Asymptotic Zero Distribution of Biorthogonal Polynomials☆

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Abstract

Let $\psi : [0,1] \to \mathbb{R}$ be a strictly increasing continuous function. Let P_n be a polynomial of degree n determined by the biorthogonality conditions

$$\int_0^1 P_n(x)\,\psi(x)^j\,dx = 0, \quad j = 0, 1, \ldots, n-1.$$

We study the distribution of zeros of P_n as $n \to \infty$, and related potential theory.

1. Introduction and Results

Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function, with inverse $\psi^{[-1]}$. Then we may uniquely determine a monic polynomial P_n of degree n by the biorthogonality conditions

$$\int_0^1 P_n(x)\,\psi(x)^j\,dx = \begin{cases} 0, & j = 0, 1, 2, \ldots, n-1, \\ I_n \neq 0, & j = n \end{cases}. \tag{1}$$

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P_n will have n simple zeros in $(0,1)$, so we may write

$$P_n(x) = \prod_{j=1}^{n} (x - x_{jn}). \tag{2}$$

The proof of this is the same as for classical orthogonal polynomials. Our goal in this paper is to investigate the zero distribution of P_n as $n \to \infty$. Accordingly, we define the zero counting measures

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{jn}}, \tag{3}$$

that place mass $\frac{1}{n}$ at each of the zeros of P_n, and want to describe the weak limit(s) of μ_n as $n \to \infty$.

This topic was initiated by the second author, in the course of his investigations on convergence acceleration [8], [24], and numerical integration of singular integrands. He considered [21], [22], [23]

$$\psi(x) = \log x, \quad x \in (0,1)$$

and found that the corresponding biorthogonal polynomials are

$$P_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left(\frac{j+1}{n+1}\right)^j x^j.$$

The latter are now often called the *Sidi polynomials*, and one may represent them as a contour integral. Using steepest descent, the strong asymptotics of P_n, and their zero distribution, were established in [14]. Asymptotics for more general polynomials of this type were analyzed by Elbert [7]. Extensions, asymptotics, and applications in numerical integration, and convergence acceleration have been considered in [15], [16], [25], [26]. Biorthogonal polynomials of a more general form have been studied in several contexts – see [5], [10], [11]. The sorts of biorthogonal polynomials used in random matrices [3], [6], [12] are mostly different, although there are some common ideas in the associated potential theory.

Herbert Stahl's interest in this topic arose after he refereed [14]. He and the first author discussed the topic at some length at a conference in honor of Paul Erdős in 1995. This led to a draft paper on zero distribution in the later 1990's, with revisions in 2001, and 2003, and this paper is the partial completion of that work. For the case $\psi(x) = x^\alpha$, $\alpha > 0$, we presented explicit formulae in [18]. Rodrigues type representations were studied in [17].

Distribution of zeros of polynomials is closely related to potential theory [1], [20], [28], and accordingly we introduce some potential theoretic concepts. We let $\mathcal{P}(\mathcal{E})$ denote the set of all probability measures with compact support contained in the set $\mathcal{E}$. For any positive Borel measure μ, we define its classical energy integral

$$I(\mu) = \iint \log \frac{1}{|x-t|} d\mu(x)\, d\mu(t), \tag{4}$$

and denote its support by supp$[\mu]$. Where appropriate, we consider these concepts for signed measures too. For any set $\mathcal{E}$ in the plane, its (inner) logarithmic capacity is

$$cap(\mathcal{E}) = \sup \left\{ e^{-I(\mu)} : \mu \in \mathcal{P}(\mathcal{E}) \right\}.$$

We say that a property holds q.e. (quasi-everywhere) if it holds outside a set of capacity 0. We use *meas* to denote linear Lebesgue measure 0. For further orientation on potential theory, see for example [13], [19], [20].

In our setting we need a new energy integral

$$J(\mu) = \iint K(x,t)\, d\mu(x)\, d\mu(t), \tag{5}$$

where

$$K(x,t) = \log \frac{1}{|x-t|} + \log \frac{1}{|\psi(x) - \psi(t)|}. \tag{6}$$

In [6], a similar energy integral was considered for $\psi(t) = e^t$, but with an external field. The minimal energy corresponding to ψ is

$$J^*(\psi) = \inf \{ J(\mu) : \mu \in \mathcal{P}([0,1]) \}. \tag{7}$$

Under mild conditions on ψ, we shall prove that there is a unique probability measure, which we denote by ν_ψ, attaining the minimum. For probability measures μ, ν, we define the classical potential

$$U^\mu(x) = \int \log \frac{1}{|x-t|} d\mu(t), \tag{8}$$

the mixed potential

$$\begin{aligned} W^{\mu,\nu}(x) &= \int \log \frac{1}{|x-t|} d\mu(t) + \int \log \frac{1}{|\psi(x) - \psi(t)|} d\nu(t) && (9) \\ &= U^\mu(x) + U^{\nu \circ \psi^{[-1]}} \circ \psi(x), && (10) \end{aligned}$$

and the ψ potential

$$W^{\mu}(x) = W^{\mu,\mu}(x) = \int K(x,t)\, d\mu(t). \tag{11}$$

We note that potential theory for generalized kernels is an old topic, see for example, Chapter VI in [13]. However, there does not seem to be a comprehensive treatment covering our setting. Our most important restrictions on ψ are contained in:

Definition 1.1. Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function, with inverse $\psi^{[-1]}$. Assume that ψ satisfies the following two conditions:

(I)
$$cap(E) = 0 \Rightarrow cap\left(\psi^{[-1]}(E)\right) = 0. \tag{12}$$

(II) For each $\varepsilon > 0$, there exists $\delta > 0$ such that
$$meas(E) \leq \delta \Rightarrow meas\left(\psi^{[-1]}(E)\right) \leq \varepsilon. \tag{13}$$

Then we say that ψ preserves smallness of sets.

The conditions (I), (II) are satisfied if ψ satisfies a local lower Lipschitz condition. By this we mean that we can write $[0,1]$ as a countable union of intervals $[a,b]$ such that in $[a,b]$, there exist C, $\alpha > 0$ depending on a, b, with
$$|\psi(x) - \psi(t)| \geq C\, |t-x|^{\alpha}, x, t \in [a,b].$$
We can apply Theorem 5.3.1 in [19, p. 137] to ψ^{-1} to deduce (12).

Using classical methods, we shall prove:

Theorem 1.2. *Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. Define the minimal energy $J^* = J^*(\psi)$ by (7). Then*

(a) J^* is finite and there exists a unique probability measure ν_ψ on $[0,1]$ such that
$$J(\nu_\psi) = J^*. \tag{14}$$

(b)
$$W^{\nu_\psi} \geq J^* \text{ q.e. in } [0,1]. \tag{15}$$

In particular, this is true at each point of continuity of W^{ν_ψ}.

