^{-\delta}\right). \tag{3}$$

Then for $n \geq 1$ and $P \in \mathcal{P}_{n-1}$, we have

$$\int_{-\infty}^{\infty} |PW|^p(x)\,(1+|x|)^{rp}\,dx \leq C\left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})\,|PW|^p(x_{jn})\,(1+|x_{jn}|)^{Rp}\right)^{1/p}, \tag{4}$$

where C is independent of n and P.

Remarks
(a) If $W(x) = W_\beta(x) = \exp\left(-|x|^\beta\right), \beta > 1$, (3) reduces to

$$\frac{\beta}{6}\left(1 - \frac{4}{p}\right) + r - \min\left\{1 - \frac{1}{p}, R\right\} < 0. \tag{5}$$

This is the same condition as in Theorem 1.1(b) of [9, p. 529], except that there we allowed ≤ 0, at least for $R \neq 1 - \frac{1}{p}$. In this weaker form (5) was shown to be necessary for (4) to hold, but sufficient only if $\beta = 2, 4, 6, \ldots$.
(b) In Theorem 1.2 of [9, p. 530], it was shown that if (4) holds for $p > 4$, then necessarily for some $\varepsilon > 0$,

$$n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} a_n^{r-\min\left\{1-\frac{1}{p}, R+\varepsilon\right\}} = O(1).$$

Thus our result is very close to sharp. Moreover, it was shown that if we assume the bound

$$\sup_{x\in\mathbb{R}} |p_{n+1}(x) - p_{n-1}(x)| W(x) \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{-1/4} \leq C a_n^{-1/2}, \qquad (6)$$

with C independent of P, n, then (4) holds if (2) does, and if also

$$n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} a_n^{r-\min\left\{1-\frac{1}{p}, R\right\}} = \begin{cases} O(1), & \text{if } R \neq 1 - \frac{1}{p}; \\ O\left((\log n)^{-R}\right), & \text{if } R = 1 - \frac{1}{p}. \end{cases} \qquad (7)$$

The bound (6) was known only for $W(x) = \exp\left(-|x|^\beta\right), \beta = 2, 4, 6, ...$ at the time of writing of [9]. It almost certainly follows now for all $\beta > 1$, from the detailed asymptotics of Kriecherbauer and McLaughlin [6] for p_n.

The essential feature of this paper is that one does not need the bound (6) to establish (4), provided one uses König's method.

(c) In Theorem 1.2 of [9] the case $p < 4$ was also handled, but then the only restrictions on r, R were

$$r < 1 - \frac{1}{p}; r \leq R; R > -\frac{1}{p}.$$

The problems for $p \geq 4$ arise from the behaviour of p_n near $\pm a_n$: very roughly, $|p_n W|(x)$ behaves in magnitude like $|1 - |x|/a_n|^{-1/4}$, and the latter is not integrable over $[-a_n, a_n]$ for $p \geq 4$. We note that the latter problem may be alleviated by adding two extra interpolation points, an idea of Szabados. See [21], [14], [1].

(d) As a corollary, we can deduce mean convergence of Lagrange interpolation for functions that are Riemann integrable in each finite interval, and that are of suitably restricted growth for large $|x|$. Similar results were proved in [13] for continuous functions, but under a slightly weaker growth restriction (there $\varepsilon = 0$ was allowed in (9)):

Corollary 1.3

Let $W \in \mathcal{F}$ and let $p \geq 4$, $\Delta \in \mathbb{R}, \alpha > 0$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is bounded and Riemann integrable in each finite interval, and that

$$\lim_{|x|\to\infty} |fW|(x)(1 + |x|)^\alpha = 0. \qquad (8)$$

Assume that for some $\varepsilon > 0$,

$$n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} a_n^{\frac{1}{p}-\min\{1,\alpha\}-\Delta} = O\left(n^{-\varepsilon}\right), n \to \infty. \qquad (9)$$

Then

$$\lim_{n\to\infty} \| (f(x) - L_n[f](x)) W(x)(1 + |x|)^{-\Delta} \|_{L_p(\mathbb{R})} = 0. \qquad (10)$$

This paper is organised as follows: we state technical lemmas in Section 2, and then prove Theorem 1.2 in Section 3. We prove Corollary 1.3 in Section 4.

In the sequel $C, C_1, C_2, ...$ denote positive constants independent of $x, n, P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C \neq C(\lambda)$ to indicate that C does not depend on a parameter λ. The same symbol does not necessarily represent the same constant in different occurrences. We use $\sim$ in the following sense: if $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences of nonzero real numbers, we write

$$b_n \sim c_n$$

if there exist $C_1, C_2 > 0$ independent of n such that

$$C_1 \leq b_n/c_n \leq C_2, n \geq 1.$$

2 Technical Preliminaries for Theorem 1.2

In this section, we present technical lemmas required for the proof of Theorem 1.2, most of which were proved elsewhere. At a first reading, the reader may wish to skip this section. Throughout, we assume that $W \in \mathcal{F}$.

We begin with a well known lemma giving conditions for boundedness of integral operators on L_p :

Lemma 2.1
Let $1 < p < \infty$ and $q := \frac{p}{p-1}$. Let (Ω, μ) be a measure space and $S, R : \Omega^2 \to \mathbb{R}$. For $\mu-$measurable $f : \Omega \to \mathbb{R}$, define

$$T[f](u) := \int_\Omega S(u,v) f(v) d\mu(v).$$

Assume that

$$\sup_{u\in\Omega} \int_\Omega \left|S(u,v)\right| \left|R(u,v)\right|^q d\mu(v) \quad \leq \quad M; \tag{11}$$

$$\sup_{v\in\Omega} \int_\Omega \left|S(u,v)\right| \left|R(u,v)\right|^{-p} d\mu(u) \quad \leq \quad M; \tag{12}$$

Then T is a bounded operator from $L_p(d\mu)$ to $L_p(d\mu)$, more precisely,

$$\|T\|_{L_p(d\mu)\to L_p(d\mu)} \leq M.$$

That is, for every μ measurable f,

$$\left(\int |T[f]|^p \, d\mu \right)^{1/p} \leq M \left(\int |f|^p \, d\mu \right)^{1/p}.$$

