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ON MARCINKIEWICZ-ZYGMUND INEQUALITIES AT HERMITE ZEROS AND THEIR AIRY FUNCTION COUSINS

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Abstract. We establish forward and converse Marcinkiewicz-Zygmund Inequalities at the zeros $\{a_j\}_{j\geq 1}$ of the Airy function $Ai(x)$, such as

$$A\frac{\pi^2}{6}\sum_{k=1}^{\infty}\frac{|f(a_k)|^p}{Ai'(a_k)^2}\leq\int_{-\infty}^{\infty}|f(t)|^p\,dt\leq B\frac{\pi^2}{6}\sum_{k=1}^{\infty}\frac{|f(a_k)|^p}{Ai'(a_k)^2}$$

under appropriate conditions on the entire function f and p. The constants A and B are those appearing in Marcinkiewicz-Zygmund inequalities at zeros of Hermite polynomials. Scaling limits are used to pass from the latter to the former.

1. Introduction

There is a close relationship between the Plancherel-Polya and Marcinkiewicz-Zygmund inequalities. The former [9, p. 152] assert that for $1<p<\infty$, and entire functions f of exponential type at most π,

$$A_p\sum_{k=-\infty}^{\infty}|f(k)|^p\leq\int_{-\infty}^{\infty}|f|^p\leq B_p\sum_{j=-\infty}^{\infty}|f(k)|^p, \tag{1.1}$$

provided either the series or integral is finite. For $0<p\leq 1$, the left-hand inequality is still true, but the right-hand inequality requires additional restrictions [2]. We assume that B_p is taken as small as possible, and A_p as large as possible. The Marcinkiewicz-Zygmund inequalities assert [35, Vol. II, p. 30] that for $p>1, n\geq 1$, and polynomials P of degree $\leq n-1$,

$$\frac{A_p'}{n}\sum_{k=1}^{n}\left|P\left(e^{2\pi ik/n}\right)\right|^p\leq\int_0^1\left|P\left(e^{2\pi it}\right)\right|^p dt\leq\frac{B_p'}{n}\sum_{k=1}^{n}\left|P\left(e^{2\pi ik/n}\right)\right|^p. \tag{1.2}$$

Here too, A_p' and B_p' are independent of n and P, and the left-hand inequality is also true for $0<p\leq 1$ [15]. The author [16] proved that the inequalities (1.1) and (1.2) are equivalent, in the sense that each implies the other. Moreover, the sharp constants are the same:

Theorem A
For $0<p<\infty$, $A_p=A_p'$ and for $1<p<\infty$, $B_p=B_p'$.

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These inequalities are useful in studying convergence of Fourier series, Lagrange interpolation, in number theory, and weighted approximation. They have been extended to many settings, and there are a great many methods to prove them [5], [8], [13], [15], [20], [19], [22], [23], [24], [25], [30], [33], [34]. The sharp constants in (1.1) and (1.2) are unknown, except for the case $p=2$, where of course we have equality rather than inequality, so that $A_2=B_2=A_2'=B_2'=1$ [9, p. 150]. It is certainly of interest to say more about these constants.

In a recent paper, we explored the connections between Marcinkiewicz-Zygmund inequalities at zeros of Jacobi polynomials, and Polya-Plancherel type inequalities at zeros of Bessel functions. Let $\alpha,\beta>-1$ and

$$w^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta},\ x\in(-1,1).$$

For $n\geq 1$, let $P_n^{\alpha,\beta}$ denote the standard Jacobi polynomial of degree n, so that it has degree n, satisfies the orthogonality conditions

$$\int_{-1}^{1}P_n^{\alpha,\beta}(x)\,x^k w^{\alpha,\beta}(x)\,dx=0,\ 0\leq k<n,$$

and is normalized by $P_n^{\alpha,\beta}(1)=\binom{n+\alpha}{n}$. Let

$$x_{nn}<x_{n-1,n}<...<x_{1n}$$

denote the zeros of $P_n^{\alpha,\beta}$. Let $\{\lambda_{kn}\}$ denote the weights in the Gauss quadrature for $w^{\alpha,\beta}$, so that for all polynomials P of degree $\leq 2n-1$,

$$\int_{-1}^{1}Pw^{\alpha,\beta}=\sum_{k=1}^{n}\lambda_{kn}P(x_{kn}).$$

There is a classical analogue of (1.2), established for special α,β by Richard Askey, and for all $\alpha,\beta>-1$ (and for more general "generalized Jacobi weights") by P. Nevai, and his collaborators [15], [20], [27], [29], with later work by König and Nielsen [8], and for doubling weights by Mastroianni and Totik [23]. The following special case follows from Theorem 5 in [20, eqn. (1.19), p. 534]:

Theorem B
Let α,β,τ,σ satisfy $\alpha,\beta,\alpha+\sigma,\beta+\tau>-1$. Let $p>0$. For $n\geq 1$, let $\{x_{kn}\}$ denote the zeros of the Jacobi polynomial $P_n^{\alpha,\beta}$ and $\{\lambda_{kn}\}$ denote the corresponding Gauss quadrature weights. There exists $A>0$ such that for $n\geq 1$, and polynomials P of degree $\leq n-1$,

$$A\sum_{k=1}^{n}\lambda_{kn}\left|P(x_{kn})\right|^{p}(1-x_{kn})^{\sigma}(1+x_{kn})^{\tau}\leq\int_{-1}^{1}\left|P(x)\right|^{p}(1-x)^{\alpha+\sigma}(1+x)^{\beta+\tau}dx.$$

(1.3)

The converse inequality is much more delicate, and in particular holds only for $p>1$, and even then only for special cases of the parameters. It too was investigated by P. Nevai, with later work by Yuan Xu [33], [34], König and Nielsen [8]. König and Nielsen gave the exact range of p for which

$$\int_{-1}^{1}\left|P(x)\right|^{p}(1-x)^{\alpha}(1+x)^{\beta}dx\leq B\sum_{k=1}^{n}\lambda_{kn}\left|P(x_{kn})\right|^{p}, \quad (1.4)$$

holds with B independent of n and P. Let

$$\begin{aligned} \mu(\alpha,\beta) &= \max\left\{1, 4\frac{\alpha+1}{2\alpha+5}, 4\frac{\beta+1}{2\beta+5}\right\}; \\ m(\alpha,\beta) &= \max\left\{1, 4\frac{\alpha+1}{2\alpha+3}, 4\frac{\beta+1}{2\beta+3}\right\}; \\ M(\alpha,\beta) &= \frac{m(\alpha,\beta)}{m(\alpha,\beta)-1}. \end{aligned} \tag{1.5}$$

Then (1.4) holds for all n and P iff

$$\mu(\alpha,\beta) < p < M(\alpha,\beta). \tag{1.6}$$

The most general sufficient condition for a converse quadrature inequality is due to Yuan Xu [33, pp. 881-882]. When we restrict to Jacobi weights, with the same weight on both sides, the inequality takes the following form:

Theorem C
Let $\alpha, \beta, \tau, \sigma$ satisfy $\alpha, \beta, \alpha+\sigma, \beta+\tau > -1$. Let $p > 1$, $q = \frac{p}{p-1}$, and assume that

$$\frac{p}{2}\left(\alpha+\frac{1}{2}\right) - (\alpha+1) < \sigma < (p-1)(\alpha+1) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\}. \tag{1.7}$$

$$\frac{p}{2}\left(\beta+\frac{1}{2}\right) - (\beta+1) < \tau < (p-1)(\beta+1) - \max\left\{0, \frac{p}{2}\left(\beta+\frac{1}{2}\right)\right\}. \tag{1.8}$$

Then there exists $B > 0$ such that for $n \geq 1$, and polynomials P of degree $\leq n-1$,

$$\int_{-1}^{1} |P(x)|^p (1-x)^{\alpha+\sigma} (1+x)^{\beta+\tau}\, dx \leq B \sum_{k=1}^{n} \lambda_{kn} |P(x_{kn})|^p (1-x_{kn})^{\sigma} (1+x_{kn})^{\tau}. \tag{1.9}$$

Inequalities of the type (1.9) for doubling weights have been established by Mastroianni and Totik [23] under the additional condition that one needs to restrict the degree of P in (1.9) further, such as $\deg(P) \leq \eta n$ for some $\eta \in (0,1)$ depending on the particular doubling weight.

