UNIVERSALITY LIMITS INVOLVING ORTHOGONAL POLYNOMIALS ON AN ARC OF THE UNIT CIRCLE

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Abstract. We establish universality limits for measures on a subarc of the unit circle. Assume that μ is a regular measure on such an arc, in the sense of Stahl, Totik, and Ullmann, and is absolutely continuous in an open arc containing some point $z_0 = e^{i\theta_0}$. Assume, moreover, that μ' is positive and continuous at z_0. Then universality for μ holds at z_0, in the sense that the reproducing kernel $K_n(z,t)$ for μ satisfies

$$\lim_{n\to\infty} \frac{K_n\left(z_0 \exp\left(\frac{2\pi i s}{n}\right), z_0 \exp\left(\frac{2\pi i \bar{t}}{n}\right)\right)}{K_n(z_0, z_0)} = e^{i\pi(s-t)} S\left((s-t)\, T(\theta_0)\right),$$

uniformly for s, t in compact subsets of the plane, where $S(z) = \frac{\sin \pi z}{\pi z}$ is the sinc kernel, and $T/2\pi$ is the equilibrium density for the arc.

1. Introduction and Results[1]

In the theory of random Hermitian matrices, arising from scattering theory in physics, universality limits play an important role. They can be reduced to scaling limits for reproducing kernels involving orthogonal polynomials, which makes the analysis feasible. This has been completed in a very wide array of settings [2], [3], [4], [8], [9], [10], [12], [13], [14], [15], [20], [23].

In a recent paper, Eli Levin and the first author established universality limits for measures on the unit circle [9]. In this paper, we consider instead subarcs of the unit circle. Our analysis depends heavily on the work of Leonid Golinskii, who provided a detailed exposition for Szegő-Bernstein theory for such arcs, and deduced asymptotics of orthogonal polynomials and their Christoffel functions [6], [7]. In turn, Golinskii's work depended heavily on work of Akhiezer [1].

Let $\alpha \in (0, \pi)$ and let our arc be

$$\Delta_\alpha = \left\{ e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha] \right\}.$$

Let μ be a finite positive Borel measure on Δ_α (or equivalently on $[\alpha, 2\pi - \alpha]$) with infinitely many points in its support. Then we may define orthonormal

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polynomials

$$\phi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} \phi_n(z) \overline{\phi_m(z)} d\mu(\theta) = \delta_{mn},$$

where $z = e^{i\theta}$.

We shall usually assume that μ is *regular* in the sense of Stahl, Totik and Ullmann [21], so that

$$\lim_{n\to\infty} \kappa_n^{1/n} = \frac{1}{\cos\frac{\alpha}{2}}. \tag{1.1}$$

Here $\cos\frac{\alpha}{2}$ is the logarithmic capacity of Δ_α. A simple sufficient condition for regularity is that $\mu' > 0$ a.e. in $[\alpha, 2\pi - \alpha)$, but there are pure jump and pure singularly continuous measures that are regular.

The nth reproducing kernel for μ is

$$K_n(z, u) = \sum_{j=0}^{n-1} \phi_j(z) \overline{\phi_j(u)}. \tag{1.2}$$

To state our results, we need some auxiliary functions: for $\theta \in [\alpha, 2\pi - \alpha]$, we define $\lambda(\theta) \in [0, \pi]$ by the equation

$$\cos\lambda(\theta) = \frac{\cos\frac{\theta}{2}}{\cos\frac{\alpha}{2}}. \tag{1.3}$$

Observe that λ is a strictly increasing continuous function of θ, that maps $[\alpha, 2\pi - \alpha]$ onto $[0, \pi]$. We also let

$$T(\theta) = \frac{\sin\frac{\theta}{2}}{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{\theta}{2}}}. \tag{1.4}$$

$T(\theta)/(2\pi)$ is the density of the equilibrium measure for Δ_α in the sense of potential theory. Finally, we need the sinc kernel:

$$S(z) = \frac{\sin\pi z}{\pi z}. \tag{1.5}$$

Our main result is:

Theorem 1.1
Let $\alpha \in (0, \pi)$, and let μ be a finite positive Borel measure on $[\alpha, 2\pi - \alpha]$ that is regular. Let $J \subset (\alpha, 2\pi - \alpha)$ be compact, and be such that μ is absolutely continuous in an open set containing J. Assume moreover, that μ' is positive and continuous at each point of J. Then uniformly for $\theta_0 \in J$ and s, t in compact subsets of the complex plane $\mathbb{C}$, we have

$$\lim_{m\to\infty} \frac{K_m\left(e^{i\left(\theta_0 + \frac{2\pi s}{m}\right)}, e^{i\left(\theta_0 + \frac{2\pi \bar{t}}{m}\right)}\right)}{K_m\left(e^{i\theta_0}, e^{i\theta_0}\right)} = e^{i\pi(s-t)} S\left((s-t)\, T(\theta_0)\right).$$

(1.6)

Remarks

(a) In the case $\alpha \to 0+$, we see that $T(\theta) \to 1$ and the right-hand side of (1.6) reduces to $e^{i\pi(s-t)} S(s-t)$, which is the result of Levin and Lubinsky [9].

(b) If J consists of just a single point θ_0, then the hypothesis is that μ is absolutely continuous in some neighborhood $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ of θ_0, while $\mu'(\theta_0) > 0$ and μ' is continuous at θ_0.

(c) As in [9], [13], the main idea in this paper is a localization principle, and a comparison inequality.

(d) As in [11], this limit has implications for the spacing of the zeros of the reproducing kernel. It is known that for $|a| = 1$, the zeros of $K_n(\cdot, a)$ lie on the unit circle [18, Thm. 2.2.12, p. 129].