(c)

$$W^{\nu_\psi} \leq J^* \text{ in supp} [\nu_\psi] . \tag{16}$$

and

$$W^{\nu_\psi} = J^* \text{ q.e. in supp} [\nu_\psi] . \tag{17}$$

(d) ν_ψ *is absolutely continuous with respect to linear Lebesgue measure on* $[0,1]$. *Moreover, there are constants* C_1 *and* C_2 *depending only on* ψ, *such that for all compact* $\mathcal{K} \subset [0,1]$,

$$\nu_\psi (K) \leq \frac{C_1}{|\log cap\mathcal{K}|} \leq \frac{C_2}{|\log meas (\mathcal{K})|}. \tag{18}$$

(e) There exists $\varepsilon > 0$ *such that*

$$[0, \varepsilon] \cup [1 - \varepsilon, 1] \subset \text{supp} [\nu_\psi] . \tag{19}$$

Let

$$I_n = \int_0^1 P_n (t) \psi (t)^n \, dt, \quad n \geq 1. \tag{20}$$

Theorem 1.3. *Let* $\psi : [0,1] \to [\psi (0) , \psi (1)]$ *be a strictly increasing continuous function that preserves smallness of sets. Let* $\{P_n\}$ *be the corresponding biorthogonal polynomials, with zero counting measures* $\{\mu_n\}$. *If*

$$\text{supp} [\nu_\psi] = [0,1] , \tag{21}$$

then the zero counting measures $\{\mu_n\}$ *of* (P_n) *satisfy*

$$\mu_n \xrightarrow{*} \nu_\psi, n \to \infty \tag{22}$$

and

$$\lim_{n\to\infty} I_n^{1/n} = \exp (-J^*) . \tag{23}$$

The weak convergence (22) is defined in the usual way:

$$\lim_{n\to\infty} \int_0^1 f (t) \, d\mu_n (t) = \int_0^1 f (t) \, d\nu_\psi (t) ,$$

for every continuous function $f : [0,1] \to \mathbb{R}$. We can replace (21) by the more implicit, but more general, assumption that supp$[\nu_\psi]$ contains the support of every weak limit of every subsequence of (μ_n). We can at least prove it when the kernel K, and hence the potential W^{ν_ψ}, satisfies a convexity condition:

Theorem 1.4. *Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. In addition assume that ψ is twice continuously differentiable in $(0,1)$ and either*

(a) for $x, t \in (0,1)$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2} K(x,t) > 0, \tag{24}$$

or

(b) for $x, t \in (\psi(0), \psi(1))$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2}\left[K\left(\psi^{[-1]}(x), \psi^{[-1]}(t)\right)\right] > 0. \tag{25}$$

Then

$$\operatorname{supp}[\nu_\psi] = [0,1]. \tag{26}$$

Example. Let $\alpha > 0$ and

$$\psi(x) = x^\alpha, \quad x \in [0,1].$$

Then either (25) or (26) holds and hence (21) holds. We show this separately for $\alpha \geq 1$ and for $\alpha < 1$.

Case I $\alpha \geq 1$
We shall show that the hypotheses of Theorem 1.4 (a) are fulfilled. A straightforward calculation gives that

$$\begin{aligned}\Delta(x,t) &:= (x-t)^2 (\psi(x) - \psi(t))^2 \frac{\partial^2}{\partial x^2} K(x,t) \\ &= (x^\alpha - t^\alpha)^2 + \left(\alpha x^{\alpha-1}\right)^2 (x-t)^2 - \alpha(\alpha-1) x^{\alpha-2} (x^\alpha - t^\alpha)(x-t)^2.\end{aligned}$$

Writing $s = tx$, we see that

$$\Delta(x,t) = x^{2\alpha} H(s),$$

where

$$H(s) := (1-s^\alpha)^2 + \alpha^2 (1-s)^2 - \alpha(\alpha-1)(1-s^\alpha)(1-s)^2. \tag{27}$$

For $s > 1$, all three terms in the right-hand side of (27) are positive, so $H(s) > 0$. If $0 \leq s < 1$, we see that

$$\begin{aligned} H(s) &= (1-s^\alpha)^2 + \alpha(1-s)^2 \{\alpha - (\alpha-1)(1-s^\alpha)\} \\ &\geq (1-s^\alpha)^2 + \alpha(1-s)^2 > 0. \end{aligned}$$

In summary, if $\alpha > 1$, we have for all $x \in [0,1]$ and $s \in [0,\infty) \backslash \{1\}$,

$$\Delta(x, sx) > 0$$

so the hypotheses (24) is fulfilled.

Case II $\alpha < 1$
Here

$$\psi^{[-1]}(x) = x^{1/\alpha}$$

and

$$K\left(\psi^{[-1]}(x), \psi^{[-1]}(t)\right) = \log \frac{1}{\left|x^{1/\alpha} - t^{1/\alpha}\right|} + \log \frac{1}{|x-t|},$$

which is exactly the case $1/\alpha > 1$ treated above, so we see that the hypothesis (25) is fulfilled.

Instead of placing an implict assumption on the support of ν_ψ, we can place an implicit assumption on the zeros of $\{P_n\}$, and obtain a unique weak limit:

Theorem 1.5. *Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. Let $\mathcal{K} \subset [0,1]$ be compact. Assume that every weak limit of every subsequence of the zero counting measures $\{\mu_n\}$ has support $\mathcal{K}$. Then there is a unique probability measure μ on K such that*

$$\mu_n \xrightarrow{*} \mu, n \to \infty, \tag{28}$$

and a unique positive number A such that

$$\lim_{n\to\infty} I_n^{1/n} = A, \tag{29}$$

Here μ is absolutely continuous with respect to linear Lebesgue measure, and is the unique solution of the integral equation

$$W^\mu(x) = \text{Constant}, \quad \text{q.e. } x \in \mathcal{K}, \tag{30}$$

Moreover, then

$$W^\mu(x) = \log \frac{1}{A}, \quad \text{q.e. } x \in \mathcal{K}.$$

We note that in [6], a related integral equation to (30) appears. We shall also need the *dual polynomials* Q_n such that $Q_n \circ \psi$ are biorthogonal to powers of x. Thus we define Q_n to be a monic polynomial of degree n determined by the conditions

$$\int_0^1 Q_n \circ \psi(t)\, t^j dt = 0, \tag{31}$$

$j = 0, 1, 2, \ldots, n-1$. Because of this biorthogonality condition,

$$\int_0^1 Q_n \circ \psi(t)\, t^n dt = \int_0^1 Q_n \circ \psi(t)\, P_n(t)\, dt = \int_0^1 P_n(t)\, \psi(t)^n\, dt.$$

That is,

$$I_n = \int_0^1 P_n(t)\, \psi(t)^n\, dt = \int_0^1 Q_n \circ \psi(t)\, t^n dt. \tag{32}$$

The orthogonality conditions ensure that $Q_n \circ \psi$ has n distinct zeros $\{y_{jn}\}$ in $(0,1)$, so we can write

$$Q_n \circ \psi(t) = \prod_{j=1}^{n} (\psi(t) - \psi(y_{jn}))\,. \tag{33}$$

Let

$$\nu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{y_{jn}}. \tag{34}$$

We shall prove

Theorem 1.6. *Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets, and assume* (21)*. We have as $n \to \infty$,*

$$\nu_n \xrightarrow{*} \nu_\psi.$$

We also prove the following extremal property for weak subsequential limits of $\{\mu_n\}$.