Now let $\alpha > -1$ and define the Bessel function of order α,

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(k+\alpha+1)} \tag{1.10}$$

and

$$J_\alpha^*(z) = J_\alpha(z)/z^\alpha, \tag{1.11}$$

which has the advantage of being an entire function for all $\alpha > -1$. J_α^* has real simple zeros, and we denote the positive zeros by

$$0 < j_1 < j_2 < \ldots$$

while for $k \geq 1$,

$$j_{-k} = -j_k.$$

The connection between Jacobi polynomials and Bessel functions is given by the classical Mehler-Heine asymptotic, which holds uniformly for z in compact subsets of $\mathbb{C}$ [32, p. 192]:

$$\lim_{n\to\infty} n^{-\alpha} P_n^{\alpha,\beta}\left(1-\frac{1}{2}\left(\frac{z}{n}\right)^2\right) = \lim_{n\to\infty} n^{-\alpha} P_n^{\alpha,\beta}\left(\cos\frac{z}{n}\right) = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) = 2^\alpha J_\alpha^*(z). \tag{1.12}$$

There is an extensive literature dealing with quadrature sums and Lagrange interpolation at the $\{j_k\}$. In particular, there is the quadrature formula [6, p. 49]

$$\int_{-\infty}^{\infty} |x|^{2\alpha+1} f(x)\, dx = \frac{2}{\tau^{2\alpha+2}} \sum_{k=-\infty,k\neq 0}^{\infty} \frac{1}{\left|J_\alpha^{*\prime}(j_k)\right|^2} f\left(\frac{j_k}{\tau}\right),$$

valid for all entire functions f of exponential type at most 2τ, for which the integral on the left-hand side is finite. That same paper contains the following converse Marcinkiewicz-Zygmund type inequality: let $\alpha \geq -\frac{1}{2}$ and $p > 1$; or $-1 < \alpha < -\frac{1}{2}$ and $1 < p < \frac{2}{|1+2\alpha|}$. Then for entire functions f of exponential type $\leq \tau$ for which $|x|^{\alpha+\frac{1}{2}} f(x) \in L_p(\mathbb{R}\setminus(-\delta,\delta))$, for some $\delta > 0$, [6, Lemma 14, p. 58; Lemma 13, p. 57]

$$\int_{-\infty}^{\infty} \left| |x|^{\alpha+\frac{1}{2}} f(x) \right|^p dx \leq \frac{B^*}{\tau} \sum_{k=-\infty,k\neq 0}^{\infty} \left| \frac{1}{\tau^{\alpha+\frac{1}{2}} J_\alpha^{*\prime}(j_k)} f\left(\frac{j_k}{\tau}\right) \right|^p. \tag{1.13}$$

Here B^* depends on α and p. In the converse direction, since $j_{k+1} - j_k$ is bounded below by a positive constant for all k, classical inequalities from the theory of entire functions [9, p. 150] show that

$$\sum_{k=-\infty,k\neq 0}^{\infty} |f(j_k)|^p \leq C \int_{-\infty}^{\infty} |f(x)|^p\, dx$$

for entire functions of finite exponential type for which the right-hand side is finite.

While Grozev and Rahman note the analogous nature of Lagrange interpolation at zeros of Jacobi polynomials and Bessel functions, and also the Mehler-Heine formula, their proofs proceed purely from properties of Bessel functions. In [17, Thms. 1.1, 1.3, pp. 227-228], the author used inequalities like (1.3) to pass to analogues for Bessel functions using scaling limits of the form (1.12), keeping the same constants, much as was done in [16]: Let $L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$ denote the space of all even entire functions f of exponential type ≤ 1 with

$$\int_0^\infty |f(t)|^p\, t^{2\alpha+2\sigma+1} dt < \infty.$$

Theorem D

Assume that $p > 0$, $\alpha, \beta, \alpha+\sigma, \beta+\tau > -1$, and

$$-p\left(\frac{\alpha}{2}+\frac{5}{4}\right) + \alpha + \sigma + 1 < 0.$$

Let A be as in Theorem B. Then

$$2A \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \leq \int_0^\infty |f(t)|^p\, t^{2\alpha+2\sigma+1} dt,$$

for all $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$.

Theorem E
Assume that $p > 1$, $\alpha, \beta, \alpha+\sigma, \beta+\tau > -1$, *and that (1.7) and (1.8) hold. Let* B *be as in Theorem C. Then for* $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$, *we have*

$$\int_0^\infty |f(t)|^p\, t^{2\alpha+2\sigma+1} dt \leq 2B \sum_{k=1}^\infty j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p . \tag{1.14}$$

In particular this holds for $\sigma = \tau = 0$ *if* p *satisfies (1.6) with* $\beta = \alpha$. *Moreover, for any* α, β, p, *it is possible to choose* σ *and* τ *satisfying (1.7), (1.8) so that this last inequality also holds.*

A very recent paper of Littmann [13] provides far reaching extensions of the inequalities of Grozev and Rahman to Hermite-Biehler weights, so that $t^{2\alpha+2\sigma+1}$ is replaced by $1/|E|^p$, where E is a Hermite-Biehler function, that is, an entire function E satisfying $|E(z)| > |E(\bar{z})|$ for $\operatorname{Re} z > 0$. Moreover, the zeros of Bessel functions are replaced by the zeros of $B(z) = \frac{i}{2}\left(E(z) - \overline{E(\bar{z})}\right)$. Littmann then uses these to establish weighted mean convergence of certain interpolation operators for classes of entire functions.

In this paper, we shall use Marcinkiewicz-Zygmund inequalities at zeros of Hermite polynomials, to derive Plancherel-Polya type inequalities at zeros of Airy functions. We begin with our notation. Throughout,

$$W(x) = \exp\left(-\frac{1}{2}x^2\right), x \in \mathbb{R}, \tag{1.15}$$

is the Hermite weight, and $\{p_n\}$ are the orthonormal Hermite polynomials, so that

$$\int_{-\infty}^\infty p_n p_m W^2 = \delta_{mn}. \tag{1.16}$$

The classical Hermite polynomial is of course denoted by H_n. The relationship between p_n and H_n is given by [32, p. 105, (5.5.1)]

$$p_n = \pi^{-1/4} 2^{-n/2} (n!)^{-1/2} H_n. \tag{1.17}$$

The leading coefficient of p_n is [32, p. 106, (5.5.6)]

$$\gamma_n = \pi^{-1/4} 2^{n/2} (n!)^{-1/2} . \tag{1.18}$$

In the sequel, $\{x_{jn}\}$ denote the zeros of the Hermite polynomials in decreasing order:

$$-\infty < x_{nn} < x_{n-1,n} < ... < x_{2n} < x_{1n} < \infty,$$

while $\{\lambda_{jn}\}$ denote the weights in the Gauss quadrature formula: for polynomials P of degree $\leq 2n-1$,

$$\int_{-\infty}^\infty P W^2 = \sum_{j=1}^n \lambda_{jn} P(x_{jn}) .$$

There is an extensive literature on Marcinkiewicz-Zygmund inequalities at zeros of Hermite polynomials, as well as for orthonormal polynomials for more general exponential weights [3], [4], [7], [14], [21], [28], [29]. We shall use the following forward and converse inequalities [14, p. 529], [21, p. 287]:

Theorem F
Let $1 \leq p < \infty$. Let $r, R \in \mathbb{R}$ and $S > 0$.
(a) Then there exists $A > 0$ such that for $n \geq 1$, and polynomials P of degree at most $n + Sn^{1/3}$,
(1.19)
$$\sum_{j=1}^{n} \lambda_{jn} \left|P\left(x_{jn}\right)\right|^{p} W^{p-2}\left(x_{jn}\right)\left(1+\left|x_{jn}\right|\right)^{Rp} \leq A \int_{-\infty}^{\infty} \left|(PW)(x)\left(1+|x|\right)^{R}\right|^{p} dx.$$
(b) Assume that
$$r < 1 - \frac{1}{p};\ r \leq R;\ R > -\frac{1}{p}. \tag{1.20}$$
In addition if $p = 4$, we assume that $r < R$, while if $p > 4$, we assume that
$$r - \min\left\{R, 1 - \frac{1}{p}\right\} + \frac{1}{3}\left(1 - \frac{4}{p}\right)\left\{\begin{array}{ll} \leq 0, & \text{if } R \neq 1 - \frac{1}{p} \\ < 0, & \text{if } R = 1 - \frac{1}{p} \end{array}\right. . \tag{1.21}$$
Then there exists $B > 0$ such that for $n \geq 1$, and polynomials P of degree $\leq n-1$,
(1.22)
$$\int_{-\infty}^{\infty} \left|(PW)(x)\left(1+|x|\right)^{r}\right|^{p} dx \leq B \sum_{j=1}^{n} \lambda_{jn} \left|P\left(x_{jn}\right)\right|^{p} W^{p-2}\left(x_{jn}\right)\left(1+\left|x_{jn}\right|\right)^{Rp}.$$

Recall that the Airy function Ai is given on the real line by [1, 10.4.32, p. 447]
$$Ai(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos\left(\frac{1}{3}t^{3} + xt\right) dt.$$
The Airy function Ai is an entire function of order $\frac{3}{2}$, with only real negative zeros $\{a_j\}$, where
$$0 > a_1 > a_2 > a_3 > \ldots .$$
These are often denoted by $\{i_j\}$ rather than $\{a_j\}$. Ai satisfies the differential equation
$$Ai''(z) - zAi(z) = 0.$$
The Airy kernel $\mathbb{A}i(\cdot,\cdot)$, much used in random matrix theory, is defined [12] by
$$\mathbb{A}i(a,b) = \left\{\begin{array}{ll} \frac{Ai(a)Ai'(b) - Ai'(a)Ai(b)}{a-b}, & a \neq b, \\ Ai'(a)^{2} - aAi(a)^{2}, & a = b. \end{array}\right. .$$
Observe that
$$\mathcal{L}_j(z) = \frac{\mathbb{A}i(z, a_j)}{\mathbb{A}i(a_j, a_j)} = \frac{Ai(z)}{Ai'(a_j)(z - a_j)},$$
is the Airy analogue of a fundamental of Lagrange interpolation, satisfying
$$\mathcal{L}_j(a_k) = \delta_{jk}.$$
There is an analogue of sampling series and Lagrange interpolation series involving $\{\mathcal{L}_j\}$:

Definition 1.1
Let $\mathcal{G}$ be the class of all functions $g : \mathbb{C} \to \mathbb{C}$ with the following properties:

(a) g is an entire function of order at most $\frac{3}{2}$;
(b) There exists $L > 0$ such that for $\delta \in (0, \pi)$, some $C_\delta > 0$, and all $z \in \mathbb{C}$ with $|\arg z| \leq \pi - \delta$,

$$|g(z)| \leq C_\delta (1 + |z|)^L \left| \exp\left(-\frac{2}{3} z^{\frac{3}{2}}\right) \right|;$$

(c)

$$\sum_{j=1}^{\infty} \frac{|g(a_j)|^2}{|a_j|^{1/2}} < \infty. \tag{1.23}$$

In [12, Corollary 1.3, p. 429], it was shown that each $g \in \mathcal{G}$ admits the locally uniformly convergent expansion

$$g(z) = \sum_{j=1}^{\infty} g(a_j) \frac{\mathbb{A}i(z, a_j)}{\mathbb{A}i(a_j, a_j)} = \sum_{j=1}^{\infty} g(a_j) \mathcal{L}_j(z).$$

We let

$$S_M[g] = \sum_{j=1}^{M} g(a_j) \mathcal{L}_j, \ \ M \geq 1, \tag{1.24}$$

denote the Mth partial sum of this expansion. Moreover, for $f, g \in \mathcal{G}$, there is the quadrature formula [12, Corollary 1.4, p. 429]

$$\int_{-\infty}^{\infty} f(x) g(x) \, dx = \sum_{j=1}^{\infty} \frac{(fg)(a_j)}{\mathbb{A}i(a_j, a_j)}.$$

In particular,

$$\int_{-\infty}^{\infty} g^2(x) \, dx = \sum_{j=1}^{\infty} \frac{|g(a_j)|^2}{\mathbb{A}i(a_j, a_j)},$$

and the series on the right converges because of (1.23), and the fact that $\mathbb{A}i(a_j, a_j) = Ai'(a_j)^2$ grows like $j^{1/3}$ - see Lemma 2.2.

Lagrange interpolation at zeros of Airy functions was considered in [18]. We shall need a class of functions that are limits in L_p of the partial sums of the Airy series expansion:

Definition 1.2
Let $0 < p < \infty$ and $f \in L_p(\mathbb{R})$. We write $f \in \mathcal{G}_p$ if

$$\lim_{M \to \infty} \|f - S_M[f]\|_{L_p(\mathbb{R})} = 0.$$

The relationship between Hermite polynomials and Airy functions lies in the asymptotic [32, p. 201],

$$e^{-x^2/2} H_n(x) = 3^{1/3} \pi^{-3/4} 2^{n/2+1/4} (n!)^{1/2} n^{-1/12} \{Ai(-t) + o(1)\} \tag{1.25}$$

as $n \to \infty$, uniformly for

$$x = \sqrt{2n}(1 - 6^{-1/3} (2n)^{-2/3} t), \tag{1.26}$$

and t in compact subsets of $\mathbb{C}$. This follows from the formulation in [32] because of the uniformity. Using this and part (a) of Theorem F with $R = r = 0$, we shall prove:

Theorem 1.3
Let $p \geq 1$. Let A be the constant in (1.19) with $R = r = 0$ there.
(a) Then for $f \in \mathcal{G}_p$, we have

$$\sum_{k=1}^{\infty} \frac{|f(a_k)|^p}{Ai'(a_k)^2} \leq A \frac{6}{\pi^2} \int_{-\infty}^{\infty} |f(t)|^p \, dt. \tag{1.27}$$

(b) In particular, if $p \geq 2$, $f \in \mathcal{G}$ and for some $C > 0$, $\beta > \frac{1}{4}$, we have

$$|f(x)| \leq C (1 + |x|)^{-\beta}, x \in \mathbb{R}, \tag{1.28}$$

then (1.27) is true.

Remark
We expect that (1.27) also holds for $0 < p < 1$, but this would require (1.19) for such p, and that does not seem to appear in the literature.

Using part (b) of Theorem F, we shall prove:

Theorem 1.4
Let $1 < p < 4$. Let B be the constant in (1.22) with $R = r = 0$ there.
(a) For $f \in \mathcal{G}_p$, we have

$$\frac{6}{\pi^2} \int_{-\infty}^{\infty} |f(t)|^p \, dt \leq B \sum_{k=1}^{\infty} \frac{|f(a_k)|^p}{Ai'(a_k)^2}. \tag{1.29}$$

(b) In particular, if $f \in \mathcal{G}$ and

$$\sum_{k=1}^{\infty} \frac{|f(a_k)|^p}{k^{1/3}} < \infty, \tag{1.30}$$

then (1.29) is true.

In the sequel, $C, C_1, C_2, ...$ denote constants independent of n, z, x, t, and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. $[x]$ denotes the greatest integer $\leq x$. Given two sequences $\{x_n\}, \{y_n\}$ of non-zeros real numbers, we write

$$x_n \sim y_n$$

if there exist constants C_1 and C_2 such that

$$C_1 \leq x_n / y_n \leq C_2$$

for $n \geq 1$. Similar notation is used for functions and sequences of functions. We establish some basic estimates and then prove Theorems 1.3 and 1.4 in Section 2.