Proof
See [3, Lemma 2.5, p.745] or [4] for a proof. $\square$

Our next two lemmas deal with Hilbert transforms. Recall that if $g \in L_1(\mathbb{R})$, we may define for a.e. x, its Hilbert transform

$$H[g](x) := PV \int_{-\infty}^{\infty} \frac{g(t)}{x-t} dt := \lim_{\varepsilon \to 0+} \int_{\{t: |t-x| \geq \varepsilon\}} \frac{g(t)}{x-t} dt.$$

We set

$$\begin{aligned}
x_{0n} &: = x_{1n}\left(1+n^{-2/3}\right); x_{n+1,n} := x_{nn}\left(1+n^{-2/3}\right); && (13)\\
I_{jn} &: = (x_{jn}, x_{j-1,n}); |I_{jn}| := x_{j-1,n} - x_{jn}; && (14)\\
\chi_{jn} &: = \chi_{I_{jn}} := \text{characteristic function of } I_{jn}; && (15)\\
h(x) &: = 1+|x|; && (16)\\
\phi_n(x) &: = \left|1-\frac{|x|}{a_n}\right| + n^{-2/3}; && (17)\\
\psi_n(x) &: = \left|1-\frac{|x|}{a_n}\right|. && (18)
\end{aligned}$$

For fixed $R \in \mathbb{R}$, we set

$$f_{jn}(x) := h(x_{jn})^{-R}\,\phi_n(x)^{-1/4} \times \begin{cases} |I_{jn}|^{-1}, & |x-x_{jn}| \leq 2|I_{jn}|; \\ \frac{|I_{jn}|}{|x-x_{jn}|}\left\{\frac{1}{|x-x_{jn}|}+\frac{1}{h(x_{jn})}\right\}, & |x-x_{jn}| > 2|I_{jn}|. \end{cases} \quad (19)$$

König's method involves replacing $\frac{1}{x-x_{jn}}$ by $H[\chi_{I_{jn}}]$. This is achieved with the aid of the following lemma:

Lemma 2.2
Uniformly for $n \geq 1, 1 \leq k \leq n, x \in [x_{nn}, x_{1n}]$,

$$\tau_{kn}(x) := a_n^{1/2}\,|p_n W|(x)\left|\frac{h^{-R}(x_{kn})}{x-x_{kn}} - \frac{H[\chi_{kn}h^{-R}](x)}{|I_{kn}|}\right| \leq C f_{kn}(x). \quad (20)$$

Proof
See [9, Lemma 5.2, pp. 542-545]. □

The following lemma deals with properties of Hilbert transforms:

Lemma 2.3
Let $b \in \mathbb{R}$ and $p > 1$ with

$$-\frac{1}{p} < b < 1-\frac{1}{p}; 0 \leq a < \frac{1}{p}. \quad (21)$$

Then for $g : \mathbb{R} \to \mathbb{R}$ supported on $[-2a_n, 2a_n]$,

$$\| H[g]\, h^b \psi_n^{-a} \|_{L_p[-2a_n,2a_n]} \leq C \| g h^b \psi_n^{-a} \|_{L_p[-2a_n,2a_n]}, \quad (22)$$

with $C \neq C(g,n)$.
Proof
This essentially goes back to Muckenhoupt and results on A_p weights [18], [20]. However, because of the dependence of (22) on n, we provide some of the details. First let us show how to get rid of the dependence on n. We use the fact that

$$\| H[g](t)\, |1-|t||^{-a} \|_{L_p(\mathbb{R})} \leq C \| g(t)\, |1-|t||^{-a} \|_{L_p(\mathbb{R})}, \quad (23)$$

with $C \neq C(g)$. This follows from our restrictions on a, which ensure that $|1-|t||^{-a}$ is an A_p weight. See [18] or [20]. Now for a given g, set

$$g_n(u) := g(a_n u), n \geq 1.$$

Applying (23) to g_n, and making a substitution $t = x/a_n$ gives

$$\int_{-\infty}^{\infty} \left| H\left[g_n\right]\left(\frac{x}{a_n}\right)\right|^p \left|1 - \frac{|x|}{a_n}\right|^{-ap} dx \leq C \int_{-\infty}^{\infty} \left| g_n\left(\frac{x}{a_n}\right)\right|^p \left|1 - \frac{|x|}{a_n}\right|^{-ap} dx,$$

with the same C as above (so that $C \neq C(g,n)$). Since

$$H\left[g_n\right]\left(\frac{x}{a_n}\right) = H\left[g\right](x)\,;\, g_n\left(\frac{x}{a_n}\right) = g(x)\,,$$

we obtain for some $C \neq C(g,n)$,

$$\| H[g]\,\psi_n^{-a} \|_{L_p(\mathbb{R})} \leq C \| g\psi_n^{-a} \|_{L_p(\mathbb{R})}\,. \tag{24}$$

Next, an old result of Muckenhoupt [18, Lemma 8, p. 440] shows that under our restrictions on b,

$$\| H[g]\,h^b \|_{L_p(\mathbb{R})} \leq C \| gh^b \|_{L_p(\mathbb{R})}, \tag{25}$$

with $C \neq C(g)$. We now "glue" these last two inequalities together in a straightforward fashion. Firstly,

$$\begin{aligned} \| \quad & H[g]\,h^b\psi_n^{-a} \|_{L_p[-2a_n,2a_n]} \\ \leq \quad & C\left\{ \| H[g]\,h^b \|_{L_p\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]} + a_n^b \| H[g]\,\psi_n^{-a} \|_{L_p\left([-2a_n,2a_n]\setminus\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]\right)} \right\}, \end{aligned} \tag{26}$$

with $C \neq C(n,g)$. The first term in the right-hand side of (26) may be estimated by (25) as

$$\begin{aligned} \| \quad & H[g]\,h^b \|_{L_p\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]} \\ \leq \quad & C \| gh^b \|_{L_p(\mathbb{R})} \leq C \| g\psi_n^{-a}h^b \|_{L_p(\mathbb{R})}, \end{aligned} \tag{27}$$

since $a \leq 0$. To estimate the second term in (26), we split

$$\begin{aligned} g &= g\chi_{\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]} + g\chi_{[-2a_n,2a_n]\setminus\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]} \\ &= : g_1 + g_2. \end{aligned}$$