Theorem 1.7. *Let $\psi : [0,1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. Assume that $\mathcal{S}$ is an infinite subsequence of positive integers such that as $n \to \infty$ through $\mathcal{S}$,*

$$\mu_n \xrightarrow{*} \mu; \tag{35}$$

$$\nu_n \xrightarrow{*} \nu; \tag{36}$$

and

$$I_n^{1/n} \to A, \tag{37}$$

where $A \in \mathbb{R}$ *and* $\mu, \nu \in \mathcal{P}([0,1])$. *Then*

$$A \leq \exp\left(-\sup_{\beta \in \mathcal{P}([0,1])} \inf_{[0,1]} W^{\mu,\beta}\right) \tag{38}$$

and

$$A \leq \exp\left(-\sup_{\alpha \in \mathcal{P}([0,1])} \inf_{[0,1]} W^{\alpha,\nu}\right). \tag{39}$$

Remarks. (a) This extremal property is very close to a characterization of equilibrium measures for external fields. For example, with ν as above, let Q be the external field

$$Q = U^{\nu \circ \psi^{[-1]}} \circ \psi \text{ on } [0,1].$$

Then the second inequality above says

$$A \leq \exp\left(-\sup_{\alpha \in \mathcal{P}([0,1])} \inf_{[0,1]} (U^{\alpha} + Q)\right).$$

This is reminiscent of one characterization of the equilibrium measure for the external field Q [20, Theorem I.3.1, p. 43].

(b) Herbert Stahl sketched a proof that when ψ is strictly increasing and piecewise linear, then (21) holds [27]. His expectation was that this and a limiting argument could establish (21) very generally.

(c) There are two principal issues left unresolved in this paper, that seem worthy of further study:

(I) Find general hypotheses for $\operatorname{supp}[\nu_\psi] = [0,1]$.

(II) Find an explicit representation of the solution μ' of the integral equation (30), that is of

$$\int_0^1 \log|x-t|\,\mu'(t)\,dt + \int_0^1 \log|\psi(x)-\psi(t)|\,\mu'(t)\,dt = \text{Constant}, \quad x \in [0,1].$$

The usual methods (differentiating, and solving a Cauchy singular integral equation) do not seem to work, even when ψ is analytic.

Next we show that if ψ is constant in an interval, then the support of the equilibrium measure should avoid that interval, as do most of the zeros of $\{P_n\}$:

Example. Let

$$\psi(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{2}\right] \\ 1, & x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Then it is not difficult to see that the equilibrium measure ν_ψ must have support $[0, \frac{1}{2}]$. Indeed if μ is a probability measure that has positive measure on $[a, b] \subset (\frac{1}{2}, 1)$, then as

$$\log \frac{1}{|\psi(x) - \psi(t)|} = \infty, \quad x, t \in [a, b],$$

so

$$J(\mu) = \infty.$$

Consequently,

$$J^* = \inf \left[2I(\mu) + \log \frac{1}{2}\right],$$

where the inf is now taken over all $\mu \in \mathcal{P}\left(\left[0, \frac{1}{2}\right]\right)$. Then ν_ψ is the classical equilibrium measure for $\left[0, \frac{1}{2}\right]$, namely

$$\nu'_\psi(x) = \frac{1}{\pi\sqrt{x\left(\frac{1}{2} - x\right)}}, \quad x \in \left[0, \frac{1}{2}\right],$$

and

$$J^* = 2\log 8 + \log \frac{1}{2} = \log 32.$$

In this case, we can also almost explicitly determine P_n. The biorthogonality conditions give for π of degree at most $n-1$,

$$\int_0^{1/2} P_n(x)\,\pi(2x)\,dx + \pi(1)\int_{1/2}^1 P_n(x)\,dx = 0.$$

In particular, this is true for $\pi \equiv 1$, so

$$\int_{1/2}^1 P_n(x)\,dx = -\int_0^{1/2} P_n(x)\,dx,$$

and we obtain for any π of degree at most $n-1$,

$$\int_0^{1/2} P_n(x)\left(\pi(2x) - \pi(1)\right)dx = 0.$$

Then for every polynomial S of degree $\leq n-2$,

$$\int_0^{1/2} P_n(x)\, S(x)\,(1-2x)\,dx = 0, \tag{40}$$

which forces P_n to have at least $n-1$ distinct zeros in $[0, \frac{1}{2}]$. Then every weak limit of every subsequence of $\{\mu_n\}$ has support in $[0, \frac{1}{2}]$.

This paper is organized as follows: in Section 2, we present a principle of descent, and a lower envelope theorem, and the proof of Theorem 1.2. In Section 3, we prove Theorems 1.3–1.7. Throughout the sequel, we assume that $\psi : [0,1] \to [\psi(0), \psi(1)]$ is a strictly increasing continuous function that preserves smallness of sets.

We close this section with some extra notation. Define the *companion polynomial* to P_n, namely

$$R_n(x) = \prod_{j=1}^{n} (x - \psi(x_{jn})). \tag{41}$$

It has the property that $R_n \circ \psi$ has the same zeros as P_n. Hence

$$P_n(x)\, R_n \circ \psi(x) \geq 0 \text{ in } [0,1]. \tag{42}$$

Analogous to R_n, we define

$$S_n(t) = \prod_{j=1}^{n} (t - y_{jn}), \tag{43}$$

so that

$$S_n(t)\, Q_n \circ \psi(t) \geq 0, \quad t \in [0,1]. \tag{44}$$

Observe that I_n of (20) satisfies

$$I_n = \int_0^1 P_n(x)\, R_n \circ \psi(x)\,dx = \int_0^1 Q_n \circ \psi(x)\, S_n(x)\,dx > 0. \tag{45}$$

2. Proof of Theorem 1.2

We begin by noting that for any positive measures α, β, $W^{\alpha,\beta}$ is lower semicontinuous, since a potential of any positive measure is, while ψ and $\psi^{[-1]}$ are continuous. We start with

Lemma 2.1 (The Principle of Descent). *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be finite positive Borel measures on* $[0,1]$ *such that*

$$\lim_{n\to\infty} \alpha_n([0,1]) = 1 = \lim_{n\to\infty} \beta_n([0,1]).$$

Assume moreover that as $n \to \infty$,

$$\alpha_n \xrightarrow{*} \alpha;$$
$$\beta_n \xrightarrow{*} \beta.$$

(a) If $\{x_n\} \subset [0,1]$ *and* $x_n \to x_0$, $n \to \infty$, *then*

$$\liminf_{n\to\infty} W^{\alpha_n,\beta_n}(x_n) \geq W^{\alpha,\beta}(x_0).$$

(b) If $\mathcal{K} \subset [0,1]$ *is compact and*

$$W^{\alpha,\beta} \geq \lambda \text{ in } \mathcal{K},$$

then uniformly in $\mathcal{K}$,

$$\liminf_{n\to\infty} W^{\alpha_n,\beta_n}(x) \geq \lambda.$$

Proof. (a) By the classical principle of descent,

$$\liminf_{n\to\infty} U^{\alpha_n}(x_n) \geq U^{\alpha}(x_0),$$

see for example, [20, Theorem I.6.8, p. 70]. Next, we see from the classical principle of descent and continuity of ψ, $\psi^{[-1]}$ that

$$\liminf_{n\to\infty} U^{\beta_n\circ\psi^{[-1]}} \circ \psi(x_n) \geq U^{\beta\circ\psi^{[-1]}} \circ \psi(x_0).$$

Combining these two gives the result.