Corollary 1.2

Assume the hypotheses of Theorem 1.1, and let $\theta_0 \in J$. For $k \geq 1$, let $e^{i\theta_{kn}}$ denote the kth closest zero of $K_n\left(e^{i\theta_0}, \cdot\right)$ to θ_0, with $\theta_{kn} > \theta_0$, while $e^{i\theta_{-kn}}$ denotes the kth closest zero to θ_0, with $\theta_{-kn} < \theta_0$. Then for large enough n, $\theta_{\pm kn}$ exists, the zero $e^{i\theta_{\pm kn}}$ is simple, and

$$\lim_{n\to\infty} n\left(\theta_{\pm kn} - \theta_0\right) = \frac{\pm 2\pi k}{T(\theta_0)}. \tag{1.7}$$

This result should be compared to the 'clock theorems' in [11], [19], the estimates in [16], and earlier work of Freud [5, p. 266].

We can also deduce asymptotics for derivatives of the reproducing kernel: we let

$$K_n^{(j,k)}(z,z) = \sum_{m=0}^{n-1} \phi_m^{(j)}(z)\, \overline{\phi_m^{(k)}(z)}.$$

Corollary 1.3

Assume the hypotheses of Theorem 1.1, and let $\theta_0 \in J, z_0 = e^{i\theta_0}$. For $j, k \geq 0$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{z_0^{j-k}}{n^{j+k}} \frac{K_n^{(j,k)}(z_0,z_0)}{K_n(z_0,z_0)} \\ (1.8) = \quad &\frac{1}{T(\theta_0)(j+k+1)} \left[\left(\frac{1+T(\theta_0)}{2}\right)^{j+k+1} - \left(\frac{1-T(\theta_0)}{2}\right)^{j+k+1}\right]. \end{aligned}$$

In the sequel $C, C_1, C_2, ...$ denote constants independent of n, z, u, θ, s, t. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. $[x]$ denotes the greatest integer $\leq x$. For sequences $\{c_n\}, \{d_n\}$ of non-zero real numbers, we write $c_n \sim d_n$ if there exist positive constants C_1, C_2 independent of n such that

$$C_1 \leq c_n/d_n \leq C_2.$$

Given measures μ^*, $\mu^{\#}$, we use $K_n^*, K_n^{\#}$ to denote their respective reproducing kernels. Similarly superscripts $*, \#$ are used to distinguish other quantities associated with them.

We denote the nth Christoffel function for the measure μ by

(1.9)
$$\Omega_n\left(e^{i\theta}\right) = 1/K_n\left(e^{i\theta}, e^{i\theta}\right) = \min_{\deg(P)\leq n-1}\left(\frac{1}{2\pi}\int_\alpha^{2\pi-\alpha}\left|P\left(e^{it}\right)\right|^2 d\mu\left(t\right)\right)/\left|P\left(e^{i\theta}\right)\right|^2.$$

The paper is organised as follows. In Section 2, we prove some of the results for a special weight considered by Leonid Golinskii in [6]. In Section 3, we prove Theorem 1.1 and Corollaries 1.2 and 1.3.

2. A Special Weight on the Arc

In this section, we consider the measure $d\mu\left(\theta\right) = W\left(\theta\right) d\theta$, where

$$W\left(\theta\right) = \frac{\sin\frac{\alpha}{2}}{2\sin\frac{\theta}{2}\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{\theta}{2}}}, \theta \in \left[\alpha, 2\pi - \alpha\right]. \tag{2.1}$$

This is the special case $\Omega = 1$ of the weights considered by Leonid Golinskii [6, p. 237]. Golinskii [6, p. 244] provided a detailed derivation of explicit formulae for the corresponding orthonormal polynomials $\{\phi_n\}$: for $n \geq 1$,

$$\phi_n\left(e^{i\theta}\right) = A\left(\theta\right) e^{in\left(\frac{\theta}{2} - \lambda(\theta)\right)} + B\left(\theta\right) e^{in\left(\frac{\theta}{2} + \lambda(\theta)\right)} \tag{2.2}$$

where $\lambda\left(\theta\right) \in [0, \pi]$ is determined by (1.3). Moreover,

$$A\left(\theta\right) = \left\{e^{-i\lambda(\theta)}\sqrt{1 - \sin\frac{\alpha}{2}} - e^{-i\frac{\theta}{2}}\sqrt{1 + \sin\frac{\alpha}{2}}\right\}\frac{g_-\left(e^{i\theta}\right)}{2i\sin\left(\frac{\theta}{2}\right)}; \tag{2.3}$$

$$B\left(\theta\right) = \left\{e^{i\lambda(\theta)}\sqrt{1 - \sin\frac{\alpha}{2}} - e^{-i\frac{\theta}{2}}\sqrt{1 + \sin\frac{\alpha}{2}}\right\}\frac{g_+\left(e^{i\theta}\right)}{2i\sin\left(\frac{\theta}{2}\right)}; \tag{2.4}$$

and all we shall need to know about $g_\pm$ is that [6, p. 241, (35), (38)] they are continuous and

$$\left|g_\pm\left(e^{i\theta}\right)\right| = \sqrt{\frac{2\sin^2\left(\theta/2\right)}{\sin\left(\alpha/2\right)}}. \tag{2.5}$$

Note that in [6], we chose $\Omega = 1$ and $\rho_\Omega\left(e^{i\theta}\right) = \frac{\sin(\alpha/2)}{2\sin^2(\theta/2)}$. We shall assume that (2.2) holds even for $n = 0$ as this makes no difference to our asymptotics. We prove