Now by (24),

$$\begin{aligned} a_n^b \quad \| \quad & H[g_2]\,\psi_n^{-a} \|_{L_p\left([-2a_n,2a_n]\setminus\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]\right)} \\ \leq \quad & Ca_n^b \| g_2\psi_n^{-a} \|_{L_p(\mathbb{R})} \\ = \quad & Ca_n^b \| g\psi_n^{-a} \|_{L_p\left([-2a_n,2a_n]\setminus\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]\right)} \\ \leq \quad & C \| g\psi_n^{-a}h^b \|_{L_p(\mathbb{R})}, \end{aligned} \tag{28}$$

since $h \sim a_n^b$ in $[-2a_n,2a_n] \setminus \left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]$. Next, for $|x| \geq \frac{1}{2}a_n$,

$$\begin{aligned} \left|H[g_1](x)\right| &\leq Ca_n^{-1} \int_{-\frac{1}{4}a_n}^{\frac{1}{4}a_n} |g(t)|\,dt \\ &\leq Ca_n^{-1} \| gh^b \|_{L_p\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]} \| h^{-b} \|_{L_q\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]}, \end{aligned}$$

with $C \neq C(n,g)$. Here $q = p/(p-1)$ is the dual parameter for p. Then

$$\begin{aligned} a_n^b & \parallel H[g_1]\,\psi_n^{-a} \parallel_{L_p\left([-2a_n,2a_n]\setminus\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]\right)} \\ \leq & \; Ca_n^{b-1+\frac{1}{p}} \parallel gh^b \parallel_{L_p\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]} \parallel h^{-b} \parallel_{L_q\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]}. \end{aligned}$$

Here $b < 1 - \frac{1}{p} = \frac{1}{q}$, so

$$a_n^{b-1+\frac{1}{p}} \parallel h^{-b} \parallel_{L_q\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]} \sim a_n^{b-\frac{1}{q}} \left(a_n^{-bq+1}\right)^{\frac{1}{q}} = 1,$$

and then

$$\begin{aligned} a_n^b & \parallel H[g_1]\,\psi_n^{-a} \parallel_{L_p\left([-2a_n,2a_n]\setminus\left[-\frac{1}{2}a_n,\frac{1}{2}a_n\right]\right)} \\ \leq & \; C \parallel g\psi_n^{-a}h^b \parallel_{L_p\left[-\frac{1}{4}a_n,\frac{1}{4}a_n\right]}, \end{aligned}$$

again since $a \leq 0$. This last inequality, (26), (27) and (28) give (22). $\square$

Our final lemma in this section lists some bounds involving the orthonormal polynomials $\{p_n\}$:

Lemma 2.4
(a) For $n \geq 1$ and $x \in \mathbb{R}$,

$$|p_n W|(x) \leq C a_n^{-1/2} \phi_n(x)^{-1/4}. \tag{29}$$

(b) Let $a, r, R \in \mathbb{R}$ with

$$r \leq R; 0 \leq a \leq \frac{1}{4}; \tag{30}$$

and

$$n^{\frac{1}{6}-\frac{2a}{3}} a_n^{r-R} \leq C, n \geq 1. \tag{31}$$

Then for $n \geq 1$ and $x \in \mathbb{R}$,

$$a_n^{1/2} |p_n W h^r|(x) \leq C\phi_n^{-a}(x)\, h^R(x) \leq C\psi_n^{-a}(x)\, h^R(x). \tag{32}$$

(c) For n large enough, and uniformly for $1 \leq j, k \leq n$,

$$\begin{aligned} \lambda_{kn} W^{-2}(x_{kn}) &\sim x_{k-1,n} - x_{kn} = |I_{kn}|; && (33) \\ |p_n' W|(x_{kn}) &\sim a_n^{-1/2} |I_{kn}|^{-1} \psi_n^{-1/4}(x_{kn}); && (34) \end{aligned}$$

$$\begin{aligned} 1 - x_{1n}/a_n &\sim n^{-2/3}; 1 + x_{nn}/a_n \sim n^{-2/3}; && (35) \\ |I_{kn}| &\sim |I_{k+1,n}| \sim \frac{a_n}{n}\phi_n(x_{kn})^{-1/2} \sim \frac{a_n}{n}\psi_n(x_{kn})^{-1/2}; && (36) \\ \psi_n(x_{kn}) &\sim \psi_n(x_{k+1,n}) \sim \phi_n(x_{k+1,n}) \sim \phi_n(x_{kn}) \text{ if } k < n; && (37) \\ h(x_{kn}) &\sim h(x_{k+1,n}); && (38) \\ |x - x_{kn}| &\sim |x_{jn} - x_{kn}|, x \in I_{jn}, \\ \text{if } |x_{jn} - x_{kn}| &\geq 2|I_{jn}| \text{ or } |j-k| \geq 2. && (39) \end{aligned}$$

(d) Let $b \in \mathbb{R}, L > 0$. Then there exists $n_0 = n_0(L)$ such that for $n \geq n_0$ and $P \in \mathcal{P}_n$, we have

$$\| PWh^b \|_{L_p(\mathbb{R})} \leq C \| PWh^b \|_{L_p\left[-a_n\left(1-Ln^{-2/3}\right), a_n\left(1-Ln^{-2/3}\right)\right]} . \tag{40}$$

(e) For $n \geq 1$,

$$C_2 n^{1/A} \geq a_n \geq C_1 n^{1/B}. \tag{41}$$

Proof
(a) This follows from Corollary 1.4 in [7, p. 467].
(b) From (a), it suffices to show that

$$\phi_n(x)^{a-\frac{1}{4}} h^{r-R}(x) \leq C. \tag{42}$$

But $a - \frac{1}{4} \leq 0$, and $r - R \leq 0$, so we see that this last inequality holds iff it holds for $x = a_n$, and at $x = a_n$, (42) reduces to (31). Since $\phi_n \geq \psi_n$, the second inequality in (32) also follows.
(c) Firstly (33) follows from Theorem 1.1 and Corollary 1.2 in [7, pp. 465-6]; next (34) follows from Corollary 1.3 in [7], while a slightly weaker form of (35), (36) follows from Corollary 1.2 in [7]. For (35), (36) as stated, see [8]. For (37), see (11.10) in [7, p.521], and then (38), (39) follow easily.
(d) See Lemma 2.1(d) in [9, p.533].
(e) See Lemma 5.2(b) in [7, p.478]. □

3 Proof of Theorem 1.2

Throughout we assume that $W \in \mathcal{F}$, that the hypotheses of Theorem 1.2 hold, and assume the notation of Section 2. We shall break the proof of Theorem 1.2 into several steps. Note that if (4) of Theorem 1.2 holds for a given r, then it holds for smaller r, and if it holds for given R, then it holds for larger R. Moreover, recall (2). We may then initially assume that

$$-\frac{1}{p} < R < 1 - \frac{1}{p}, \tag{43}$$

while preserving (3). Indeed, if $R \geq 1 - \frac{1}{p}$, we may replace R by $R_1 = 1 - \frac{1}{p} - \delta_1$, with $\delta_1 > 0$ small enough, while keeping $R_1 > r; R_1 > -\frac{1}{p}$. Note here that a_n does grow like a positive power of n.