2. Proof of Theorems 1.3 and 1.4

We start with properties of Hermite polynomials. Throughout $\{p_n\}$ denote the orthonormal Hermite polynomials satisfying (1.16), with leading coefficient γ_n, and with zeros $\{x_{jn}\}$. In the sequel, we let

$$\psi_n(x) = \left|1 - \frac{|x|}{\sqrt{2n}}\right| + n^{-2/3}.$$

We also let

$$K_n(x,y)=\sum_{j=0}^{n-1}p_j(x)\,p_j(y)$$

denote the nth reproducing kernel, and

$$\lambda_n(x)=1/K_n(x,x)$$

denote the nth Christoffel function. In particular, $\lambda_{jn}=\lambda_n(x_{jn})$. The jth fundamental polynomial at the zeros of $p_n(x)$ is

$$\ell_{jn}(x)=\frac{p_n(x)}{p_n'(x_{jn})(x-x_{jn})}.$$

It is also admits the identity

$$\ell_{jn}(x)=\lambda_{jn}K_n(x,x_{jn}). \tag{2.1}$$

Lemma 2.1
(a)

$$\frac{\gamma_{n-1}}{\gamma_n}=\sqrt{\frac{n}{2}}. \tag{2.2}$$

(b) For each fixed j, as $n\to\infty$,

$$x_{jn}=\sqrt{2n}(1-6^{-1/3}(2n)^{2/3}\{|a_j|+o(1)\}). \tag{2.3}$$

(c) Uniformly for t in compact subsets of $\mathbb{C}$, and for

$$x=\sqrt{2n}\left(1-6^{-1/3}(2n)^{-2/3}t\right), \tag{2.4}$$

we have

$$(p_nW)(x)=3^{1/3}\pi^{-1}2^{1/4}n^{-1/12}\{Ai(-t)+o(1)\}. \tag{2.5}$$

(d) For each fixed j, as $n\to\infty$,

$$(p_n'W)(x_{jn})=3^{2/3}\pi^{-1}2^{3/4}n^{1/12}\{Ai'(a_j)+o(1)\}. \tag{2.6}$$

(e) For each fixed j, and uniformly for t in compact subsets of $\mathbb{C}$, and x of the form (2.4)

$$\lim_{n\to\infty}(\ell_{jn}W)(x)\,W^{-1}(x_{jn})=\mathcal{L}_j(-t). \tag{2.7}$$

(f) For all $1\le j\le n$ and all $x\in\mathbb{R}$,

$$|\ell_{jn}W|(x)\,W^{-1}(x_{jn})\le C\left(\frac{\psi_n(x)}{\psi_n(x_{jn})}\right)^{1/4}\frac{1}{1+n^{1/2}\psi_n(x)^{1/2}|x-x_{jn}|}. \tag{2.8}$$

(g) In particular for fixed j, and $n\ge n_0(j)$ and all $x\in\mathbb{R}$,

$$|\ell_{jn}W|(x)\,W^{-1}(x_{jn})\le C\frac{n^{1/6}\psi_n(x)^{1/4}}{1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right|}. \tag{2.9}$$

(h) For each fixed j,

$$\lambda_{jn}^{-1}W^2(x_{jn})=3^{4/3}\pi^{-2}2^{3/2}n^{1/6}Ai'(a_j)^2(1+o(1)). \tag{2.10}$$

Proof

(a) This follows from (1.18).

(b) See [32, p. 132, (6.32.5)]. We note that Szego uses $Ai(-x)$ as the Airy function, so there zeros are positive there. Moreover there the symbol i_j is used for $|a_j|$.

(c) This follows from (1.25) and (1.17).

(d) Because of the uniform convergence, we can differentiate the relation (2.5) : uniformly for t in compact sets,

$$W(x)\{-xp_n(x)+p_n'(x)\}\frac{dx}{dt}=3^{1/3}\pi^{-1}2^{1/4}n^{-1/12}\{-Ai'(-t)+o(1)\}$$

so setting $x=x_{jn}$ and using (2.4), we obtain (2.6).

(e) From (2.3-2.6),

$$\begin{aligned}(\ell_{jn}W)(x)W^{-1}(x_{jn}) &= \frac{(p_nW)(x)}{(p_n'W)(x_{jn})(x-x_{jn})}\\ &= \frac{3^{1/3}\pi^{-1}2^{1/4}n^{-1/12}\{Ai(-t)+o(1)\}}{3^{2/3}\pi^{-1}2^{3/4}n^{1/12}\{Ai'(a_j)+o(1)\}\left(-6^{-1/3}(2n+1)^{-1/6}(t-|a_j|+o(1))\right)}\\ &= \frac{Ai(-t)}{Ai'(a_j)(-t-a_j)}(1+o(1))=\mathcal{L}_j(-t)+o(1).\end{aligned}$$

(f) We note the following estimates [10, p. 465-467]: uniformly for $n\geq 1$ and $x\in\mathbb{R}$,

$$n^{1/4}|p_n(x)|W(x)\leq C\psi_n(x)^{-1/4}. \tag{2.11}$$

Note that for the Hermite weight, the Mhaskar-Rakhmanov number is $a_n=\sqrt{2n}$. We have uniformly for $n\geq 1$ and $x\in\left[-\sqrt{2n},\sqrt{2n}\right]$,

$$\lambda_n(x)\sim\frac{W^2(x)}{\sqrt{n}}\psi_n(x)^{-1/2}, \tag{2.12}$$

while for all $x\in(-\infty,\infty)$,

$$\lambda_n(x)\geq C\frac{W^2(x)}{\sqrt{n}}\psi_n(x)^{1/2}. \tag{2.13}$$

Also uniformly for $1\leq k\leq n$,

$$|p_{n-1}W|(x_{kn})\sim n^{-1/4}\psi_n(x_{kn})^{-1/4} \tag{2.14}$$

and

$$|p_n'W|(x_{kn})\sim n^{1/4}\psi_n(x_{kn})^{1/4}. \tag{2.15}$$

Hence

$$|\ell_{jn}W|(x)W^{-1}(x_{jn})=\frac{|p_nW|(x)}{|p_n'W|(x_{jn})|x-x_{jn}|}\leq C\frac{n^{-1/4}\psi_n(x)^{-1/4}}{n^{1/4}\psi_n(x_{jn})^{1/4}|x-x_{jn}|}.$$

Next by Cauchy-Schwarz, and then (2.12), (2.13),

$$\begin{aligned}|\ell_{jn}W|(x)W^{-1}(x_{jn}) &= \lambda_{jn}W^{-1}(x_{jn})W(x)|K_n(x,x_{jn})|\\ &\leq \lambda_{jn}W^{-1}(x_{jn})W(x)(K_n(x,x)K_n(x_{jn},x_{jn}))^{1/2}\\ &= \left(\lambda_{jn}W^{-2}(x_{jn})\right)^{1/2}\left(\lambda_n(x)W^{-2}(x)\right)^{-1/2}\\ &\leq C\psi_n(x_{jn})^{-1/4}\psi_n(x)^{1/4}.\end{aligned}$$

Thus combining the two estimates,

$$\left|\ell_{jn}W\right|(x)\,W^{-1}(x_{jn}) \leq C\left(\frac{\psi_n(x)}{\psi_n(x_{jn})}\right)^{1/4}\min\left\{1, \frac{1}{n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right|}\right\},$$

which can be recast as (2.8).