Theorem 2.1
Let $d\mu(\theta) = W(\theta)\,d\theta$ be given by (2.1), and let K_m denote its mth reproducing kernel. Then uniformly for s, t in compact subsets of the complex plane,

$$\lim_{m\to\infty} \frac{K_m\left(e^{i\left(\theta_0+\frac{2\pi s}{m}\right)}, e^{i\left(\theta_0+\frac{2\pi \bar{t}}{m}\right)}\right)}{K_m\left(e^{i\theta_0}, e^{i\theta_0}\right)} = e^{i\pi(s-t)} S\left((s-t)\,T(\theta_0)\right).$$

We begin with real s, t:

Lemma 2.2
Let $\theta_0 \in (\alpha, 2\pi - \alpha)$, $s, t \in \mathbb{R}$, and for $m \geq 1$, define $\theta = \theta(m), \phi = \phi(m)$ by

$$\theta = \theta_0 + \frac{2\pi s}{m}; \phi = \theta_0 + \frac{2\pi t}{m}. \tag{2.6}$$

(a) Uniformly for s, t in compact subsets of the real line,

$$\lim_{m\to\infty} m\left(\lambda(\theta) - \lambda(\phi)\right) = T(\theta_0)\,\pi(s-t). \tag{2.7}$$

(b) Uniformly for s, t in compact subsets of the real line,

$$\begin{aligned} & \lim_{m\to\infty} \frac{1}{m} K_m\left(e^{i\theta}, e^{i\phi}\right) \\ = \; & \left|A(\theta_0)\right|^2 e^{i\pi\left(\frac{s-t}{2}\right)(1-T(\theta_0))} S\left(\left(\frac{s-t}{2}\right)(1-T(\theta_0))\right) \\ & + \left|B(\theta_0)\right|^2 e^{i\pi\left(\frac{s-t}{2}\right)(1+T(\theta_0))} S\left(\left(\frac{s-t}{2}\right)(1+T(\theta_0))\right). \end{aligned} \tag{2.8}$$

(c)

$$\left|A(\theta_0)\right|^2 = \frac{\sin^2\frac{\theta_0}{2}}{\sin\frac{\alpha}{2}}\left(1 - T(\theta_0)^{-1}\right); \tag{2.9}$$

$$\left|B(\theta_0)\right|^2 = \frac{\sin^2\frac{\theta_0}{2}}{\sin\frac{\alpha}{2}}\left(1 + T(\theta_0)^{-1}\right). \tag{2.10}$$

(d) Uniformly for s in compact subsets of the real line,

$$\lim_{m\to\infty} \frac{1}{m} K_m\left(e^{i\theta}, e^{i\theta}\right) = \frac{2\sin^2\frac{\theta_0}{2}}{\sin\frac{\alpha}{2}}. \tag{2.11}$$

and

$$\lim_{m\to\infty} \frac{1}{m} K_m\left(e^{i\theta}, e^{i\theta}\right) W(\theta) = T(\theta_0). \tag{2.12}$$

(e) Uniformly for θ_0 in compact subsets of $(\alpha, 2\pi - \alpha)$, and s, t in compact subsets of $\mathbb{C}$,

$$\lim_{m\to\infty} \frac{K_m\left(e^{i\theta}, e^{i\phi}\right)}{K_m\left(e^{i\theta_0}, e^{i\theta_0}\right)} = e^{i\pi(s-t)} S\left((s-t)\,T(\theta_0)\right). \tag{2.13}$$

Proof
(a) Now by the definition (1.3) of λ,

$$\left(\cos\frac{\alpha}{2}\right)\left[\cos\lambda\left(\theta\right)-\cos\lambda\left(\phi\right)\right]=\cos\frac{\theta}{2}-\cos\frac{\phi}{2}.$$

Hence

$$\begin{aligned}&\left(\cos\frac{\alpha}{2}\right)\sin\left(\frac{\lambda\left(\theta\right)-\lambda\left(\phi\right)}{2}\right)\sin\left(\frac{\lambda\left(\theta\right)+\lambda\left(\phi\right)}{2}\right)\\ =\ &\sin\left(\frac{\theta-\phi}{4}\right)\sin\left(\frac{\theta+\phi}{4}\right).\end{aligned}$$

Since λ is continuous in $(\alpha, 2\pi-\alpha)$, and $\theta,\phi\to\theta_0$ as $m\to\infty$, we deduce that

$$\begin{aligned}&\left(\cos\frac{\alpha}{2}\right)\frac{m\left[\lambda\left(\theta\right)-\lambda\left(\phi\right)\right]}{2}\sin\left(\lambda\left(\theta_0\right)\right)\\ =\ &m\left(\frac{\theta-\phi}{4}\right)\sin\frac{\theta_0}{2}+o\left(1\right)\\ =\ &\pi\frac{s-t}{2}\sin\frac{\theta_0}{2}+o\left(1\right).\end{aligned}\tag{2.14}$$

Finally,

$$\begin{aligned}&\left(\cos\frac{\alpha}{2}\right)\sin\left(\lambda\left(\theta_0\right)\right)\\ =\ &\left(\cos\frac{\alpha}{2}\right)\sqrt{1-\cos^2\lambda\left(\theta_0\right)}\\ =\ &\sqrt{\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta_0}{2}}.\end{aligned}$$