(b) This follows easily from (a). If (b) fails, we can choose a sequence (x_n) in K with limit $x_0 \in K$ such that

$$\liminf_{n\to\infty} W^{\alpha_n,\beta_n}(x_n) < \lambda \leq W^{\alpha,\beta}(x_0).$$

□

Recall our notation $W^{\alpha_n} = W^{\alpha_n,\alpha_n}$. We now establish

Lemma 2.2 (Lower Envelope Theorem)**.** *Assume the hypotheses of Lemma* 2.1*. Then for q.e.* $x \in [0,1]$*,*

$$\liminf_{n\to\infty, n\in\mathcal{S}} W^{\alpha_n}(x) = W^{\alpha}(x).$$

Proof. We already know from Lemma 2.1 (the principle of descent) that everywhere in $[0,1]$,

$$\liminf_{n\to\infty, n\in\mathcal{S}} W^{\alpha_n}(x) \geq W^{\alpha}(x).$$

Suppose the result is false. Then there exists $\varepsilon > 0$, and a (Borel) set S of positive capacity such that

$$\liminf_{n\to\infty, n\in\mathcal{S}} W^{\alpha_n}(x) \geq W^{\alpha}(x) + \varepsilon \text{ in } S. \tag{46}$$

Because Borel sets are inner regular, and even more, capacitable, we may assume that S is compact. Then there exists a probability measure ω with support in S such that U^{ω} is continuous in $\mathbb{C}$. See, for example, [20, Corollary I.6.11, p. 74]. As ψ and $\psi^{[-1]}$ are continuous,

$$W^{\omega} = U^{\omega} + U^{\omega\circ\psi^{[-1]}} \circ \psi$$

is also continuous in $[0,1]$. Then by Fubini's Theorem and weak convergence

$$\begin{aligned}\liminf_{n\to\infty, n\in\mathcal{S}} \int W^{\alpha_n} d\omega &= \liminf_{n\to\infty, n\in\mathcal{S}} \int W^{\omega} d\alpha_n \\ &= \int W^{\omega} d\alpha = \int W^{\alpha} d\omega.\end{aligned}$$

Here since $K(x,t)$ is bounded below in $[0,1]$, we may continue this using (46) and Fatou's Lemma as

$$\begin{aligned}&= \int (W^{\alpha} + \varepsilon)\, d\omega - \varepsilon \\ &\leq \int \left(\liminf_{n\to\infty, n\in\mathcal{S}} W^{\alpha_n}\right) d\omega - \varepsilon \\ &\leq \liminf_{n\to\infty, n\in\mathcal{S}} \int W^{\alpha_n} d\omega - \varepsilon.\end{aligned}$$

So we have a contradiction. □

Next, we show that J^* is finite, establishing part of Theorem 1.2(a):

Lemma 2.3. *J^* is finite.*

Proof. This is really a consequence of Cartan's Lemma for potentials. Let $\mu = meas$ denote Lebesgue measure on $[0,1]$. Then for $x \in [0,1]$,

$$U^{\mu}(x) = \int_0^1 \log \frac{1}{|x-t|} dt \leq 2 \int_0^1 \log \frac{1}{s} ds$$

and U^{μ} is continuous. Now consider the unit measure $\mu \circ \psi^{[-1]}$. By Cartan's Lemma [9, p. 366], if $\varepsilon > 0$ and

$$\mathcal{A}^{\varepsilon} = \left\{ y \in \mathbb{R} : U^{\mu \circ \psi^{[-1]}}(y) > \log \frac{1}{\varepsilon} \right\},$$

then

$$\mu(\mathcal{A}^{\varepsilon}) \leq 3e\varepsilon.$$

With a suitably small choice of ε, we then have by the hypothesis (13),

$$\mu\left(\psi^{[-1]}(\mathcal{A}^{\varepsilon})\right) \leq \frac{1}{2}.$$

With this choice of ε, let

$$\mathcal{B} = [0,1] \backslash \psi^{[-1]}(\mathcal{A}^{\varepsilon}),$$

a closed set. Let

$$\nu = \frac{\mu_{|\mathcal{B}}}{\mu(\mathcal{B})}.$$

As $\mu(\mathcal{B}) \geq \frac{1}{2}$, ν is a well defined probability measure. Moreover, $x \in \mathcal{B} \Rightarrow \psi(x) \notin \mathcal{A}^{\varepsilon}$, and

$$\begin{aligned} U^{\nu \circ \psi^{[-1]}} \circ \psi(x) &= \frac{1}{\mu(B)} \left[U^{\mu \circ \psi^{[-1]}} \circ \psi(x) - U^{\mu_{|[0,1] \backslash \mathcal{B}} \circ \psi^{[-1]}} \circ \psi(x) \right] \\ &\leq \frac{1}{\mu(B)} \left[\log \frac{1}{\varepsilon} + \log\left(2 \parallel \psi \parallel_{L_{\infty}[0,1]}\right) \right] =: C_0 < \infty. \end{aligned}$$

Then

$$J^* \leq J(\nu) \leq I(\nu) + C_0 < \infty.$$

$\square$

Proof of Theorem 1.2. (a) We can choose a sequence $\{\alpha_n\}$ of probability measures on $[0,1]$ such that

$$\lim_{n\to\infty} J(\alpha_n) = J^*.$$

By Helly's Theorem, we can choose a subsequence converging weakly to some probability measure α on $[0,1]$, and by relabelling, we may assume that the full sequence $\{\alpha_n\}$ converges weakly to α. Then $\left\{\alpha_n \circ \psi^{[-1]}\right\}$ converges weakly to $\alpha \circ \psi^{[-1]}$. By the classical principle of descent

$$\liminf_{n\to\infty} I(\alpha_n) \geq I(\alpha)$$

and

$$\liminf_{n\to\infty} I\left(\alpha_n \circ \psi^{[-1]}\right) \geq I\left(\alpha \circ \psi^{[-1]}\right),$$

or equivalently,

$$\begin{aligned}\liminf_{n\to\infty} \int\int \log \frac{1}{|\psi(x) - \psi(t)|} d\alpha_n(x)\, d\alpha_n(t)& \\ \geq \int\int \log \frac{1}{|\psi(x) - \psi(t)|} d\alpha(x)\, d\alpha(t)&.\end{aligned}$$