(g) First note that as $\left|1-\frac{x_{jn}}{\sqrt{2n}}\right| \leq Cn^{-2/3}$, we have $\psi_n(x_{jn}) \sim n^{-2/3}$. We have to show that uniformly in n and for $x \in \mathbb{R}$,

$$1+n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right| \sim 1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right|. \tag{2.16}$$

Let L be some large positive number. If firstly $\left|x-\sqrt{2n}\right| \geq L\sqrt{2n}n^{-2/3}$, then from (2.3),

$$\left|\frac{x-x_{jn}}{x-\sqrt{2n}}-1\right| = \frac{\left|x_{jn}-\sqrt{2n}\right|}{\left|x-\sqrt{2n}\right|} \leq \frac{C\sqrt{2n}n^{-2/3}}{L\sqrt{2n}n^{-2/3}}$$

so that

$$\left|\frac{x-x_{jn}}{x-\sqrt{2n}}\right| \leq 1+C/L,$$

so that

$$1+n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right| \leq C\left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right|\right).$$

Also, for some C_1 independent of L,

$$\begin{aligned}
&1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right| \\
\leq\;& 1+n^{1/2}\psi_n(x)^{1/2}\left(\left|x-x_{jn}\right|+C_1\sqrt{2n}n^{-2/3}\right) \\
\leq\;& \left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right|\right)+n^{1/2}\psi_n(x)^{1/2}\frac{C_1}{L}\left|x-\sqrt{2n}\right| \\
\leq\;& \left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right|\right)+\frac{C_1}{L}\left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right|\right)
\end{aligned}$$

so that

$$\left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right|\right)\left(1-\frac{C_1}{L}\right) \leq \left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right|\right).$$

Then we have (2.16) if L is large enough. Next, if $\left|x-\sqrt{2n}\right| < L\sqrt{2n}n^{-2/3}$, $\psi_n(x) \sim n^{-2/3}$ and then

$$\begin{aligned}
1 \;&\leq\; 1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right| \\
&\leq\; 1+Cn^{1/2}n^{-1/3}\sqrt{2n}n^{-2/3} \\
&\leq\; C_2 \leq C_2\left(1+n^{1/2}\psi_n(x)^{1/2}\left|x-x_{jn}\right|\right).
\end{aligned}$$

Again we have (2.16).

(h) We use the confluent form of the Christoffel-Darboux formula:

$$\lambda_{jn}^{-1} = \frac{\gamma_{n-1}}{\gamma_n}p_n'(x_{jn})\,p_{n-1}(x_{jn})$$

Here since [32, p. 106, (5.5.10)], $H_n'(x) = 2nH_{n-1}(x)$ so from (1.17),

$$p_n'(x) = \sqrt{2n}p_{n-1}(x).$$

Together with (2.2) this gives

$$\lambda_{jn}^{-1} = p_n' \left(x_{jn}\right)^2 .$$

Then (2.10) follows from (2.6). ■

Next, we record some estimates involving the Airy function:

Lemma 2.2
(a) For $x \in [0, \infty)$,

$$|Ai\,(x)| \leq C\,(1+x)^{-1/4} \exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right); \tag{2.17}$$

$$|Ai\,(-x)| \leq C\,(1+x)^{-1/4} . \tag{2.18}$$

(b) As $x \to \infty$,

$$Ai'\,(-x) = -\pi^{-1/2}x^{1/4}\left[\cos\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right) + O\left(x^{-3/2}\right)\right]. \tag{2.19}$$

(2.20)

$$Ai'\,(a_j) = (-1)^{j-1}\,\pi^{-1/2}\left(\frac{3\pi}{8}\,(4j-1)\right)^{1/6}\left(1+O\left(j^{-2}\right)\right) = (-1)^{j-1}\,\pi^{-1/2}\,|a_j|^{1/4}\,(1+o\,(1)) .$$

(c)

$$a_j = -\left[3\pi\,(4j-1)\,/8\right]^{2/3}\left(1+O\left(\frac{1}{j^2}\right)\right) = -\left(\frac{3\pi j}{2}\right)^{2/3}(1+o\,(1)) . \tag{2.21}$$

(d)

$$|a_j| - |a_{j-1}| = \pi\,|a_j|^{-1/2}\,(1+o\,(1)) . \tag{2.22}$$

(d) For $j \geq 1$ and $t \in [0, \infty)$,

$$|\mathcal{L}_j\,(t)| \leq Cj^{-5/6}\,(1+t)^{-1/4} \exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right) \tag{2.23}$$

and

$$|\mathcal{L}_j\,(-t)| \leq \frac{C}{1+(1+t)^{1/4}\,|a_j|^{1/4}\,|t-|a_j||}. \tag{2.24}$$

Proof
(a) The following asymptotics and estimates for Airy functions are listed on pages 448-449 of [1]: see (10.4.59-61) there.

$$Ai\,(x) = \frac{1}{2\pi^{1/2}}x^{-1/4}\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)(1+o\,(1)\,, x \to \infty;$$

$$Ai\,(-x) = \pi^{-1/2}x^{-1/4}\left[\sin\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right) + O\left(x^{-\frac{3}{2}}\right)\right],\ x \to \infty.$$

Then (2.17) and (2.18) follow as Ai is entire.
(b), (c), (d) The zeros $\{a_j\}$ of Ai satisfy [1, p. 450, (10.4.94,96)]

$$a_j = -\left[3\pi\,(4j-1)\,/8\right]^{2/3}\left(1+O\left(\frac{1}{j^2}\right)\right) = -\left(\frac{3\pi j}{2}\right)^{2/3}(1+o\,(1)) .$$

$$Ai'(a_j) = (-1)^{j-1}\pi^{-1/2}\left(\frac{3\pi}{8}(4j-1)\right)^{1/6}\left(1+O\left(j^{-2}\right)\right) = (-1)^{j-1}\pi^{-1/2}|a_j|^{1/4}(1+o(1)).$$

Then (2.22) also follows, as was shown in [12, p. 431, eqn. (2.7)].
(d) We first prove (2.24). For $t \in [0,\infty)$,

$$\begin{aligned}|\mathcal{L}_j(-t)| &= \left|\frac{Ai(-t)}{Ai'(a_j)(-t-a_j)}\right| \\ &\leq \frac{C(1+t)^{-1/4}}{j^{1/6}\,|t-|a_j||}\end{aligned}$$

by (2.18), (2.20). If $(1+t)^{1/4} j^{1/6}|t-|a_j|| \geq \frac{1}{2}|a_1|$, we then obtain (2.24). In the contrary case,

$$\begin{aligned}&(1+t)^{1/4} j^{1/6}|t-|a_j|| < \frac{1}{2}|a_1| \\ &\Rightarrow |t-|a_j|| < \frac{1}{2}|a_1| \leq \frac{1}{2}|a_j|.\end{aligned}$$

We then for some ξ between $-t$ and a_j, from (2.19),

$$|\mathcal{L}_j(t)| = \left|\frac{Ai'(\xi)}{Ai'(a_j)}\right| \leq C\left(\frac{|\xi|}{|a_j|}\right)^{1/4} \leq C.$$

We again obtain (2.24). Next, for $t \in (0,\infty)$, we have from (2.18), (2.20),

$$\begin{aligned}|\mathcal{L}_j(t)| &= \left|\frac{Ai(t)}{Ai'(a_j)(t-a_j)}\right| \\ &\leq \frac{C(1+t)^{-1/4}}{j^{1/6}|a_j|}\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right) \\ &\leq Cj^{-5/6}(1+t)^{-1/4}\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right).\end{aligned}$$

■

Next, we record a restricted range inequality:

Lemma 2.3
Let $\eta \in (0,1)$, $0 < p < \infty$. There exists B, n_0 such that for $n \geq n_0$ and polynomials P of degree $\leq n + n^{1/3}$,

$$\|PW\|_{L_p(\mathbb{R})} \leq (1+\eta)\|PW\|_{L_p[-D_n,D_n]}, \tag{2.25}$$

where

$$D_n = \sqrt{2n}\left(1+Bn^{-2/3}\right).$$

Proof
It suffices to prove that

$$\|PW\|_{L_p(\mathbb{R}\setminus[-D_n,D_n])} \leq \eta\|PW\|_{L_p[-D_n,D_n]}. \tag{2.26}$$