Step 1: Express PW as a sum of two terms
Let $P \in \mathcal{P}_{n-1}$. For $1 \leq k \leq n$, set

$$y_{kn} := a_n^{-1/2} \frac{(PWh^R)(x_{kn})}{(p_n' W)(x_{kn})}.$$

In view of (34), we have uniformly in k and n,

$$|y_{kn}| \sim |I_{kn}| \, \psi_n^{1/4}(x_{kn}) \left| PWh^R \right| (x_{kn}). \tag{44}$$

We write

$$
\begin{aligned}
(PWh^r)(x) &= (L_n[P]Wh^r)(x) \\
&= a_n^{1/2}(p_n Wh^r)(x)\sum_{k=1}^{n} y_{kn}\left[\frac{h^{-R}(x_{kn})}{x-x_{kn}} - \frac{H\left[\chi_{kn}h^{-R}\right](x)}{|I_{kn}|}\right] \\
&\quad + a_n^{1/2}(p_n Wh^r)(x) H\left[\sum_{k=1}^{n} y_{kn}\frac{\chi_{kn}h^{-R}}{|I_{kn}|}\right](x) \\
&= : J_1(x) + J_2(x). \qquad (45)
\end{aligned}
$$

Note that in view of the behaviour of the smallest and largest zeros (see (35)) and the infinite-finite range inequality (40), we have for some C independent of P and n,

$$
\|PWh^r\|_{L_p(\mathbb{R})} \le C\,\|PWh^r\|_{L_p[x_{nn},x_{1n}]} \le C\left(\|J_1\|_{L_p[x_{nn},x_{1n}]} + \|J_2\|_{L_p[x_{nn},x_{1n}]}\right). \qquad (46)
$$

Step 2: Estimate J_2

(We begin with J_2 as it is easier to handle). For some small enough $\delta_1 > 0$, we let

$$
a = \frac{1}{p} - \delta_1 .
$$

Since a_n grows faster than some power of n, (3) then implies (31). Thus we may apply Lemma 2.4(b) and then Lemma 2.3 to deduce that

$$
\begin{aligned}
\|J_2\|_{L_p[x_{nn},x_{1n}]} &\le C\left\|\psi_n^{-a}h^R H\left[\sum_{k=1}^{n} y_{kn}\frac{\chi_{kn}h^{-R}}{|I_{kn}|}\right]\right\|_{L_p[x_{nn},x_{1n}]} \\
&\le C\left\|\psi_n^{-a}h^R \sum_{k=1}^{n} y_{kn}\frac{\chi_{kn}h^{-R}}{|I_{kn}|}\right\|_{L_p[x_{nn},x_{0n}]} \\
&\le C\left[\sum_{k=1}^{n}\left\{\frac{|y_{kn}|}{|I_{kn}|}\right\}^p \int_{I_{kn}} \psi_n^{-ap}\right]^{1/p} \\
&\le C\left[\sum_{k=1}^{n}\left\{\psi_n^{1/4}(x_{kn})\left|PWh^R\right|(x_{kn})\right\}^p |I_{kn}|\,\psi_n^{-ap}(x_{kn})\right]^{1/p} \\
&\le C\left[\sum_{k=1}^{n}|I_{kn}|\left|PWh^R\right|(x_{kn})^p\right]^{1/p}. \qquad (47)
\end{aligned}
$$

In the second last step, we used (37) and (44) and in the last step, we used the fact that $1/4 - a \ge 0$, so that $\psi_n^{1/4-a}$ is bounded above independent of n, x.

Step 3: Estimation of J_1, Part 1

By Lemma 2.2, with the notation there,

$$
|J_1(x)| \le \left|\sum_{k=1}^{n}|y_{kn}|\,\tau_{kn}(x)h^r(x)\right| \le C\sum_{k=1}^{n}|y_{kn}|\,f_{kn}(x)h^r(x).
$$

Here by (19), and (36) to (39), uniformly in n and $j \ne k$,

$$
f_{kn}(x) \sim f_{kn}(x_{jn}) \text{ in } I_{jn},
$$

so

$$\begin{aligned}
\|J_1\|_{L_p[x_{nn},x_{1n}]} &\leq C\left[\sum_{j=2}^{n}\int_{I_{jn}}\left[\sum_{k=1}^{n}|y_{kn}|\,f_{kn}(x)h^r(x)\right]^p(x)dx\right]^{1/p} \\
&\leq C\left[\sum_{j=2}^{n}|I_{jn}|\left[\sum_{k=1}^{n}|y_{kn}|\,f_{kn}(x_{jn})h^r(x_{jn})\right]^p\right]^{1/p} \\
&\leq C\left(S_1+S_2+S_3\right), \qquad (48)
\end{aligned}$$

where (recall (19))

$$S_1 \ : = \left[\sum_{j=2}^{n}|I_{jn}|\left[\sum_{\substack{k=1\\k\neq j}}^{n}|y_{kn}|\,\frac{h^{-R}(x_{kn})\,\phi_n(x_{jn})^{-1/4}\,|I_{kn}|\,h^r(x_{jn})}{(x_{jn}-x_{kn})^2}\right]^p\right]^{1/p} \qquad (49)$$

$$S_2 \ : = \left[\sum_{j=2}^{n}|I_{jn}|\left[\sum_{\substack{k=1\\k\neq j}}^{n}|y_{kn}|\,\frac{h^{-R}(x_{kn})\,\phi_n(x_{jn})^{-1/4}\,|I_{kn}|\,h^r(x_{jn})}{|x_{jn}-x_{kn}|\,h(x_{kn})}\right]^p\right]^{1/p} \qquad (50)$$

$$S_3 \ : = \left[\sum_{j=2}^{n}|I_{jn}|\left[|y_{jn}|\,\frac{h^{-R}(x_{jn})\,\phi_n(x_{jn})^{-1/4}\,h^r(x_{jn})}{|I_{jn}|}\right]^p\right]^{1/p}. \qquad (51)$$

Using our estimate (44) for y_{jn} and (37), we obtain

$$\begin{aligned}
S_3 &\leq C\left[\sum_{j=1}^{n}|I_{jn}|\,|PWh^r|\,(x_{jn})^p\right]^{1/p} \\
&\leq C\left[\sum_{j=1}^{n}|I_{jn}|\,\left|PWh^R\right|(x_{jn})^p\right]^{1/p}, \qquad (52)
\end{aligned}$$

as $r\leq R$.