See, for example, [20, Thm. I.6.8, p. 70]. Combining these, we have

$$J^* = \liminf_{n\to\infty} J(\alpha_n) \geq J(\alpha),$$

so α achieves the inf, and is an equilibrium distribution. If β is another such distribution, then the parallelogram law

$$J\left(\frac{1}{2}(\alpha+\beta)\right) + J\left(\frac{1}{2}(\alpha-\beta)\right) = \frac{1}{2}(J(\alpha) + J(\beta)) = J^*,$$

gives

$$J\left(\frac{1}{2}(\alpha-\beta)\right) = J^* - J\left(\frac{1}{2}(\alpha+\beta)\right) \leq 0,$$

as $\frac{1}{2}(\alpha+\beta)$ is also a probability measure on $[0,1]$. Here

$$J\left(\frac{1}{2}(\alpha-\beta)\right) = I\left(\frac{1}{2}(\alpha-\beta)\right) + I\left(\frac{1}{2}\left(\alpha\circ\psi^{[-1]} - \beta\circ\psi^{[-1]}\right)\right),$$

and both terms on the right-hand side are non-negative as both measures inside the energy integrals on the right have total mass 0. See [20, Lemma I.1.8, p. 29]. Hence

$$I\left(\frac{1}{2}(\alpha-\beta)\right) = 0,$$

so $\alpha = \beta$ [20, Lemma I.1.8, p. 29].
(b) Suppose the result is false. Then for some large enough integer n_0,

$$E_1 := \left\{ x \in [0,1] : W^{\nu_\psi}(x) \leq J^* - \frac{1}{n_0} \right\},$$

has positive capacity and is compact, since W^{ν_ψ} is lower semi-continuous. But,

$$\int W^{\nu_\psi} d\nu_\psi = J(\nu_\psi) = J^*,$$

so there exists a compact subset E_2 disjoint from E_1 such that

$$W^{\nu_\psi}(x) > J^* - \frac{1}{2n_0}, \quad x \in E_2,$$

and

$$m = \nu_\psi(E_2) > 0.$$

Now as E_1 is a compact set of positive capacity, we can find a positive measure σ on E_1, with support in E_1, such that U^σ is continuous in the plane [20, Cor. I.6.11, p. 74]. Then $U^{\sigma \circ \psi^{[-1]}}$ is also continuous in $[\psi(0), \psi(1)]$, so W^σ is continuous in $[0,1]$. We may also assume that

$$\sigma(E_1) = m.$$

Define a signed measure σ_1 on $[0,1]$, by

$$\sigma_1 := \begin{cases} \sigma & \text{in } E_1 \\ -\nu_\psi & \text{in } E_2 \\ 0 & \text{elsewhere} \end{cases}.$$

Here if $\eta \in (0,1)$,

$$\begin{aligned} J(\nu_\psi + \eta\sigma_1) &= J(\nu_\psi) + 2\eta \int W^{\nu_\psi} d\sigma_1 + \eta^2 J(\sigma_1) \\ &\leq J(\nu_\psi) + 2\eta \left\{ \int_{E_1} \left[J^* - \frac{1}{n_0} \right] d\sigma + \int_{E_2} \left[J^* - \frac{1}{2n_0} \right] d(-\nu_\psi) \right\} + \eta^2 J(\sigma_1) \\ &= J(\nu_\psi) + 2\eta m \left\{ \left[J^* - \frac{1}{n_0} \right] - \left[J^* - \frac{1}{2n_0} \right] \right\} + \eta^2 J(\sigma_1) \\ &= J(\nu_\psi) - \frac{\eta m}{n_0} + \eta^2 J(\sigma_1) < J(\nu_\psi), \end{aligned}$$

for small $\eta > 0$. As σ_1 has total mass 0, so $\nu_\psi + \eta\sigma_1$ has total mass 1, and we see from the identity

$$\nu_\psi + \eta\sigma_1 = (1-\eta)\,\nu_{\psi|E_2} + \nu_{\psi|[0,1]\setminus E_2} + \eta\sigma$$

that it is non-negative. Then we have a contradiction to the minimality of $J(\nu_\psi)$.

(c) Let $x_0 \in \operatorname{supp}[\nu_\psi]$ and suppose that

$$W^{\nu_\psi}(x_0) > J^*.$$

By lower semi-continuity of W^{ν_ψ}, there exists $\varepsilon > 0$ and closed $[a,b]$ containing x_0 such that

$$W^{\nu_\psi}(x) > J^* + \varepsilon, \quad x \in [a,b].$$

We know too that

$$W^{\nu_\psi}(x) \geq J^* \text{ for q.e. } x \in \operatorname{supp}[\nu_\psi].$$

Here as J^* is finite, so $I(\nu_\psi)$ must be finite (recall that $K(x,t)$ is bounded below). Then ν_ψ vanishes on sets of capacity 0, so this last inequality holds ν_ψ a.e. (cf. [19, Theorem 3.2.3, p. 56]). Then

$$\begin{aligned} J^* &= J(\nu_\psi) = \left(\int_a^b + \int_{[0,1]\setminus[a,b]} \right) W^{\nu_\psi}(x)\, d\nu_\psi(x) \\ &\geq (J^* + \varepsilon)\,\nu_\psi([a,b]) + J^*\nu_\psi([0,1]\setminus[a,b]) \\ &= J^* + \varepsilon\nu_\psi([a,b]), \end{aligned}$$

a contradiction.

(d) If $cap(\mathcal{K}) = 0$, then as $I(\nu_\psi) < \infty$, we have also $\nu_\psi(\mathcal{K}) = 0$, and the inequality (18) is immediate. So assume that $\mathcal{K} \subset$supp$[\nu_\psi]$ has positive capacity, and let ω be the equilibrium measure for $\mathcal{K}$. We may also assume that $\mathcal{K} \subset \operatorname{supp}[\nu_\psi]$, since

$$\nu_\psi(\mathcal{K}) = \nu_\psi(\mathcal{K} \cap \operatorname{supp}[\nu_\psi]).$$

Now, there exists a positive constant C_0 such that

$$K(x,t) \geq -C_0, \quad x,t \in [0,1].$$

Then by (c), for $x \in \mathcal{K}$,

$$\begin{aligned}\int_{\mathcal{K}} K(x,t)\, d\nu_\psi(t) &\leq J^* - \int_{[0,1]\setminus\mathcal{K}} K(x,t)\, d\nu_\psi(t) \\ &\leq J^* + C_0\end{aligned}$$

and hence for $x \in \mathcal{K}$,

$$\int_{\mathcal{K}} \log \frac{1}{|x-t|} d\nu_\psi(t) \leq J^* + C_0 + \log\left(2\|\psi\|_{L_\infty[0,1]}\right) =: C_1. \qquad (47)$$