Substituting this into (2.14) gives the result.
(b) From (2.2),

$$\begin{aligned}&\frac{1}{m}K_m\left(e^{i\theta},e^{i\phi}\right)\\ =\ &\frac{1}{m}\sum_{n=0}^{m-1}\left[A\left(\theta\right)e^{in\left(\frac{\theta}{2}-\lambda(\theta)\right)}+B\left(\theta\right)e^{in\left(\frac{\theta}{2}+\lambda(\theta)\right)}\right]\\ &\times\left[\overline{A\left(\phi\right)}e^{-in\left(\frac{\phi}{2}-\lambda(\phi)\right)}+\overline{B\left(\phi\right)}e^{-in\left(\frac{\phi}{2}+\lambda(\phi)\right)}\right]\\ =\ &\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,\end{aligned}\tag{2.15}$$

where these four sums are specified below: firstly by continuity of A,

$$\begin{aligned}
\Sigma_1 &= \frac{1}{m}\sum_{n=0}^{m-1} A(\theta)\overline{A(\phi)} e^{in\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)} \\
&= |A(\theta_0)|^2 \frac{1}{m}\sum_{n=0}^{m-1} e^{in\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)} + o(1) \\
&= |A(\theta_0)|^2 \frac{1}{m}\frac{1-e^{im\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)}}{1-e^{i\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)}} + o(1) \\
&= |A(\theta_0)|^2 e^{i\frac{m-1}{2}\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)} \frac{S\left(\frac{m}{2\pi}\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)\right)}{S\left(\frac{1}{2\pi}\left(\frac{\theta-\phi}{2}-\lambda(\theta)+\lambda(\phi)\right)\right)} + o(1) \\
&= |A(\theta_0)|^2 e^{i\pi\left(\frac{s-t}{2}\right)(1-T(\theta_0))} S\left(\frac{s-t}{2}(1-T(\theta_0))\right) + o(1),
\end{aligned}$$

by (2.7) and the continuity of S at 0, where $S(0)=1$. Similarly,

$$\begin{aligned}
\Sigma_4 &= \frac{1}{m}\sum_{n=0}^{m-1} B(\theta)\overline{B(\phi)} e^{in\left(\frac{\theta-\phi}{2}+\lambda(\theta)-\lambda(\phi)\right)} \\
&= |B(\theta_0)|^2 e^{i\pi\left(\frac{s-t}{2}\right)(1+T(\theta_0))} S\left(\frac{s-t}{2}(1+T(\theta_0))\right) + o(1).
\end{aligned}$$

Next,

$$\begin{aligned}
\Sigma_3 &= \frac{1}{m}\sum_{n=0}^{m-1} A(\theta)\overline{B(\phi)} e^{in\left(\frac{\theta-\phi}{2}-\lambda(\theta)-\lambda(\phi)\right)} \\
&= A(\theta_0)\overline{B(\theta_0)} \frac{1}{m}\sum_{n=0}^{m-1} e^{in\left(\frac{\theta-\phi}{2}-\lambda(\theta)-\lambda(\phi)\right)} + o(1) \\
&= A(\theta_0)\overline{B(\theta_0)} \frac{1}{m}\frac{1-e^{im\left(\frac{\theta-\phi}{2}-\lambda(\theta)-\lambda(\phi)\right)}}{1-e^{i\left(\frac{\theta-\phi}{2}-\lambda(\theta)-\lambda(\phi)\right)}} + o(1).
\end{aligned}$$

Here as $m\to\infty$,

$$\frac{\theta-\phi}{2}-\lambda(\theta)-\lambda(\phi) = -2\lambda(\theta_0)+o(1)$$

and $-2\lambda(\theta_0)\in(-2\pi,0)$, so the denominator $1-e^{i\left(\frac{\theta-\phi}{2}-\lambda(\theta)-\lambda(\phi)\right)}$ in Σ_3 is bounded away from 0. Thus

$$\Sigma_3 = o(1),$$

and similarly,

$$\Sigma_4 = o(1).$$

Combining the above asymptotics for Σ_j, $j=1,2,3,4$, gives the result.
(c) Now

$$\begin{aligned}
& \left| e^{-i\lambda(\theta_0)}\sqrt{1-\sin\frac{\alpha}{2}} - e^{-i\frac{\theta_0}{2}}\sqrt{1+\sin\frac{\alpha}{2}}\right|^2 \\
= & \left(1-\sin\frac{\alpha}{2}\right) + (-2)\cos\frac{\alpha}{2}\cos\left(\frac{\theta_0}{2}-\lambda(\theta_0)\right) + \left(1+\sin\frac{\alpha}{2}\right) \\
= & \ 2\left(1-\cos\frac{\alpha}{2}\left[\cos\frac{\theta_0}{2}\cos\lambda(\theta_0)+\sin\frac{\theta_0}{2}\sin\lambda(\theta_0)\right]\right) \\
= & \ 2\left(1-\cos^2\frac{\theta_0}{2}-\sin\frac{\theta_0}{2}\sqrt{\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta_0}{2}}\right),
\end{aligned}$$

by definition (1.3) of $\lambda(\theta_0)$. We continue this as

$$= 2\sin^2\frac{\theta_0}{2}\left(1-T(\theta_0)^{-1}\right).$$

Then (2.9) follows from (2.3) and (2.5). (2.10) is similar.
(d) This follows by setting $\phi=\theta$ in (2.8) and using (2.9) and (2.10).
(e) From (2.8) to (2.11),

$$\begin{aligned}
& \lim_{m\to\infty}\frac{K_m\left(e^{i\theta},e^{i\phi}\right)}{K_m\left(e^{i\theta_0},e^{i\theta_0}\right)} \\
= & \ \frac{1}{2}\left(1-T(\theta_0)^{-1}\right)e^{i\pi\left(\frac{s-t}{2}\right)(1-T(\theta_0))}S\left(\frac{s-t}{2}(1-T(\theta_0))\right) \\
& +\frac{1}{2}\left(1+T(\theta_0)^{-1}\right)e^{i\pi\left(\frac{s-t}{2}\right)(1+T(\theta_0))}S\left(\frac{s-t}{2}(1+T(\theta_0))\right).
\end{aligned}$$