Step 4: Estimation of S_1

Now it follows from (36) that

$$|I_{\ell n}|\sim\frac{a_n}{n}\phi_n^{-1/2}(x_{\ell n})\sim\frac{a_n}{n}\psi_n^{-1/2}(x_{\ell n})\Leftrightarrow\psi_n(x_{\ell n})\sim\phi_n(x_{\ell n})\sim\left(\frac{n}{a_n}|I_{\ell n}|\right)^{-2}. \qquad (53)$$

Then we see from (49) and (44) that

$$S_1\leq C\left[\sum_{j=1}^{n}\left[\sum_{k=1}^{n}b_{jk}\left|PWh^R\right|(x_{kn})\,|I_{kn}|^{1/p}\right]^p\right]^{1/p}, \qquad (54)$$

where $b_{kk}:=b_{1k}:=0$ and if $2\leq j\neq k$,

$$b_{jk}:=\frac{|I_{kn}|^{\frac{3}{2}-\frac{1}{p}}\,|I_{jn}|^{\frac{1}{2}+\frac{1}{p}}\,h^r(x_{jn})\,h^{-R}(x_{kn})}{(x_{jn}-x_{kn})^2}.$$

Defining the $n \times n$ matrix $B := (b_{jk})_{j,k=1}^n$, we see that if ℓ_p^n denotes $\mathbb{R}^n$ with the usual ℓ_p norm, then

$$S_1 \leq C\|B\|_{\ell_p^n \to \ell_p^n} \left[\sum_{k=1}^n |I_{kn}| \left| PWh^R \right|^p (x_{kn}) \right]^{1/p}.$$

If we can show that for some C_1 independent of n, that

$$\|B\|_{\ell_p^n \to \ell_p^n} \leq C_1,\ n \geq 1, \tag{55}$$

then we obtain for some $C \neq C(n, P)$,

$$S_1 \leq C \left[\sum_{k=1}^n |I_{kn}| \left| PWh^R \right|^p (x_{kn}) \right]^{1/p}. \tag{56}$$

Let us set $\Omega = \{1, 2, 3, \ldots, n\}$ in Lemma 2.1, and let us set there $\mu(\{j\}) = 1,\ 1 \leq j \leq n$, and

$$S(j,k) := b_{jk};\ R(j,k) := \left(\frac{|I_{kn}|}{|I_{jn}|} \right)^{\frac{1}{pq}} \left(\frac{h(x_{jn})^r}{h(x_{kn})^R} \right)^{\frac{1}{p}}.$$

We see that Lemma 2.1 gives (55) if we can show that

$$\begin{aligned} \sup_j \sum_{k=1}^n S(j,k) R(j,k)^q &\leq C; \\ \sup_k \sum_{j=1}^n S(j,k) R(j,k)^{-p} &\leq C, \end{aligned}$$

that is, if we recall the choice of $\{b_{jk}\}, S, R$ and that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \sup_j \sum_{\substack{k=1 \\ k \neq j}}^n \frac{|I_{kn}|^{\frac{3}{2}} |I_{jn}|^{\frac{1}{2}}}{(x_{jn} - x_{kn})^2} \left(\frac{h^r(x_{jn})}{h^R(x_{kn})} \right)^q &\leq C; \\ \sup_k \sum_{\substack{j=1 \\ j \neq k}}^n \frac{|I_{jn}|^{\frac{3}{2}} |I_{kn}|^{\frac{1}{2}}}{(x_{jn} - x_{kn})^2} &\leq C. \end{aligned} \tag{57}$$

Swopping j and k in the last inequality, it follows that we must show for $\ell = 1, 2$,

$$\sup_j \sum_{\substack{k=1 \\ k \neq j}}^n \frac{|I_{kn}|^{\frac{3}{2}} |I_{jn}|^{\frac{1}{2}}}{(x_{jn} - x_{kn})^2} \left(\frac{h^r(x_{jn})}{h^R(x_{kn})} \right)^{\sigma_\ell} \leq C;$$

where

$$\sigma_1 := q; \sigma_2 := 0.$$

Using (36-39), we can estimate the sum by an integral: we must show that for $\ell = 1, 2$,

$$\sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} \int_{\{t \in [x_{nn}, x_{1n}] : |t - x_{jn}| \geq C|I_{jn}|\}} \frac{\phi_n(t)^{-1/4}}{(t - x_{jn})^2} \left(\frac{h^r(x_{jn})}{h^R(t)} \right)^{\sigma_\ell} dt \leq C. \tag{58}$$

Split the range of integration into 3 ranges:

$$\begin{aligned}
\mathcal{S}_1 &: = \{t \in [x_{nn}, x_{1n}] : |t - x_{jn}| \geq C |I_{jn}| \text{ and } h(t) \geq 2h(x_{jn})\};\\
\mathcal{S}_2 &: = \left\{t \in [x_{nn}, x_{1n}] : |t - x_{jn}| \geq C |I_{jn}| \text{ and } h(t) \leq \frac{1}{2} h(x_{jn})\right\};\\
\mathcal{S}_3 &: = \left\{t \in [x_{nn}, x_{1n}] : |t - x_{jn}| \geq C |I_{jn}| \text{ and } \frac{1}{2} < h(t)/h(x_{jn}) < 2\right\}.
\end{aligned}$$

(I) $\int_{\mathcal{S}_1}$:
Here

$$|t - x_{jn}| \geq |(1 + |t|) - (1 + |x_{jn}|)| \geq \frac{1}{2}(1 + |t|) = \frac{1}{2} h(t),$$

so

$$\begin{aligned}
T_1 &: = \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} \int_{\mathcal{S}_1} \frac{\phi_n(t)^{-1/4}}{(t - x_{jn})^2} \left(\frac{h^r(x_{jn})}{h^R(t)}\right)^{\sigma_\ell} dt\\
&\leq C \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} h^{r\sigma_\ell}(x_{jn}) \int_{\mathcal{S}_1} \phi_n(t)^{-1/4} h^{-R\sigma_\ell - 2}(t)\, dt. \quad (59)
\end{aligned}$$