For $p \geq 1$, the triangle inequality then yields (2.25). For $p < 1$, we can use the triangle inequality on the integral inside the norm and then just reduce the size of η appropriately. Let $m = m(n) = n + n^{1/3}$. It follows from Theorem 4.2(b) in [11, p. 96] that for $B \geq 0$, P of degree $\leq m$,
(2.27)

$$\|PW\|_{L_p\left(\mathbb{R}\setminus[-\sqrt{2m}\left(1+\frac{1}{2}Bm^{-2/3}\right),\sqrt{2m}\left(1+\frac{1}{2}Bm^{-2/3}\right))\right)} \leq C_1\exp\left(-C_2B^{3/2}\right)\|PW\|_{L_p\left[-\sqrt{2m},\sqrt{2m}\right]}.$$

Here C_1 and C_2 are independent of m, P, B. Choose $B \geq 2$ so large that

$$C_1 \exp\left(-C_2 B^{3/2}\right) \leq \eta. \tag{2.28}$$

Now

$$\begin{aligned} &\sqrt{2m}\left(1+\frac{1}{2}Bm^{-2/3}\right)/D_n \\ = &\sqrt{\frac{m}{n}\frac{1+\frac{1}{2}Bm^{-2/3}}{1+Bn^{-2/3}}} \\ \leq &\sqrt{1+n^{-2/3}\frac{1+\frac{1}{2}Bn^{-2/3}}{1+Bn^{-2/3}}} \leq 1 \end{aligned}$$

for $n \geq n_0(B)$ as $B \geq 2$. Then also $\sqrt{2m}/D_n \leq 1$, and

$$\mathbb{R}\backslash[-\sqrt{2m}\left(1+\frac{1}{2}Bm^{-2/3}\right), \sqrt{2m}\left(1+\frac{1}{2}Bm^{-2/3}\right)] \supseteq \mathbb{R}\backslash\left[-D_n, D_n\right]$$

and (2.26) follows from (2.27) and (2.28). ■

Following is the main part of the proof of Theorem 1.3:

Lemma 2.4
Fix $M \geq 1$ and let

$$P(x) = \sum_{k=1}^{M} c_k \mathcal{L}_k(x). \tag{2.29}$$

Then

$$\sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2} \leq A\frac{6}{\pi^2}\int_{-\infty}^{\infty} |P(t)|^p \, dt. \tag{2.30}$$

Here A is the constant in (1.19) with $R = r = 0$.

Proof
Choose $\eta \in (0,1)$ and D_n, B as in the above lemma. Let

$$R_n(x) = U_n(x) \sum_{k=1}^{M} c_k \ell_{kn}(x) W^{-1}(x_{kn}). \tag{2.31}$$

Here we set

$$U_n(x) = \left(\frac{T_m\left(\frac{x}{D_n}\right) - T_m(1)}{m^2\left(\frac{x}{D_n} - 1\right)}\right)^L, \tag{2.32}$$

where T_m is the usual Chebyshev polynomial, L is some large enough even positive integer, and $m = \left[\frac{\varepsilon}{L}n^{1/3}\right]$, while $\varepsilon \in (0,1)$. Since R_n has degree $\leq n + n^{1/3}$, we have by Lemma 2.3, at least for large enough n, that

$$\|R_n W\|_{L_p(\mathbb{R})} \leq (1+\eta) \|R_n W\|_{L_p[-D_n, D_n]}. \tag{2.33}$$

We first estimate the norm on the right by splitting the integral inside the norm into ranges near 1 and away from 1. First let us deal with the range

$$\mathcal{I}_1 = \left[\sqrt{2n}\left(1 - 6^{-1/3}(2n)^{-2/3} R\right), D_n\right],$$

where R is some fixed (large) number. For $x \in \mathcal{I}_1$, write for $t \in [-R, 6^{1/3}2^{2/3}B]$,

$$x = \sqrt{2n}\left(1 + 6^{-1/3}(2n)^{-2/3}\, t\right). \tag{2.34}$$

To find the asymptotics for U_n, also write

$$\begin{aligned}
\frac{x}{D_n} &= \cos\frac{s}{m} \\
&\Rightarrow\ 1 - \frac{x}{D_n} = 2\sin^2\frac{s}{2m} = \frac{1}{2}\left(\frac{s}{m}\right)^2(1 + o(1)) \\
&\Rightarrow\ s = \sqrt{2m^2\left(1 - \frac{x}{D_n}\right)} + o(1) \\
&\Rightarrow\ s = \frac{\varepsilon}{L}\sqrt{2\left(B - 6^{-1/3}2^{-2/3}t\right)} + o(1).
\end{aligned}$$

Then if $\mathbb{S}(u) = \frac{\sin u}{u}$ is the sinc kernel,

$$\begin{aligned}
&\frac{T_m\left(\frac{x}{D_n}\right) - T_m(1)}{m^2\left(\frac{x}{D_n} - 1\right)} \\
=\ &\frac{\cos s - 1}{m^2\left(\frac{x}{D_n} - 1\right)} = \frac{-2\sin^2\frac{s}{2}}{-\frac{1}{2}s^2} + o(1) \\
=\ &\left(\mathbb{S}\left(\frac{s}{2}\right)\right)^2 + o(1) = \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B - 6^{-1/3}2^{-2/3}t}{2}}\right) + o(1),
\end{aligned}$$

and uniformly in such x,

$$U_n(x) = \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B - 6^{-1/3}2^{-2/3}t}{2}}\right)^L + o(1).$$

In particular, for each fixed k, as $n \to \infty$, recalling (2.3), and that $a_k < 0$,

$$U_n(x_{kn}) = \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B + 6^{-1/3}2^{-2/3}|a_k|}{2}}\right)^L + o(1). \tag{2.35}$$

Then uniformly for x in this range, from Lemma 2.1(e) and recalling (2.29),

$$\begin{aligned}
|R_n W|(x) &= \left|U_n(x)\sum_{k=1}^{M} c_k(\ell_{kn}W)(x)\, W^{-1}(x_{kn})\right| \\
&= \left|\mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B - 6^{-1/3}2^{-2/3}t}{2}}\right)^L P(-t)\right| + o(1).
\end{aligned} \tag{2.36}$$

Then as $|\mathbb{S}(u)| \le 1$,

$$\begin{aligned}
&\int_{\mathcal{I}_1} |R_n W|^p(x)\, dx \\
\le\ &6^{-1/3}(2n)^{-1/6}\left(\int_{-R}^{6^{1/3}2^{2/3}B} |P(-t)|^p\, dt + o(1)\right).
\end{aligned}$$

(2.37)

Next, for $x \in [-D_n, D_n]$,

$$\begin{aligned} |U_n(x)| &\leq \left(\min\left\{1, \frac{2}{\left|m^2\left(\frac{x}{D_n}-1\right)\right|}\right\}\right)^L \\ &\leq \frac{C}{\left(1+m^2\left|\frac{x}{D_n}-1\right|\right)^L} \\ &\leq Cn^{-2L/3}\frac{1}{\left(n^{-2/3}+\left|\frac{x}{a_n}-1\right|\right)^L} \end{aligned}$$

by straightforward estimation. Here C depends on ε. Then from Lemma 2.1(g),

$$|R_n(x)W(x)| \leq Cn^{-2L/3}\frac{1}{\left(n^{-2/3}+\left|\frac{x}{a_n}-1\right|\right)^L}\frac{n^{1/6}\psi_n(x)^{1/4}}{1+n^{1/2}\psi_n(x)^{1/2}|x-a_n|}. \tag{2.38}$$