This can be continued as

$$\begin{aligned}
& \frac{1}{2T(\theta_0)}\frac{e^{i\pi\left(\frac{s-t}{2}\right)}}{\pi\left(\frac{s-t}{2}\right)}\left[\begin{array}{c} -e^{-i\pi\frac{s-t}{2}T(\theta_0)}\sin\left(\pi\left(\frac{s-t}{2}\right)(1-T(\theta_0))\right) \\ +e^{i\pi\frac{s-t}{2}T(\theta_0)}\sin\left(\pi\left(\frac{s-t}{2}\right)(1+T(\theta_0))\right)\end{array}\right] \\
= & \ \frac{1}{2T(\theta_0)}\frac{e^{i\pi\left(\frac{s-t}{2}\right)}}{\pi\left(\frac{s-t}{2}\right)}\left[\begin{array}{c} \sin\left(\pi\frac{s-t}{2}\right)\cos\left(\pi\frac{s-t}{2}T(\theta_0)\right)\left\{e^{i\pi\frac{s-t}{2}T(\theta_0)}-e^{-i\pi\frac{s-t}{2}T(\theta_0)}\right\} \\ +\cos\left(\pi\frac{s-t}{2}\right)\sin\left(\pi\frac{s-t}{2}T(\theta_0)\right)\left\{e^{i\pi\frac{s-t}{2}T(\theta_0)}+e^{-i\pi\frac{s-t}{2}T(\theta_0)}\right\}\end{array}\right] \\
= & \ \frac{1}{T(\theta_0)}\frac{e^{i\pi\left(\frac{s-t}{2}\right)}}{\pi\left(\frac{s-t}{2}\right)}\cos\left(\pi\frac{s-t}{2}T(\theta_0)\right)\sin\left(\pi\frac{s-t}{2}T(\theta_0)\right)\left[\cos\left(\pi\frac{s-t}{2}\right)+i\sin\left(\pi\frac{s-t}{2}\right)\right] \\
= & \ \frac{1}{T(\theta_0)}\frac{e^{i\pi(s-t)}}{\pi(s-t)}\sin\left(\pi(s-t)T(\theta_0)\right) \\
= & \ e^{i\pi(s-t)}S\left((s-t)T(\theta_0)\right).
\end{aligned}$$

■

Proof of Theorem 2.1

We already have the result for real s,t. For $m \geq 1$, let

$$f_m(s,t) = \frac{K_m\left(e^{i\left(\theta_0+\frac{2\pi s}{m}\right)}, e^{i\left(\theta_0+\frac{2\pi \bar{t}}{m}\right)}\right)}{K_m\left(e^{i\theta_0}, e^{i\theta_0}\right)}. \tag{2.16}$$

This is a polynomial in $e^{i2\pi s/m}$ and $e^{-i2\pi t/m}$. We shall show that $\{f_m\}$ is uniformly bounded for s,t in compact subsets of $\mathbb{C}$: that is, given $r > 0$, there exists C such that

$$\sup_{m\geq 1} \sup_{|s|,|t|\leq r} |f_m(s,t)| \leq C. \tag{2.17}$$

Thus $\{f_m\}$ is a normal family. In as much as the limit (2.13) holds for real s,t, and the right-hand side of (2.13) is an entire function of s,t, it then follows from the principle of analytic continuation that the limit holds uniformly for s,t in compact subsets of the plane.

To prove (2.17), along standard lines, we note first from (2.15) that for $\theta, \phi \in (\alpha, 2\pi - \alpha)$,

$$\begin{aligned} & \frac{1}{m}\left|K_m\left(e^{i\theta}, e^{i\phi}\right)\right| \\ \leq \quad & (|A(\theta)| + |B(\theta)|) \times (|A(\phi)| + |B(\phi)|). \end{aligned}$$

Let J be a compact subinterval of $(\alpha, 2\pi - \alpha)$, and $\Delta_J = \left\{e^{i\theta} : \theta \in J\right\}$. The last inequality, and continuity of A, B on J shows that

$$\sup_{z,u\in\Delta_J} \frac{1}{m}|K_m(z,u)| \leq C.$$

Let G denote the Green's function for $\mathbb{C}\backslash J$ with pole at ∞. From the Bernstein-Walsh inequality [17, p. 156], it follows that for all $z,u \in \mathbb{C}$,

$$\frac{1}{m}|K_m(z,u)| \leq Ce^{m(G(z)+G(u))}.$$

Moreover, $G(z) = 0$ for $z \in \Delta_J$, and because Δ_J is a "smooth" arc,

$$\left|G\left(ze^{iu}\right)\right| \leq C_1|u|,$$

for $z \in J_1$ and $|u| \leq 1$, where J_1 is any compact subinterval of the interior of J. It follows that for $m \geq m_0(r)$,

$$\frac{1}{m}\left|K_m\left(e^{i\left(\theta_0+\frac{2\pi s}{m}\right)}, e^{i\left(\theta_0+\frac{2\pi \bar{t}}{m}\right)}\right)\right| \leq C_1 e^{C_2(|s|+|t|)} \leq C_1 e^{2C_2 r}.$$

See Lemmas 6.1 and 6.2 in [9, pp. 556-557] for more details. Finally, by (2.11), $K_m\left(e^{i\theta_0}, e^{i\theta_0}\right) \geq Cm$. So we have (2.17) and the result. ■

3. Proof of Theorem 1.1

We begin with asymptotics for Christoffel functions:

Lemma 3.1
Let μ be a regular measure on $[\alpha, 2\pi-\alpha]$. Assume that μ is absolutely continuous in an open set containing a compact set $J \subset (\alpha, 2\pi-\alpha)$, and at each point of J, μ' is positive and continuous. Let $A>0$. Then uniformly for $a \in [-A, A]$, and $\theta \in J$,

$$\lim_{n\to\infty} n\Omega_n\left(\exp\left(i\left(\theta+\frac{a}{n}\right)\right)\right)=\mu'(\theta)/T(\theta). \tag{3.1}$$