We claim that for $\ell = 1, 2$, the exponent $-R\sigma_\ell - 2 < -1$. Indeed for $\ell = 1$,

$$-R\sigma_\ell - 2 = -Rq - 2 < \frac{q}{p} - 2 < -1,$$

by (43) and since $p \geq 4 \Rightarrow q < p$. Also for $\ell = 2$,

$$-R\sigma_\ell - 2 = -2.$$

Since the range $\mathcal{S}_1$ involves a range of t with $|t| \geq h(x_{jn})$, and since $\phi_n(t)^{-1/4}$ is bounded above except in a neighbourhood of $\pm a_n$, while $-\frac{1}{4} > -1$, we may split the integral in (59) into that part with $|t| \in \left[\frac{1}{2} a_n, 2a_n\right]$, and the rest, and then estimate each integral to deduce that

$$T_1 \leq C \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} h^{(r-R)\sigma_\ell - 1}(x_{jn}).$$

For j such that $|x_{jn}| \leq \frac{1}{2} a_n$, $\phi_n(x_{jn}) \sim 1$, while $(r - R)\sigma_\ell - 1 \leq 0$, $\ell = 1, 2$, so we see that the term in the last right-hand side is bounded. For j such that $|x_{jn}| \geq \frac{1}{2} a_n$, we see that the term in the last right-hand side is bounded by

$$\begin{aligned}
& C \frac{a_n}{n} n^{\frac{1}{6}} a_n^{(r-R)\sigma_\ell - 1}\\
= \;& C n^{-\frac{5}{6}} a_n^{(r-R)\sigma_\ell} \leq C,
\end{aligned}$$

again as $r \leq R$ and $\sigma_\ell \geq 0$. Thus

$$T_1 \leq C. \quad (60)$$

(II) $\int_{\mathcal{S}_2}$:
Here we see, much as for S_1, that

$$|t - x_{jn}| \geq \frac{1}{2} h(x_{jn}),$$

so

$$\begin{aligned} T_2 \quad : \quad &= \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} \int_{\mathcal{S}_2} \frac{\phi_n(t)^{-1/4}}{(t-x_{jn})^2} \left(\frac{h^r(x_{jn})}{h^R(t)} \right)^{\sigma_\ell} dt \\ &\leq C \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} h^{r\sigma_\ell - 2}(x_{jn}) \int_{\mathcal{S}_2} \phi_n(t)^{-1/4} h^{-R\sigma_\ell}(t)\, dt. \end{aligned}$$

Here if $\ell = 1, R\sigma_\ell = Rq < 1$, (by (43)) while if $\ell = 2$, $R\sigma_\ell = 0$. It follows that

$$\begin{aligned} &\int_{\mathcal{S}_2} \phi_n(t)^{-1/4} h^{-R\sigma_\ell}(t)\, dt \\ \leq\ & C \int_0^{\frac{1}{2}h(x_{jn})} \phi_n(t)^{-1/4} h^{-R\sigma_\ell}(t)\, dt \\ \leq\ & Ch(x_{jn})^{1-R\sigma_\ell}. \end{aligned}$$

(As above, one splits the range of integration into that part for which $|t| \in \left[\frac{1}{2}a_n, 2a_n\right]$ and the rest; of course the former range may be empty). Then

$$T_2 \leq C \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} h^{(r-R)\sigma_\ell - 1}(x_{jn}) \leq C, \tag{61}$$

exactly as for T_1.
(III) $\int_{\mathcal{S}_3}$:
Here $h(t) \sim h(x_{jn})$, so

$$\begin{aligned} T_3 \quad : \quad &= \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} \int_{\mathcal{S}_3} \frac{\phi_n(t)^{-1/4}}{(t-x_{jn})^2} \left(\frac{h^r(x_{jn})}{h^R(t)} \right)^{\sigma_\ell} dt \\ &\leq C \sup_j \frac{a_n}{n} \phi_n(x_{jn})^{-1/4} h^{(r-R)\sigma_\ell}(x_{jn}) \int_{\mathcal{S}_3} \frac{\left|1 - \frac{|t|}{a_n}\right|^{-1/4}}{(t-x_{jn})^2} dt. \end{aligned} \tag{62}$$

Let us estimate the integral assuming that $x_{jn} \geq 0$. We see by the substitution $1 - \frac{t}{a_n} = \left(1 - \frac{x_{jn}}{a_n}\right) s$ that

$$\begin{aligned} &\int_{\mathcal{S}_3} \frac{\left|1 - \frac{|t|}{a_n}\right|^{-1/4}}{(t-x_{jn})^2} dt \\ \leq\ & Ca_n^{-1} \left|1 - \frac{x_{jn}}{a_n}\right|^{-5/4} \int_{\left\{s: |s-1| \geq C|I_{jn}| / \left(a_n \left|1 - \frac{x_{jn}}{a_n}\right|\right)\right\}} \frac{|s|^{-1/4}}{(s-1)^2} ds \\ \leq\ & Ca_n^{-1} \left|1 - \frac{x_{jn}}{a_n}\right|^{-5/4} \int_{\left\{s: |s-1| \geq Cn^{-1} \left|1 - \frac{x_{jn}}{a_n}\right|^{-3/2}\right\}} \frac{|s|^{-1/4}}{(s-1)^2} ds \\ \leq\ & Ca_n^{-1} n \left|1 - \frac{x_{jn}}{a_n}\right|^{1/4}. \end{aligned}$$

In the second last step, we used (36). Substituting in (62) and using $(r-R)\sigma_\ell \leq 0$, we obtain

$$T_3 \leq C.$$

This last estimate, (60) and (61) give (58) and hence (55). Then we have finished estimation of S_1.