Here C_1 is independent of $\mathcal{K}, x$. Now

$$U^\omega(t) = \log \frac{1}{cap\mathcal{K}}$$

for q.e. $t \in \mathcal{K}$ and since ν_ψ vanishes on sets of capacity zero, this also holds for ν_ψ a.e. $t \in \mathcal{K}$. Integrating (47) with respect to $d\omega(x)$ and using Fubini's theorem, gives

$$\int_{\mathcal{K}} U^\omega(t)\, d\nu_\psi(t) \leq C_1$$

and hence

$$\nu_\psi(\mathcal{K}) \log \frac{1}{cap\mathcal{K}} \leq C_1.$$

This gives the first inequality in (18), and then well known inequalities relating *cap* and *meas* give the second. In particular, that inequality implies the absolute continuity of μ with respect to linear Lebesgue measure.
(e) Suppose that $0 \notin \operatorname{supp}[\nu_\psi]$. Let $c > 0$ be the closest point in the support of ν_ψ to 0. Then for $x \in [0, \frac{c}{2}]$, and for all $t \in [c, 1]$, we have from the strict monotonicity of ψ that

$$K(x,t) < K(c,t),$$

so for such x,

$$\begin{aligned}W^{\nu_\psi}(x) &= \int_c^1 K(x,t)\, d\nu_\psi(t) \\ &< \int_c^1 K(c,t)\, d\nu_\psi(t) = W^{\nu_\psi}(c) \leq J^*.\end{aligned}$$

Thus in spite of the continuity of W^{ν_ψ} in $[0, c)$,

$$W^{\nu_\psi} < J^* \text{ in } \left[0, \frac{c}{2}\right],$$

contradicting (b). Absolute continuity of ν_ψ then shows that for some $\varepsilon > 0$, we have $[0, \varepsilon] \subset \operatorname{supp}[\nu_\psi]$. Similarly we can show that for some $\varepsilon > 0$, $[1-\varepsilon, 1] \subset \operatorname{supp}[\nu_\psi]$. $\square$

3. Proof of Theorems 1.3–1.7

Recall that μ_n and ν_n were defined respectively by (3) and (34). Throughout this section, we assume that $\mathcal{S}$ is an infinite subsequence of positive integers such that as $n \to \infty$ through $\mathcal{S}$,

$$\mu_n \xrightarrow{*} \mu; \tag{48}$$

$$\nu_n \xrightarrow{*} \nu; \tag{49}$$

and

$$I_n^{1/n} \to A, \tag{50}$$

where $A \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}([0,1])$. In the sequel we make frequent use of identities such as

$$|P_n(x)|^{1/n} = \exp\left(-U^{\mu_n}(x)\right)$$

and

$$|P_n(x) R_n \circ \psi(x)|^{1/n} = \exp\left(-W^{\mu_n}(x)\right).$$

We begin with

Lemma 3.1 (An upper bound for W^{μ}). *(a) With the hypotheses above, let $[a,b] \subset [0,1]$ and assume that $[a,b]$ contains two zeros of P_n for infinitely many $n \in \mathcal{S}$. Then*

$$\inf_{[a,b]} W^{\mu} \le \log \frac{1}{A}.$$

(b) In particular, if x_0 is a limit of two zeros of P_n as $n \to \infty$ through $\mathcal{S}$, or $x_0 \in supp\,[\mu]$, then

$$W^{\mu}(x_0) \le \log \frac{1}{A}.$$

Proof. (a) We may assume (by passing to a subsequence) that for all $n \in \mathcal{S}$, P_n has two zeros in $[a,b]$. Assume on the contrary, that for some $\varepsilon > 0$,

$$\inf_{[a,b]} W^{\mu} > \log \frac{1}{A} + \varepsilon. \tag{51}$$

Let x_n, y_n be two zeros of P_n in $[a,b]$ and let

$$R_n^*(x) = R_n(x) / \left[(x - \psi(x_n))(x - \psi(y_n))\right].$$

Then we see that

$$P_n(x)\, R_n^* \circ \psi(x) \geq 0, \quad x \in [0,1] \setminus [a,b],$$

and

$$0 \leq P_n(x)\, R_n \circ \psi(x) \leq |P_n(x)\, R_n^* \circ \psi(x)| \left(4\|\psi\|_{L_\infty[0,1]}\right)^2, \quad x \in [0,1].$$

Moreover, as R_n^* has the same asymptotic zero distribution as R_n, we see from Lemma 2.1 and (51) that

$$\begin{aligned} \limsup_{n\to\infty, n\in\mathcal{S}} |P_n(x)\, R_n^* \circ \psi(x)|^{1/n} &\leq \exp\left(-W^{\mu,\mu}(x)\right) \\ &= \exp\left(-W^{\mu}(x)\right) \leq Ae^{-\varepsilon}, \end{aligned}$$

uniformly in $[a,b]$. Then by biorthogonality, and positivity of $P_n(x)\, R_n^* \circ \psi(x)$ outside $[a,b]$,

$$\begin{aligned} &\limsup_{n\to\infty, n\in\mathcal{S}} \left(\int_{[0,1]\setminus[a,b]} |P_n(x)\, R_n^* \circ \psi(x)|\, dx\right)^{1/n} \\ &= \limsup_{n\to\infty, n\in\mathcal{S}} \left|\int_{[a,b]} P_n(x)\, R_n^* \circ \psi(x)\, dx\right|^{1/n} \leq Ae^{-\varepsilon}. \end{aligned}$$

Of course Lemma 2.1(b) also gives

$$\limsup_{n\to\infty, n\in\mathcal{S}} \left(\int_{[a,b]} |P_n(x)\, R_n^* \circ \psi(x)|\, dx\right)^{1/n} \leq Ae^{-\varepsilon},$$

so

$$\begin{aligned} A &= \limsup_{n\to\infty, n\in\mathcal{S}} I_n^{1/n} \\ &\leq \limsup_{n\to\infty, n\in\mathcal{S}} \left(4\|\psi\|_{L_\infty[0,1]}\right)^{2/n} \left(\int_0^1 |P_n(x)\, R_n^* \circ \psi(x)|\, dx\right)^{1/n} \\ &\leq Ae^{-\varepsilon}. \end{aligned}$$

This contradiction gives the result.
(b) This follows from (a), and lower semicontinuity of W^μ. □

Lemma 3.2 (A Lower bound for W^μ). *At each point of continuity of W^μ in $[0,1]$, we have*

$$W^\mu \geq \log \frac{1}{A}. \tag{52}$$

In particular, this inequality holds q.e. in $[0,1]$.

Proof. Assume that $a \in [0,1]$ is a point of continuity of W^{μ}, but for some $\varepsilon > 0$,

$$W^{\mu}(a) \leq \log \frac{1}{A} - 2\varepsilon.$$

Then there exists an interval $[a,b]$ containing a, such that

$$W^{\mu}(x) \leq \log \frac{1}{A} - \varepsilon, \quad x \in [a,b].$$

By the lower envelope theorem (Lemma 2.2)

$$\begin{aligned}
&\limsup_{n\to\infty, n\in\mathcal{S}} \left(P_n(x) R_n \circ \psi(x)\right)^{1/n} \\
&\quad = \exp\left(-\liminf_{n\to\infty, n\in\mathcal{S}} W^{\mu_n}(x)\right) = \exp\left(-W^{\mu}(x)\right) \geq A e^{\varepsilon}
\end{aligned}$$

for q.e. $x \in [a,b]$. Let

$$\mathcal{T}_n = \left\{x \in [a,b] : \left(P_n(x) R_n \circ \psi(x)\right)^{1/n} \geq A e^{\varepsilon/2}\right\}.$$

Then for each $m \geq 1$,

$$\bigcup_{n=m}^{\infty} \mathcal{T}_n$$

contains q.e. $x \in [a,b]$, so has linear Lebesgue measure $b-a$. Then for infinitely many n, $\mathcal{T}_n$ has linear Lebesgue measure at least n^{-2}, so

$$\begin{aligned}
I_n^{1/n} &\geq \left(\int_{\mathcal{T}_n} P_n(x) R_n \circ \psi(x)\, dx\right)^{1/n} \\
&\geq n^{-2/n} A e^{\varepsilon/2}
\end{aligned}$$

so

$$A = \limsup_{n\to\infty, n\in\mathcal{S}} I_n^{1/n} \geq A e^{\varepsilon/2},$$

a contradiction.