Of course here C depends on the particular P and ε, but not on n nor R nor x. Then

$$\begin{aligned} &\int_{[-D_n,D_n]\setminus\mathcal{I}_1}|R_nW|(x)^p\,dx \\ \leq\ & Cn^{-2Lp/3+p/6}\int_{-D_n}^{\sqrt{2n}\left(1-6^{-1/3}(2n)^{-2/3}R\right)}\left[\frac{1}{\left(n^{-2/3}+\left|\frac{x}{\sqrt{2n}}-1\right|\right)^L}\frac{n^{1/6}\psi_n(x)^{1/4}}{1+n^{1/2}\psi_n(x)^{1/2}\left|x-\sqrt{2n}\right|}\right]^p dx \\ \leq\ & Cn^{-2Lp/3+p/6+1/2}\int_{-\left(1+Bn^{-2/3}\right)}^{1-6^{-1/3}(2n)^{-2/3}R}\left[\frac{1}{\left(n^{-2/3}+|y-1|\right)^L}\frac{\left(|1-|y||+n^{-2/3}\right)^{1/4}}{1+n\left(|1-|y||+n^{-2/3}\right)^{1/2}|y-1|}\right]^p dy \\ \leq\ & Cn^{-2Lp/3+p/6+1/2}\left\{\begin{array}{l} \int_{-\left(1+Bn^{-2/3}\right)}^{0}\left[\frac{\left(|1-|y||+n^{-2/3}\right)^{1/4}}{1+n\left(|1-|y||+n^{-2/3}\right)^{1/2}}\right]^p dy \\ \quad+\int_0^{1-6^{-1/3}(2n)^{-2/3}R}\left[\frac{1}{n|y-1|^{L+5/4}}\right]^p dy \end{array}\right\} \\ \leq\ & Cn^{-2Lp/3+p/6+1/2}\left\{n^{-2/3}\int_{-B}^{n^{2/3}}\left[\frac{n^{-1/6}\left(|s|+1\right)^{1/4}}{1+n^{2/3}\left(|s|+1\right)^{1/2}}\right]^p ds+n^{-p}\left(Rn^{-2/3}\right)^{1-(L+5/4)p}\right\} \\ \leq\ & Cn^{-2Lp/3+p/6+1/2}\left\{n^{-2/3-5p/6}\int_{-B}^{n^{2/3}}\frac{1}{\left(|s|+1\right)^{p/4}}ds+n^{-p}\left(Rn^{-2/3}\right)^{1-(L+5/4)p}\right\} \\ \leq\ & Cn^{-2Lp/3+p/6+1/2}\left\{n^{-5p/6}+n^{-p}\left(Rn^{-2/3}\right)^{1-(L+5/4)p}\right\} \\ \leq\ & Cn^{-2Lp/3-2p/3+1/2}+Cn^{-1/6}R^{1-(L+5/4)p}. \end{aligned}$$

Assuming that L is large enough so that

$$-2Lp/3-2p/3+1/2<-1/6$$

and

$$1-(L+5/4)p<-1,$$

we have

$$\int_{[-D_n,D_n]\setminus\mathcal{I}_1}|R_nW|(x)^p\,dx \leq o\left(n^{-1/6}\right)+Cn^{-1/6}R^{-1}.$$

Then combined with (2.37) and (2.33) this gives

$$\begin{aligned}
&(1+\eta)^{-p} \int_{-\infty}^{\infty} |R_n W|^p \\
\leq\ & 6^{-1/3} (2n)^{-1/6} \int_{-R}^{6^{1/3}2^{2/3}B} |P(t)|^p \, dt + o\left(n^{-1/6}\right) + Cn^{-1/6} R^{-1}.
\end{aligned}$$

(2.39)

Next from (2.10), and (2.35-36), for each fixed k, as $P(a_k) = c_k$,

$$\begin{aligned}
&\lambda_{kn} W^{-2}(x_{kn}) |R_n W(x_{kn})|^p \\
=\ & \left[3^{4/3}\pi^{-2}2^{3/2}n^{1/6} Ai'(a_k)^2\right]^{-1} \left\{ \left| \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B + 6^{-1/3}2^{-2/3}|a_k|}{2}}\right)\right|^{Lp} |P(a_k)|^p + o(1) \right\}
\end{aligned}$$

so

$$\begin{aligned}
&\sum_{k=1}^{M} \lambda_{kn} W^{-2}(x_{kn}) |R_n W(x_{kn})|^p \\
=\ & \left[3^{4/3}\pi^{-2}2^{3/2}n^{1/6}\right]^{-1} \left\{ \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2} \left| \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B + 6^{-1/3}2^{-2/3}|a_k|}{2}}\right)\right|^{Lp} + o(1) \right\}.
\end{aligned}$$

(2.40)

Together with (1.19) and (2.39), this gives as $n \to \infty$,

$$\begin{aligned}
&(1+\eta)^{-p} \left[3^{4/3}\pi^{-2}2^{3/2}\right]^{-1} \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2} \left| \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B + 6^{-1/3}2^{-2/3}|a_k|}{2}}\right)\right|^{Lp} \\
\leq\ & 6^{-1/3}2^{-1/6} A \int_{-R}^{6^{1/3}2^{2/3}B} |P(t)|^p \, dt + CR^{-1}.
\end{aligned}$$

Here B, ε are independent of R. We let $R \to \infty$ and obtain

$$\begin{aligned}
&(1+\eta)^{-p} \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2} \left| \mathbb{S}\left(\frac{\varepsilon}{L}\sqrt{\frac{B + 6^{-1/3}2^{-2/3}|a_k|}{2}}\right)\right|^{Lp} \\
\leq\ & 6\pi^{-2} A \int_{-\infty}^{6^{1/3}2^{1/6}B} |P(t)|^p \, dt.
\end{aligned}$$

Now let $\varepsilon \to 0+$:

$$(1+\eta)^{-p} \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2} \leq 6\pi^{-2} A \int_{-\infty}^{\infty} |P(t)|^p \, dt.$$

Finally we can let $\eta \to 0$:

$$\sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2} \leq 6\pi^{-2} A \int_{-\infty}^{\infty} |P(t)|^p \, dt.$$

■

Proof of Theorem 1.3 (a)

Recall that $S_M[f]$ is the partial sum defined in (1.24). As $f \in \mathcal{G}_p$,

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} |f(t) - S_M[f](t)|^p \, dt = 0.$$

Then for a fixed positive integer L, and by Lemma 2.4, and as $S_M[f](a_k) = f(a_k)$ for $k \le M$,

$$\begin{aligned}
\left(\sum_{k=1}^{L} \frac{|f(a_k)|^p}{Ai'(a_k)^2} \right)^{1/p} &= \lim_{M\to\infty} \left(\sum_{k=1}^{L} \frac{|S_M[f](a_k)|^p}{Ai'(a_k)^2} \right)^{1/p} \\
&\le \limsup_{M\to\infty} \left(\sum_{k=1}^{M} \frac{|S_M[f](a_k)|^p}{Ai'(a_k)^2} \right)^{1/p} \\
&\le \left(\frac{6}{\pi^2} A \right)^{1/p} \limsup_{M\to\infty} \left(\int_{-\infty}^{\infty} |S_M[f](t)|^p \, dt \right)^{1/p} \\
&\le \left(\frac{6}{\pi^2} A \right)^{1/p} \limsup_{M\to\infty} \left\{ \left(\int_{-\infty}^{\infty} |S_M[f](t) - f(t)|^p \, dt \right)^{1/p} + \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p} \right\} \\
&= \left(\frac{6}{\pi^2} A \right)^{1/p} \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p}.
\end{aligned}$$

Now let $L \to \infty$. ■

For Theorem 1.3(b), we need :

Lemma 2.5

Assume that for some $\beta > \frac{1}{4}$, we have

$$|f(x)| \le C(1+|x|)^{-\beta}, x \in (-\infty, 0). \tag{2.41}$$

Then for $M \ge 1$, and all $t \in (-\infty, 0]$,

$$|S_M[f]|(t) \le C(1+|t|)^{-\beta} \log(2+|t|). \tag{2.42}$$

For $t \in (0, \infty)$,

$$|S_M[f]|(t) \le C(1+t)^{-1/4} \exp\left(-\frac{2}{3} t^{\frac{3}{2}} \right). \tag{2.43}$$