Step 5: Estimation of S_2

Proceeding as for S_1, we see from (50) that

$$S_2 \leq C \left[\sum_{j=1}^{n} \left[\sum_{k=1}^{n} \hat{b}_{jk} \left| PWh^R \right| (x_{kn}) \left| I_{kn} \right|^{1/p} \right]^p \right]^{1/p}, \tag{63}$$

where $\hat{b}_{kk} := \hat{b}_{1k} := 0$ and if $2 \leq j \neq k$,

$$\hat{b}_{jk} := \frac{\left| I_{kn} \right|^{\frac{3}{2} - \frac{1}{p}} \left| I_{jn} \right|^{\frac{1}{2} + \frac{1}{p}} h^r (x_{jn}) h^{-R} (x_{kn})}{\left| x_{jn} - x_{kn} \right| h (x_{kn})} = b_{jk} \frac{\left| x_{jn} - x_{kn} \right|}{h (x_{kn})}.$$

We may dispense with a large part of the sum in (63) by noting that

$$h (x_{jn}) \leq 4h (x_{kn}) \Rightarrow \left| x_{jn} - x_{kn} \right| \leq 5h (x_{kn}) \Rightarrow \left| \hat{b}_{jk} \right| \leq 5 \left| b_{jk} \right|.$$

Then setting

$$b^*_{jk} := \hat{b}_{jk} \text{ if } h (x_{jn}) > 4h (x_{kn})$$

and $b^*_{jk} := 0$ otherwise, and $B^* := \left(b^*_{jk} \right)$, we see that

$$\begin{aligned} S_2 &\leq CS_1 + C \left[\sum_{j=1}^{n} \left| I_{jn} \right| \left[\sum_{k=1}^{n} b^*_{jk} \left| PWh^R \right| (x_{kn}) \left| I_{kn} \right|^{1/p} \right]^p \right]^{1/p} \\ &\leq C \left\{ 1 + \left\| B^* \right\|_{\ell_p^n \to \ell_p^n} \right\} \left[\sum_{k=1}^{n} \left| I_{kn} \right| \left| PWh^R \right|^p (x_{kn}) \right]^{1/p}. \end{aligned} \tag{64}$$

Now we use Lemma 2.1 as before to show that

$$\left\| B^* \right\|_{\ell_p^n \to \ell_p^n} \leq C_1, \ n \geq 1. \tag{65}$$

This time we choose

$$S(j,k) := b^*_{jk}; \ R(j,k) := \left(\frac{\left| I_{kn} \right|}{\left| I_{jn} \right|} \right)^{\frac{1}{pq}} \left(\frac{h(x_{jn})^r}{h(x_{kn})^R} \right)^{\frac{1}{p}} \phi_n (x_{jn})^{\frac{3}{4p}}.$$

Instead of (57), we must show now that

$$\begin{aligned} \sup_j \sum_{\substack{k=1 \\ h(x_{jn}) > 4h(x_{kn})}}^{n} \frac{\left| I_{kn} \right|^{\frac{3}{2}} \left| I_{jn} \right|^{\frac{1}{2}}}{\left| x_{jn} - x_{kn} \right| h (x_{kn})} \left(\frac{h(x_{jn})^r}{h(x_{kn})^R} \right)^q \phi_n (x_{jn})^{\frac{3q}{4p}} &\leq C; \\ \sup_k \sum_{\substack{j=1 \\ h(x_{jn}) > 4h(x_{kn})}}^{n} \frac{\left| I_{jn} \right|^{\frac{3}{2}} \left| I_{kn} \right|^{\frac{1}{2}}}{\left| x_{jn} - x_{kn} \right| h (x_{kn})} \phi_n (x_{jn})^{-\frac{3}{4}} &\leq C. \end{aligned}$$

Now $h (x_{jn}) > 4h (x_{kn}) \Rightarrow \left| x_{jn} - x_{kn} \right| \geq \frac{3}{4} h (x_{jn})$. Moreover, then, $\left| x_{kn} \right| / a_n$ cannot approach 1, so uniformly in n and k in the sums,

$$\left| I_{kn} \right| \sim \frac{a_n}{n} \phi_n^{-1/2} (x_{kn}) \sim \frac{a_n}{n}.$$

Hence we can bound the sums above by

$$T_1 := \sup_j \frac{a_n}{n} \phi_n (x_{jn})^{-\frac{1}{2}\left(\frac{1}{2}-\frac{3q}{2p}\right)} h(x_{jn})^{rq-1} \sum_{\substack{k=1 \\ h(x_{jn})>4h(x_{kn})}}^{n} |I_{kn}| \, h(x_{kn})^{-Rq-1};$$

$$T_2 := \sup_k \left(\frac{a_n}{n}\right) h(x_{kn})^{-1} \sum_{\substack{j=1 \\ h(x_{jn})>4h(x_{kn})}}^{n} |I_{jn}| \, \phi_n (x_{jn})^{-1} h(x_{jn})^{-1}.$$

Much as for S_1, we may estimate

$$\begin{aligned} T_1 &\leq C \sup_j \frac{a_n}{n} \phi_n (x_{jn})^{-\frac{1}{2}\left(\frac{1}{2}-\frac{3q}{2p}\right)} h^{rq-1}(x_{jn}) \int_0^{\frac{1}{2}h(x_{jn})} h^{-Rq-1}(t)\, dt \\ &\leq C \sup_j \frac{a_n}{n} \phi_n (x_{jn})^{-\frac{1}{2}\left(\frac{1}{2}-\frac{3q}{2p}\right)} h^{rq-1}(x_{jn}) \begin{cases} h(x_{jn})^{-Rq}, & R<0 \\ \log(1+h(x_{jn})), & R=0 \\ 1, & R>0 \end{cases}. \end{aligned} \tag{66}$$

We now analyse when the terms in this last right-hand side are largest. Note that

$$\begin{aligned} p &\geq 4 \Rightarrow \frac{q}{p} = q-1 \leq \frac{1}{3} \\ &\Rightarrow \frac{1}{2} - \frac{3q}{2p} \geq 0. \end{aligned}$$

Hence the term involving ϕ_n is largest when x_{jn} is closest to x_{1n}. Moreover, $rq < Rq < 1$, by (2) and (43), so the powers of $h(x_{jn})$ are all non-positive powers. It follows that the term in (66) is bounded when x_{jn} is, so it suffices to consider x_{jn} close to a_n. So

$$T_1 \leq C + C\frac{a_n}{n} n^{\frac{1}{3}\left(\frac{1}{2}-\frac{3q}{p}\right)} a_n^{rq-1} \times \begin{cases} a_n^{-Rq}, & R<0 \\ \log n, & R=0 \\ 1, & R>0 \end{cases}.$$