Moreover, uniformly for $n \geq n_0(A), \theta \in J$, and $a \in [-A, A]$,

$$\Omega_n\left(\exp\left(i\left(\theta+\frac{a}{n}\right)\right)\right)\sim\frac{1}{n}. \tag{3.2}$$

Remarks
(a) We emphasize that we are assuming that μ' is continuous in J when regarded as a function defined on $[\alpha, 2\pi-\alpha]$.
(b) Asymptotics for Christoffel functions associated with special measures on the arc were established by Golinskii [7]. Totik [23], [24] established asymptotics a.e. on more general arcs and curves, that include (3.1) in the case $a=0$.
(c) It follows from Totik's results and that above, that $\frac{1}{2\pi}T(\theta)$ is the density of the equilibrium measure (in the sense of potential theory) for the arc.
Proof
We already know this result for the special weight $W(\theta)\,d\theta$ of the previous section. The extension to the general case is exactly the same as for the whole unit circle in [9, pp. 549-551, proof of Theorem 3.1], so we omit the details. ■

Next, we need a comparison inequality:

Lemma 3.2
Let $r>0$ and μ, μ^* be measures on $[\alpha, 2\pi-\alpha]$, with $\mu \leq r\mu^*$. Then for all real θ, ϕ,

$$\begin{aligned}&\left|\left(K_n-\frac{1}{r}K_n^*\right)\left(e^{i\theta},e^{i\phi}\right)\right|/K_n\left(e^{i\theta},e^{i\theta}\right)\\ \leq\ &\left(\frac{K_n\left(e^{i\phi},e^{i\phi}\right)}{K_n\left(e^{i\theta},e^{i\theta}\right)}\right)^{1/2}\left[1-\frac{K_n^*\left(e^{i\theta},e^{i\theta}\right)}{rK_n\left(e^{i\theta},e^{i\theta}\right)}\right]^{1/2}.\end{aligned} \tag{3.3}$$

Proof
Let $\mu^{\#}=r\mu^*$, so that $\mu \leq \mu^{\#}$. In [9, Theorem 4.1, page 552-3], we showed

that

$$\begin{aligned}&\left|\left(K_n - K_n^{\#}\right)\left(e^{i\theta}, e^{i\phi}\right)\right| / K_n\left(e^{i\theta}, e^{i\theta}\right)\\ \leq\ &\left(\frac{K_n\left(e^{i\phi}, e^{i\phi}\right)}{K_n\left(e^{i\theta}, e^{i\theta}\right)}\right)^{1/2}\left[1 - \frac{K_n^{\#}\left(e^{i\theta}, e^{i\theta}\right)}{K_n\left(e^{i\theta}, e^{i\theta}\right)}\right]^{1/2}.\end{aligned}$$

It is easily seen from the definition of the orthonormal polynomials and reproducing kernel that

$$K_n^{\#}(z, w) = \frac{1}{r} K_n^{*}(z, w).$$

Then the result follows. ■

Proof of Theorem 1.1

Let $\varepsilon \in (0, 1)$ and $\theta_0 \in J$. By continuity of μ' and W at θ_0, we can choose $\delta > 0$ such that for $|\theta - \theta_0| \leq \delta$

$$\begin{aligned}1 - \varepsilon &\leq \frac{\mu'(\theta)}{\mu'(\theta_0)} \leq (1 - \varepsilon)^{-1};\\ 1 - \varepsilon &\leq \frac{W(\theta)}{W(\theta_0)} \leq (1 - \varepsilon)^{-1}.\end{aligned}$$

Let

$$c = (1 - \varepsilon)^{-2} \frac{\mu'(\theta_0)}{W(\theta_0)}$$

and define two new measures μ^* and $\mu^{\#}$ on $[\alpha, 2\pi - \alpha]$ by

$$d\mu^{\#}(\theta) = W(\theta)\, d\theta;$$

$$\begin{aligned}d\mu^*(\theta) &= W(\theta)\, d\theta \text{ in } |\theta - \theta_0| < \delta;\\ d\mu^*(\theta) &= W(\theta)\, d\theta + \frac{1}{c} d\mu(\theta) \text{ in } [\alpha, 2\pi - \alpha] \setminus (\theta_0 - \delta, \theta_0 + \delta).\end{aligned}$$

Then $\mu^* \geq \mu^{\#}$ and $c\mu^* \geq \mu$ in $[\alpha, 2\pi - \alpha]$. Moreover, by our asymptotics for Christoffel functions in Lemma 3.1, uniformly for s in a bounded real interval,

$$\lim_{n\to\infty} \frac{K_n^*\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right)}{K_n\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right)} = \frac{\mu'(\theta_0)}{W(\theta_0)} = c(1 - \varepsilon)^2;$$

$$\lim_{n\to\infty} \frac{K_n^*\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right)}{K_n^{\#}\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right)} = 1.$$

Moreover, uniformly for s in a bounded interval,

$$\begin{aligned}K_n\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right) &\sim K_n^{\#}\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right)\\ &\sim K_n^{*}\left(e^{i(\theta_0 + 2\pi s/n)}, e^{i(\theta_0 + 2\pi s/n)}\right) \sim n.\end{aligned}$$