Finally, we note that any logarithmic potential is continuous q.e. [13, p. 185], so U^{μ} and $U^{\mu\circ\psi^{[-1]}}$ are continuous q.e. Our hypothesis that $\psi^{[-1]}(E)$ has capacity zero whenever E does ensures that $U^{\mu\circ\psi^{[-1]}} \circ \psi$ is continuous q.e. also. Hence W^{μ} is continuous q.e. and so (52) holds q.e. in $[0,1]$. $\square$

Next, we establish lower and upper bounds for A.

Lemma 3.3. *(a) There exist constants $C_1, C_2 > 0$ depending only on ψ (and not on the subsequence $\mathcal{S}$ above) such that*

$$C_1 \geq A \geq C_2. \tag{53}$$

(b) In particular,

$$I(\mu) < \infty.$$

(c)

$$J(\mu) = \log \frac{1}{A} \tag{54}$$

and

$$W^{\mu} = \log \frac{1}{A} \text{ q.e. and a.e. } (\mu) \text{ in } \operatorname{supp}[\mu]. \tag{55}$$

(d) μ is absolutely continuous with respect to linear Lebesgue measure on $[0,1]$. Moreover, there are constants C_1 and C_2 depending only on ψ, and not on $\mathcal{S}$, such that for all compact $\mathcal{K} \subset [0,1]$,

$$\mu(K) \leq \frac{C_1}{|\log cap\mathcal{K}|} \leq \frac{C_2}{|\log meas(\mathcal{K})|}.$$

Proof. (a) Firstly as all zeros of P_n and $R_n \circ \psi$ lie in $[0,1]$, so

$$\begin{aligned} I_n &= \int_0^1 P_n(x) R_n \circ \psi(x)\, dx \\ &\leq (\operatorname{diam}\psi[0,1])^n. \end{aligned}$$

Here diam denotes the diameter of a set. So

$$A \leq \operatorname{diam}\psi[0,1].$$

In the other direction, we use Cartan's Lemma for polynomials [2, p. 175], [9, p. 366]. This asserts that if $\delta > 0$, then

$$|R_n(x)| \geq \left(\frac{\delta}{4e}\right)^n$$

outside a set $\mathcal{E}$ of linear Lebesgue measure at most δ. Then

$$|R_n \circ \psi(x)| \geq \left(\frac{\delta}{4e}\right)^n, \quad x \in [0,1] \backslash \psi^{[-1]}(\mathcal{E}).$$

By our hypothesis (13), we may choose δ so small that

$$meas\left(\mathcal{E}\right)\leq\delta\Rightarrow meas\left(\psi^{[-1]}\left(\mathcal{E}\right)\right)\leq\frac{1}{4}.$$

Next, Cartan's Lemma also shows that

$$\left|P_n\left(x\right)\right|\geq\left(\frac{1}{16e}\right)^n,\quad x\in\left[0,1\right]\backslash\mathcal{F},$$

where

$$meas\left(\mathcal{F}\right)\leq\frac{1}{4}.$$

Then

$$P_n\left(x\right)R_n\circ\psi\left(x\right)\geq\left(\frac{\delta}{64e^2}\right)^n,\quad x\in\left[0,1\right]\backslash\left(\psi^{[-1]}\left(\mathcal{E}\right)\cup\mathcal{F}\right)$$

and so

$$\begin{aligned}I_n &\geq \int_{[0,1]\backslash\left(\psi^{[-1]}(\mathcal{E})\cup\mathcal{F}\right)}P_n\left(x\right)R_n\circ\psi\left(x\right)dx\\ &\geq \left(\frac{\delta}{64e^2}\right)^n\frac{1}{2}.\end{aligned}$$

Hence

$$A\geq\frac{\delta}{64e^2}.$$

(b) Since for $x,t\in\left[0,1\right]$,

$$\log\frac{1}{\left|\psi\left(x\right)-\psi\left(t\right)\right|}\geq\log\frac{1}{2\mathrm{diam}\psi\left[0,1\right]}>-\infty,$$

so for $x\in\mathrm{supp}\left[\mu\right]$, Lemma 3.1(b) gives

$$\log\frac{1}{A}\geq W^{\mu}\left(x\right)\geq U^{\mu}\left(x\right)+\log\frac{1}{2\mathrm{diam}\psi\left[0,1\right]}.$$

Then

$$I\left(u\right)\leq\log\frac{1}{A}-\log\frac{1}{2\mathrm{diam}\psi\left[0,1\right]}.$$

(c) As μ has finite energy, it vanishes on sets of capacity zero. Then combining Lemma 3.1 and 3.2,

$$W^{\mu}=\log\frac{1}{A}\text{ both q.e. and a.e. }\left(\mu\right)\text{ in }\mathrm{supp}\left[\mu\right].$$

Then the first assertion (54) also follows.
(d) This is almost identical to that of Theorem 1.2(d), following from the fact that

$$W^{\mu} \leq \log \frac{1}{A} \text{ in } \operatorname{supp}[\mu].$$

□

Proof of Theorem 1.5. Assume that $\mathcal{S}, \mu$ and A are as in the beginning of this section. Assume that $\mathcal{S}^{\#}$, $\mu^{\#}$, $A^{\#}$ satisfy analogous hypotheses. We shall show that

$$A = A^{\#} \text{ and } \mu = \mu^{\#}.$$

Our hypothesis on the zeros shows that

$$\operatorname{supp}[\mu] = \operatorname{supp}\left[\mu^{\#}\right] = \mathcal{K}.$$

Then Lemma 3.3 shows that

$$W^{\mu} = \log \frac{1}{A} \text{ q.e. in } \mathcal{K}$$

and

$$W^{\mu^{\#}} = \log \frac{1}{A^{\#}} \text{ q.e. in } \mathcal{K}.$$

Since $I(\mu)$ and $I\left(\mu^{\#}\right)$ are finite by Lemma 3.3, these last statements also hold μ a.e. and $\mu^{\#}$ a.e. in $\mathcal{K}$. Then

$$\log \frac{1}{A} = \int W^{\mu} d\mu^{\#} = \int W^{\mu^{\#}} d\mu = \log \frac{1}{A^{\#}}.$$

It follows that there is a unique number A that is the limit of $I_n^{1/n}$ as $n \to \infty$. Next,

$$\begin{aligned} J\left(\mu - \mu^{\#}\right) &= J(\mu) + J\left(\mu^{\#}\right) - 2\int W^{\mu} d\mu^{\#} \\ &= \log \frac{1}{A} + \log \frac{1}{A} - 2\log \frac{1}{A} = 0. \end{aligned}$$