Now we use (3), which implies that

$$(\log n)\, a_n^{rq} \leq C a_n^{Rq} n^{-\frac{q}{6}\left(1-\frac{4}{p}\right)}.$$

So,

$$\begin{aligned} T_1 &\leq C + C n^{-\frac{5}{6}-\frac{q}{p}-\frac{q}{6}\left(1-\frac{4}{p}\right)} \begin{cases} 1, & R<0 \\ 1, & R=0 \\ a_n^{Rq}, & R>0 \end{cases} \\ &\leq C + C n^{\frac{1}{6}-\frac{q}{p}-\frac{q}{6}\left(1-\frac{4}{p}\right)}, \end{aligned}$$

as $Rq < 1$ and as $a_n = O(n)$. Here we may rewrite the exponent of n as

$$\begin{aligned} & q\left(\frac{1}{6}\left(\frac{1}{q}-1\right) - \frac{1}{p} + \frac{2}{3p}\right) \\ = \; & q\left(-\frac{1}{6p} - \frac{1}{p} + \frac{2}{3p}\right) = -\frac{q}{2p} < 0. \end{aligned}$$

Thus,

$$T_1 \leq C.$$

Next, we may estimate

$$\begin{aligned} T_2 &\leq C \sup_k \left(\frac{a_n}{n}\right) h\left(x_{kn}\right)^{-1} \int_{\frac{1}{4}h(x_{kn})}^{2a_n} \phi_n\left(t\right)^{-1} h\left(t\right)^{-1} dt \\ &\leq C\left(\frac{a_n}{n}\right) \cdot 1 \cdot \log n = o\left(1\right), \end{aligned}$$

by straightforward estimation. Thus T_1 and T_2 are bounded, so we have shown (65) and the proof of Theorem 1.2 is complete. □

4 Proof of Corollary 1.3

First let us set for some small enough δ_1,

$$r := -\Delta; R := \alpha - \frac{1}{p} - \delta_1.$$

Then we may reformulate (9) as

$$n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} a_n^{r-\min\left\{1-\frac{1}{p}, R+\delta_1\right\}} = O\left(n^{-\varepsilon}\right), n \geq 1.$$

If δ_1 is small enough relative to ε, it is then easy to see that (3) holds with an appropriate δ. Also then (3) necessarily implies $r < 1 - \frac{1}{p}; r < R$, while as $\alpha > 0, R > -\frac{1}{p}$, provided δ_1 is small enough. Then (2) also holds. So we may apply Theorem 1.2 with the given r, R.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that is bounded and Riemann integrable in each finite interval and satisfies (8). Let P be a polynomial. For n larger than the degree of P, we have from Theorem 1.2,

$$\begin{aligned} &\| \left(P - L_n\left[f\right]\right) W h^{-\Delta} \|_{L_p(\mathbb{R})} = \| L_n\left[P - f\right] W h^r \|_{L_p(\mathbb{R})} \\ &\leq C\left(\sum_{j=1}^{n} \lambda_{jn} W^{-2}\left(x_{jn}\right) \left(\left|f - P\right| W h^R\right)^p \left(x_{jn}\right)\right)^{1/p}, \end{aligned}$$

with C independent of n, P and f. Let $\varepsilon > 0$. Because of the rapid decay of W relative to P, and because of (8), we may choose $A > 0$ so large that

$$|(f - P)W h^\alpha|\left(x\right) \leq \varepsilon, |x| > A.$$

Then in view of (33),

$$\begin{aligned} &\sum_{|x_{jn}|>A} \lambda_{jn} W^{-2}\left(x_{jn}\right) \left(\left|f - P\right| W h^R\right)^p \left(x_{jn}\right) \\ &\leq C\varepsilon^p \sum_{|x_{jn}|>A} \left(x_{j-1,n} - x_{jn}\right) h^{(R-\alpha)p}\left(x_{jn}\right). \end{aligned}$$

It is crucial here that C is independent of ε, A. Using our choice of R, we may continue this last estimate as

$$\leq C\varepsilon^p \int_{-\infty}^{\infty} h^{-1-\delta_1 p}\left(x\right) dx, \tag{67}$$

where C is independent of n, A, ε. Next, $(|f-P|\, Wh^{\alpha})^p$ is Riemann integrable over $[-A, A]$, so we have by (33), and our spacing (36) of the zeros,

$$\begin{aligned} & \sum_{|x_{jn}| \le A} \lambda_{jn} W^{-2}(x_{jn}) \left(|f-P|\, Wh^{-\Delta}\right)^p (x_{jn}) \\ \le \quad & C \sum_{|x_{jn}| \le A} (x_{j-1,n} - x_{jn}) \left(|f-P|\, Wh^{-\Delta}\right)^p (x_{jn}) \\ \to \quad & C \int_{-A}^{A} \left|(f-P) Wh^{-\Delta}\right|^p, n \to \infty. \qquad (68) \end{aligned}$$

We emphasise that $C \neq C(A, n, P, f)$. This last inequality and (67), (68) give

$$\limsup_{n\to\infty} \| (P - L_n[f])\, Wh^{-\Delta} \|_{L_p(\mathbb{R})} \le C_1 \| (f-P)\, Wh^{-\Delta} \|_{L_p[-A,A]} + C_1 \varepsilon,$$

where C_1 is independent of A, ε, P, f. Letting $A \to \infty$ and then $\varepsilon \to 0+$ gives

$$\limsup_{n\to\infty} \| (P - L_n[f])\, Wh^{-\Delta} \|_{L_p(\mathbb{R})} \le C_1 \| (f-P)\, Wh^{-\Delta} \|_{L_p(\mathbb{R})},$$

with C_1 independent of n, P, f. The triangle inequality then gives

$$\limsup_{n\to\infty} \| (f - L_n[f])\, Wh^{-\Delta} \|_{L_p(\mathbb{R})} \le C_1 \| (f-P)\, Wh^{-\Delta} \|_{L_p(\mathbb{R})} .$$

Here our conditions on f and W guarantee that we may choose a polynomial P for which the last right-hand side is arbitrarily small: for

$$\left((f-P)\, Wh^{-\Delta}\right)(x) = o\left(h^{-(\Delta+\alpha)}(x)\right), |x| \to \infty,$$

and (9) guarantees that $\Delta + \alpha > \frac{1}{p}$. $\square$

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