Then Lemma 3.2 applied to $\mu^{\#}$ and μ^* gives, with $r = 1$, and $\theta = \theta_0 + 2\pi s/n$, $\phi = \theta_0 + 2\pi t/n$,

$$\begin{aligned}
&\left|\left(K_n^{\#} - K_n^*\right)\left(e^{i\theta}, e^{i\phi}\right)\right| / K_n^{\#}\left(e^{i\theta}, e^{i\theta}\right) \\
\leq\ & \left(\frac{K_n^{\#}\left(e^{i\phi}, e^{i\phi}\right)}{K_n^{\#}\left(e^{i\theta}, e^{i\theta}\right)}\right)^{1/2}\left[1 - \frac{K_n^*\left(e^{i\theta}, e^{i\theta}\right)}{K_n^{\#}\left(e^{i\theta}, e^{i\theta}\right)}\right]^{1/2} \\
\rightarrow\ & 0 \text{ as } n \rightarrow \infty;
\end{aligned}$$

and Lemma 3.2 applied to μ and μ^* with $r = c$ gives,

$$\begin{aligned}
&\left|\left(K_n - \frac{1}{c}K_n^*\right)\left(e^{i\theta}, e^{i\phi}\right)\right| / K_n\left(e^{i\theta}, e^{i\theta}\right) \\
\leq\ & \left(\frac{K_n\left(e^{i\phi}, e^{i\phi}\right)}{K_n\left(e^{i\theta}, e^{i\theta}\right)}\right)^{1/2}\left[1 - \frac{K_n^*\left(e^{i\theta}, e^{i\theta}\right)}{cK_n\left(e^{i\theta}, e^{i\theta}\right)}\right]^{1/2} \\
\leq\ & C\left[1 - (1-\varepsilon)^2\right]^{1/2} \leq C\left[3\varepsilon\right]^{1/2}.
\end{aligned}$$

Here C is independent of $s, t, a, b, n, \varepsilon$. Combining these last two inequalities gives, for large enough n,

$$\left|\left(cK_n - K_n^{\#}\right)\left(e^{i\theta}, e^{i\phi}\right)\right| / n \leq C\varepsilon^{1/2},$$

and recalling the definition of c, and the fact that $K_n = O(n)$, also

$$\left|\left(\frac{\mu'(\theta_0)}{W(\theta_0)}K_n - K_n^{\#}\right)\left(e^{i\theta}, e^{i\phi}\right)\right| / n \leq C\varepsilon^{1/2}.$$

Here the left-hand side is independent of ε, so we deduce

$$\limsup_{n\to\infty}\left|\left(\frac{\mu'(\theta_0)}{W(\theta_0)}K_n - K_n^{\#}\right)\left(e^{i(\theta_0+2\pi s/n)}, e^{i(\theta_0+2\pi t/n)}\right)\right| / n = 0.$$

Using Lemma 3.1 on $K_n\left(e^{i\theta_0}, e^{i\theta_0}\right)$ and $K_n^{\#}\left(e^{i\theta_0}, e^{i\theta_0}\right)$ once more, and Theorem 2.1, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{K_n\left(e^{i(\theta_0+2\pi s/n)}, e^{i(\theta_0+2\pi t/n)}\right)}{K_n\left(e^{i\theta_0}, e^{i\theta_0}\right)} \\
=\ & \lim_{n\to\infty}\frac{K_n^{\#}\left(e^{i(\theta_0+2\pi s/n)}, e^{i(\theta_0+2\pi t/n)}\right)}{K_n^{\#}\left(e^{i\theta_0}, e^{i\theta_0}\right)} = e^{i\pi(s-t)}S\left((s-t)T(\theta_0)\right).
\end{aligned}$$

The limit holds uniformly for s, t in a real interval. We still have to establish it for complex s, t. To do this, we can proceed as in the proof of Theorem 2.1: let f_m be defined by (2.16). We again need to show the uniform boundedness (2.17). But in some interval J containing θ_0, we have $\mu' \geq C$, and consequently, in a slightly smaller interval J_1,

$$\left|K_m\left(e^{i\theta}, e^{i\phi}\right)\right| \leq Cm,\ \theta, \phi \in J, m \geq 1.$$

We can now mimic the proof given in Theorem 2.1 to show the uniform boundedness (2.17), and then apply normality and analytic continuation. ■

Proof of Corollary 1.2
This is an easy consequence of Hurwitz's theorem: the function $e^{i\pi s}S\left(sT\left(\theta_0\right)\right)$ has (simple) zeros when and only when $sT\left(\theta_0\right)$ is an integer. It follows from the uniform convergence in Theorem 1.1, and Hurwitz' Theorem, that for large enough n, $K_n\left(e^{i(\theta_0+2\pi s/n)}, e^{i\theta_0}\right)$ has a simple zero $s_{\pm kn}$, with

$$\lim_{n\to\infty} s_{\pm kn}T\left(\theta_0\right) = \pm k.$$

Moreover, these are the only zeros of $K_n\left(e^{i(\theta_0+2\pi s/n)}, e^{i\theta_0}\right)$ in a bounded neighborhood of 0. Now observe that

$$\theta_{\pm kn} = \theta_0 + 2\pi s_{\pm kn}/n,$$

so

$$n\left(\theta_{\pm kn} - \theta_0\right) = 2\pi s_{\pm kn} = \frac{\pm 2\pi k}{T\left(\theta_0\right)} + o\left(1\right).$$

■

Proof of Corollary 1.3
We begin with the identity

$$S\left(x\right) = \frac{\sin \pi x}{\pi x} = \frac{1}{2}\int_0^1 \left(e^{i\pi xy} + e^{-i\pi xy}\right) dy.$$

This easily yields

$$\begin{aligned} & e^{i\pi(s-t)}S\left(\left(s-t\right)T\left(\theta_0\right)\right) \\ = \; & \frac{1}{2}\int_0^1 \left[e^{i\pi s(1+yT(\theta_0))}e^{-i\pi t(1+yT(\theta_0))} + e^{i\pi s(1-yT(\theta_0))}e^{-i\pi t(1-yT(\theta_0))}\right] dy. \end{aligned}$$