As in Theorem 1.2(a), this then gives

$$\mu = \mu^{\#}.$$

This proof also shows that μ is the unique solution of the integral equation

$$W^{\mu} = C \text{ q.e. in } \mathcal{K}.$$

□

We turn to the

Proof of Theorem 1.3. Let μ be a weak limit of some subsequence $\{\mu_n\}_{n\in\mathcal{S}}$ of $\{\mu_n\}_{n=1}^{\infty}$. We may also assume that (50) holds. From Lemma 3.3, μ has finite logarithmic energy, and from Lemma 3.2,

$$W^{\mu} \geq \log \frac{1}{A} \text{ q.e. in } [0,1].$$

Moreover, by Theorem 1.2(c) and our hypothesis (21),

$$W^{\nu_\psi} = J^* \text{ q.e. in } [0,1].$$

Then the last relations also hold μ a.e. and ν_ψ a.e., so

$$J^* = \int W^{\nu_\psi} d\mu = \int W^{\mu} d\nu_\psi \geq \log \frac{1}{A}.$$

Moreover, by Lemma 3.3(c),

$$W^{\mu} = \log \frac{1}{A} \ \mu \text{ a.e. in supp}\,[\mu]$$

so

$$J(\mu) = \int W^{\mu} d\mu = \log \frac{1}{A} \leq J^*.$$

Then necessarily

$$\log \frac{1}{A} = J(\mu) = J^*$$

and

$$\mu = \nu_\psi.$$

□

Proof of Theorem 1.4. Assume first that ψ'' is continuous in $(0,1)$ and that for each x, $t \in [0,1]$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2} K(x,t) > 0,$$

but that the support is not all of $[0,1]$. We already know that $[0,\varepsilon] \cup [1-\varepsilon,1] \subset \text{supp}\,[\nu_\psi]$ for some $\varepsilon > 0$. Then there exist $0 < a < b < 1$ such that

$$(a,b) \cap \text{supp}\,[\nu_\psi] = \varnothing. \tag{56}$$

We may assume that both

$$a, b \in \operatorname{supp}[\nu_\psi]. \tag{57}$$

Then by Theorem 1.2(c),

$$W^{\nu_\psi}(a) \leq J^* \text{ and } W^{\nu_\psi}(b) \leq J^*.$$

But in (a, b), which lies outside the support of μ, W^μ will be twice continuously differentiable, and by our hypothesis,

$$\frac{\partial^2}{\partial x^2} W^{\nu_\psi}(x) = \int \frac{\partial^2}{\partial x^2} K(x, t)\, d\nu_\psi(t) > 0.$$

The convexity of W^{ν_ψ} forces in some $(c, d) \subset (a, b)$

$$W^\mu < J^*.$$

This contradicts Theorem 1.2(b).

Next, suppose that for $x, t \in (\psi(0), \psi(1))$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2}\left[K\left(\psi^{[-1]}(x), \psi^{[-1]}(t)\right)\right] > 0.$$

Consider

$$\begin{aligned} W^{\nu_\psi} \circ \psi^{[-1]}(x) &= \int K\left(\psi^{[-1]}(x), t\right) d\nu_\psi(t) \\ &= \int K\left(\psi^{[-1]}(x), \psi^{[-1]}(s)\right) d\nu_\psi \circ \psi^{[-1]}(s). \end{aligned}$$

We have

$$W^{\nu_\psi} \circ \psi^{[-1]}(x) \leq J^* \quad \text{ if } x \in \psi(\operatorname{supp}[\nu_\psi])$$

and at each point of continuity of $W^{\nu_\psi} \circ \psi^{[-1]}$, Theorem 1.2(b) gives

$$W^{\nu_\psi} \circ \psi^{[-1]}(x) \geq J^*.$$

We also see that for $x \in [\psi(0), \psi(1)] \backslash \psi(\operatorname{supp}[\nu_\psi])$,

$$\frac{\partial^2}{\partial x^2}\left[W^{\nu_\psi} \circ \psi^{[-1]}(x)\right] = \int \frac{\partial^2}{\partial x^2}\left[K\left(\psi^{[-1]}(x), \psi^{[-1]}(s)\right)\right] d\nu_\psi \circ \psi^{[-1]}(s) > 0.$$

If $0 < a < b < 1$ and (56), (57) hold, then by Theorem 1.1(c),

$$W^{\nu_\psi} \circ \psi^{[-1]}(\psi(a)) \leq J^* \text{ and } W^{\nu_\psi} \circ \psi^{[-1]}(\psi(b)) \leq J^*$$

so in some interval

$$(c,d) \subset (\psi(a), \psi(b)),$$

the convexity gives

$$W^{\nu_\psi} \circ \psi^{[-1]} < J^*$$

But then

$$W^{\nu_\psi} < J^* \text{ in } (\psi(c), \psi(d)),$$

contradicting Theorem 1.2(b). □

Proof of Theorem 1.6. Recall from (45) that

$$I_n = \int_0^1 S_n Q_n \circ \psi$$

and

$$|S_n(x) Q_n \circ \psi(x)|^{1/n} = \exp(-W^{\nu_n}(x)).$$

Then much as in the proof of Lemma 3.1, 3.2, under the hypotheses (48)–(50), we obtain

$$W^\nu \leq \log \frac{1}{A} \text{ in } \operatorname{supp}[\nu]$$

and

$$W^\nu \geq \log \frac{1}{A} \text{ q.e. in } [0,1],$$

in particular at every point of continuity of W^ν. Then the proof of Theorem 1.3 shows that $\nu = \nu_\psi$, and the result follows. □

We next prove an inequality for I_n, assuming the hypotheses (35)–(36). Below, if α, β are probability measures on $[0,1]$, we set

$$m_{\alpha,\beta} := \inf_{[0,1]} W^{\alpha,\beta}.$$

Proof of Theorem 1.7. Let β be a probability measure on $[0,1]$. By orthogonality, for any monic polynomial Π_n of degree n, we have

$$I_n = \int_0^1 P_n(x) \Pi_n \circ \psi(x)\, dx.$$

Given a probability measure on $[0,1]$, we may choose a sequence of polynomials Π_n such that Π_n has n simple zeros in $[\psi(0), \psi(1)]$, and the corresponding zero counting measures converge weakly to $\beta \circ \psi^{[-1]}$ as $n \to \infty$.

(This follows easily as pure jump measures are dense in the set of probability measures.) As

$$W^{\mu,\beta} \geq m_{\mu,\beta} \text{ in the closed set } [0,1],$$

we obtain, by Lemma 2.1,

$$\limsup_{n\to\infty, n\in\mathcal{S}} |P_n(x)\,\Pi_n \circ \psi(x)|^{1/n} \leq \exp(-m_{\mu,\beta}),$$

uniformly in $[0,1]$. Then

$$A = \limsup_{n\to\infty, n\in\mathcal{S}} I_n^{1/n} \leq \exp(-m_{\mu,\beta}).$$

Taking sup's over all such β gives (38). The other relation follows similarly, because of the duality identity (32). □

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