Proof

From (2.41) and (2.24), followed by (2.22), for $t \ge 0$,

$$\begin{aligned}
|S_M[f]|(-t) &\le C \sum_{j=1}^{M} \frac{|a_j|^{-\beta}}{1 + (1+t)^{1/4} |a_j|^{1/4} |t - |a_j||} \\
&\le C \sum_{j=1}^{M} (|a_j| - |a_{j-1}|) \frac{|a_j|^{-\beta+1/2}}{1 + (1+t)^{1/4} |a_j|^{1/4} |t - |a_j||} \\
&\le C \int_0^{\infty} \frac{s^{-\beta+1/2}}{1 + (1+t)^{1/4} s^{1/4} |t-s|} ds.
\end{aligned}$$

If $0 \le t \le 1$, we can bound this by

$$C\int_0^2 s^{-\beta+1/2}ds + C\int_2^\infty s^{-\beta-3/4}ds \le C,$$

recall $\beta > \frac{1}{4}$. If $t \ge 1$, we can bound this by

$$\begin{aligned}
& C\int_0^\infty \frac{s^{-\beta+1/2}}{1+t^{1/4}s^{1/4}\left|t-s\right|}ds \\
= \quad & Ct^{-\beta+3/2}\int_0^\infty \frac{u^{-\beta+1/2}}{1+t^{3/2}u^{1/4}\left|u-1\right|}du \\
\le \quad & Ct^{-\beta+3/2}\left[\begin{array}{c} t^{-3/2}\int_0^{1-1/t^{3/2}} \frac{u^{-\beta+1/4}du}{|u-1|} + \int_{1-1/t^{3/2}}^{1+1/t^{3/2}} 1du \\ +t^{-3/2}\int_{1+1/t^{3/2}}^2 \frac{du}{|u-1|} + t^{-3/2}\int_2^\infty u^{-\beta-3/4}du \end{array}\right] \\
\le \quad & Ct^{-\beta}\left[\log\left(1+|t|\right)+1+\log\left(1+|t|\right)+1\right].
\end{aligned}$$

Thus we have the bound (2.42). Next, if $t \ge 0$, we obtain from (2.23) and (2.21),

$$\begin{aligned}
\left|S_M\left[f\right]\right|(-t) \quad & \le \quad C\left(1+t\right)^{-1/4}\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)\sum_{j=1}^M \left|a_j\right|^{-\beta} j^{-5/6} \\
& \le \quad C\left(1+t\right)^{-1/4}\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)\sum_{j=1}^M j^{-5/6-2\beta/3} \\
& \le \quad C\left(1+t\right)^{-1/4}\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right),
\end{aligned}$$

as $5/6+2\beta/3 > 5/6+1/6 > 1$. ■

Proof of Theorem 1.3(b)
Recall that we are assuming $p \ge 2$. If $N > M$, we have in view of the lemma and our bound on f

$$\begin{aligned}
& \int_{-\infty}^\infty \left|S_N\left[f\right]-S_M\left[f\right]\right|^p(t)\,dt \\
\le \quad & C\int_{-\infty}^\infty \left|S_N\left[f\right]-S_M\left[f\right]\right|^2(t)\,dt \\
\to \quad & 0 \text{ as } M,N\to\infty,
\end{aligned}$$

as $f \in \mathcal{G}$ implies that $S_M\left[f\right] \to f$ in $L_2\left(\mathbb{R}\right)$ as $M \to \infty$. It follows that $\left\{S_M\left[f\right]\right\}$ is Cauchy in $L_p\left(\mathbb{R}\right)$, so has a limit there. This limit must be f, as $f \in \mathcal{G}$. Then also $f \in \mathcal{G}_p$ and the result follows. ■

Lemma 2.6
Assume that (1.22) holds with $R = r = 0$. Let $P = \sum_{k=1}^M P\left(a_k\right)\mathcal{L}_k$ and $1 < p < 4$. Then

$$\int_{-\infty}^\infty \left|P\left(t\right)\right|^p dt \le B\frac{\pi^2}{6}\sum_{j=1}^M \frac{\left|P\left(a_k\right)\right|^p}{Ai'\left(a_k\right)^2}. \tag{2.44}$$

Proof

We use (1.22) with $R = r = 0$. If R_n is a polynomial of degree $\leq n-1$,

$$\int_{-\infty}^{\infty} |(R_n W)(x)|^p \, dx \leq B \sum_{j=1}^{n} \lambda_{jn} |R_n(x_{jn})|^p W^{p-2}(x_{jn}). \tag{2.45}$$

Let

$$R_n(x) = \sum_{k=1}^{M} P(a_k) \ell_{kn}(x) W^{-1}(x_{kn}).$$

Let $R > 0$ and

$$\mathcal{I}_1 = \left[\sqrt{2n}\left(1 - 6^{-1/3}(2n)^{-2/3} R\right), \sqrt{2n}(1 + 6^{-1/3}(2n)^{-2/3} R)\right].$$

From (2.7) with x of the form (2.4), we have

$$|R_n W|(x) = |P(-t)| + o(1),$$

so

$$\int_{\mathcal{I}_1} |R_n W|(x)^p \, dx = 6^{-1/3}(2n)^{-1/6}\left(\int_{-R}^{R} |P(t)|^p \, dt + o(1)\right).$$

Also, as at (2.40),

$$\begin{aligned} & \sum_{j=1}^{n} \lambda_{jn} |R_n(x_{jn})|^p W^{p-2}(x_{jn}) \\ = & \sum_{j=1}^{M} \lambda_{jn} |R_n(x_{jn})|^p W^{p-2}(x_{jn}) \\ = & (1 + o(1))\left[3^{4/3}\pi^{-2}2^{3/2}n^{1/6}\right]^{-1} \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2}. \end{aligned}$$

Then (2.45) gives

$$6^{-1/3}(2n)^{-1/6}\left(\int_{-R}^{R} |P(t)|^p \, dt + o(1)\right) \leq B(1 + o(1))\left[3^{4/3}\pi^{-2}2^{3/2}n^{1/6}\right]^{-1} \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2}.$$

or

$$\left(\int_{-R}^{R} |P(t)|^p \, dt + o(1)\right) \leq B(1 + o(1)) \frac{\pi^2}{6} \sum_{k=1}^{M} \frac{|P(a_k)|^p}{Ai'(a_k)^2}.$$

Letting $R \to \infty$ gives (2.44). ∎

Proof of Theorem 1.4

(a) Lemma 2.6 gives

$$\begin{aligned} \|f\|_{L_p(\mathbb{R})} & \leq \|f - S_M[f]\|_{L_p(\mathbb{R})} + \|S_M[f]\|_{L_p(\mathbb{R})} \\ & \leq \|f - S_M[f]\|_{L_p(\mathbb{R})} + \left(B \frac{\pi^2}{6} \sum_{k=1}^{M} \frac{|f(a_k)|^p}{Ai'(a_k)^2}\right)^{1/p} \\ & \to 0 + \left(B \frac{\pi^2}{6} \sum_{k=1}^{\infty} \frac{|f(a_k)|^p}{Ai'(a_k)^2}\right)^{1/p}, \end{aligned}$$

as $M \to \infty$.

(b) Our assumption that $f \in \mathcal{G}$ ensures that $f = \lim_{M\to\infty} S_M[f]$ uniformly in compact sets. Next, given $N > M$, we have from Lemma 2.6,

$$\begin{aligned}\int_{-\infty}^{\infty} |S_N[f] - S_M[f]|^p(t)\,dt &\leq B\frac{\pi^2}{6}\sum_{k=M+1}^{N}\frac{|f(a_k)|^p}{Ai'(a_k)^2} \\ &\leq C\sum_{k=M+1}^{\infty}\frac{|f(a_k)|^p}{k^{1/3}} \to 0,\end{aligned}$$

as $k \to \infty$ - recall (2.20) and our hypothesis (1.30). So $\{S_M[f]\}$ is Cauchy in complete $L_p(\mathbb{R})$ and as above, its limit in $L_p(\mathbb{R})$ must be f, so that (a) is applicable. ■

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