We now use the Maclaurin series for the exponential function on each term in the last line, and then integrate with respect to y. On multiplying and dividing by a suitable power of 2, we obtain

$$\begin{aligned} & e^{i\pi(s-t)}S\left(\left(s-t\right)T\left(\theta_0\right)\right) \\ = \; & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(2i\pi s\right)^j}{j!}\frac{\left(-2i\pi t\right)^k}{k!}\frac{1}{T\left(\theta_0\right)\left(j+k+1\right)}\left[\left(\frac{1+T\left(\theta_0\right)}{2}\right)^{j+k+1} - \left(\frac{1+T\left(\theta_0\right)}{2}\right)^{j+k+1}\right]. \end{aligned}$$

(3.4)

Next, the asymptotic in Theorem 1.1 can also be recast in the form

$$\lim_{n\to\infty} \frac{K_n\left(z_0\left(1+\frac{2\pi i s}{n}\right), z_0\left(1+\frac{2\pi i\bar{t}}{n}\right)\right)}{K_n\left(z_0, z_0\right)} = e^{i\pi(s-t)}S\left(\left(s-t\right)T\left(\theta_0\right)\right), \tag{3.5}$$

uniformly for s,t in compact sets. To establish this, one uses that

$$e^{2\pi is/n} = 1 + \frac{2\pi is}{n} + O\left(\frac{1}{n^2}\right),$$

together with bounds such as

$$\left|K_n^{(1,0)}(z,z)\right| \leq C_2 n^2,$$

uniformly for $|z - z_0| \leq C_1/n$, for any given $C_1 > 0$. This latter estimate may easily be deduced from Cauchy's estimates for derivatives, and the fact that $|K_n(z,z)| \leq C_3 n$ for $|z - z_0| \leq C_1/n$ - as in the proof of Theorem 2.1, this follows from the Bernstein-Walsh growth lemma for polynomials. Finally, we note that Taylor series expansion gives

$$\begin{aligned} &\frac{K_n\left(z_0\left(1+\frac{2\pi is}{n}\right), z_0\left(1+\frac{2\pi i\bar{t}}{n}\right)\right)}{K_n(z_0,z_0)} \\ = &\sum_{j,k=0}^{\infty} \frac{K_n^{(j,k)}(z_0,z_0)}{K_n(z_0,z_0)} \frac{z_0^{j-k}}{n^{j+k}} \frac{(2\pi is)^j}{j!} \frac{(-2\pi it)^k}{k!}. \end{aligned}$$

This, the Taylor series (3.4), and the uniform convergence (3.5) give the result. ■

References

[1] N. I. Akhiezer, *On Polynomials Orthogonal on a Circular Arc,* Dokl. Akad. Nauk SSSR, 130(1960), 247-250 [in Russian]; Soviet Math. Dokl., 1(1960), 31-34.

[2] J. Baik, T. Kriecherbauer, K. T-R. McLaughlin, P.D. Miller, *Uniform Asymptotics for Polynomials Orthogonal with respect to a General Class of Discrete Weights and Universality Results for Associated Ensembles*, Princeton Annals of Mathematics Studies, 2006.

[3] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[4] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides and X. Zhou, *Uniform Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory*, Communications in Pure and Applied Maths., 52(1999), 1335-1425.

[5] G. Freud, *Orthogonal Polynomials*, Akademiai Kiado, Budapest, 1971.

[6] L. Golinskii, *Akhiezer's Orthogonal Polynomials and Bernstein-Szegő method for a circular arc*, J. Approx. Theory, 95(1998), 229-263.

[7] L. Golinskii, *The Christoffel Function for Orthogonal Polynomials on a Circular Arc*, J. Approx. Theory, 101(1999), 165-174.

[8] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations from the Modified Jacobi Unitary Ensemble*, International Maths. Research Notices, 30(2002), 1575-1600.

[9] Eli Levin and D.S. Lubinsky, *Universality Limits Involving Orthogonal Polynomials on the Unit Circle,* Computational Methods and Function Theory, 7(2007), 543-561.

[10] Eli Levin and D.S. Lubinsky, *Universality Limits in the bulk for Varying Measures*, Advances in Mathematics, 219(2008), 743-779.

[11] Eli Levin and D.S. Lubinsky, *Applications of Universality Limits to Zeros and Reproducing Kernels of Orthogonal Polynomials*, Journal of Approximation Theory, 150(2008), 69-95.
[12] D.S. Lubinsky, *A New Approach to Universality Limits at the Edge of the Spectrum*, Contemporary Mathematics, 458(2008), 281-290.
[13] D.S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, Annals of Mathematics, 170(2009), 915-939.
[14] D.S. Lubinsky, *Bulk Universality Holds in Measure for Compactly Supported Measures*,J. d' Analyse de Mathematique, 116(2012), 219-253.
[15] A. Martinez-Finkelshtein, K. T.-R. McLaughlin, and E. B. Saff, *Szegö orthogonal polynomials with respect to an analytic weight: canonical representation and strong asymptotics,* Constructive Approximation 24(2006), 319-363.
[16] G. Mastroianni, V. Totik, *Uniform spacing of zeros of orthogonal polynomials,* Constr. Approx. 32 (2010), 181–192.
[17] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.
[18] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2, American Mathematical Society, Providence, 2005.
[19] B. Simon, *Fine structure of the zeros of orthogonal polynomials, I. A tale of two pictures,* Electronic Transactions on Numerical Analysis 25 (2006), 328-268.
[20] B. Simon, *Two Extensions of Lubinsky's Universality Theorem*, J. d'Analyse Mathematique, 105 (2008), 345-362.
[21] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.
[22] V. Totik, *Asymptotics for Christoffel Functions for General Measures on the Real Line*, J. d'Analyse Math., 81(2000), 283-303.
[23] V. Totik, *Universality and fine zero spacing on general sets*, Arkiv för Matematik, 47(2009), 361-391.
[24] V. Totik, *Christoffel functions on curves and domains*, Trans. Amer. Math. Soc. 362 (2010), 2053–2087.

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