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ON MARCINKIEWICZ-ZYGMUND INEQUALITIES AT JACOBI ZEROS AND THEIR BESSEL FUNCTION COUSINS

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Abstract. Marcinkiewicz-Zygmund Inequalities involving the zeros $\{x_{kn}\}$ of Jacobi polynomials for the weight $w^{\alpha,\beta}$ can take the form

$$\begin{aligned} & A\sum_{k=1}^{n}\lambda_{kn}\left\vert P\left(x_{kn}\right)\right\vert^{p}w^{\sigma,\tau}\left(x_{kn}\right)\leq\int_{-1}^{1}\left\vert P\left(x\right)\right\vert^{p}w^{\alpha+\sigma,\beta+\tau}\left(x\right)dx \\ \leq\quad & B\sum_{k=1}^{n}\lambda_{n}\left\vert P\left(x_{kn}\right)\right\vert^{p}w^{\sigma,\tau}\left(x_{kn}\right). \end{aligned}$$

Here $p>1$, P is any polynomial of degree $<n$, the λ_{kn} are Gauss quadrature weights for $w^{\alpha,\beta}$, the parameters σ,τ are appropriately chosen, and A,B are independent of n. We show how these generate analogous inequalities at zeros $\{j_k\}$ of the Bessel function J_α, with the same constants A and B :

$$\begin{aligned} & A\sum_{k=1}^{\infty}j_k^{2\sigma}J_\alpha^{*\prime}\left(j_k\right)^{-2}\left\vert f\left(j_k\right)\right\vert^{p}\leq\int_{0}^{\infty}\left\vert f\left(t\right)\right\vert^{p}t^{2\sigma+2\alpha+1}dt \\ \leq\quad & B\sum_{k=1}^{\infty}j_k^{2\sigma}J_\alpha^{*\prime}\left(j_k\right)^{-2}\left\vert f\left(j_k\right)\right\vert^{p}. \end{aligned}$$

Here f is an even entire function of exponential type ≤ 1 for which the integral in the middle converges.

1. Introduction

In a recent paper, the author studied the relationship between the classical Plancherel-Polya inequalities and the classical Marcinkiewicz-Zygmund inequalities. The former [6, p. 152] assert that for $1<p<\infty$, and entire functions f of exponential type at most π,

$$A_p\sum_{k=-\infty}^{\infty}\left\vert f\left(k\right)\right\vert^{p}\leq\int_{-\infty}^{\infty}\left\vert f\right\vert^{p}\leq B_p\sum_{j=-\infty}^{\infty}\left\vert f\left(k\right)\right\vert^{p}, \tag{1.1}$$

provided the integral is finite. For $0<p\leq 1$, the left-hand inequality is still true, but the right-hand inequality is not. We assume that B_p is taken as small as possible, and A_p as large as possible.

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The Marcinkiewicz-Zygmund inequalities assert [19, Vol. II, p. 30] that for $p > 1, n \geq 1$, and polynomials P of degree $\leq n-1$,

$$\frac{A'_p}{n} \sum_{k=1}^{n} \left| P\left(e^{2\pi ik/n}\right) \right|^p \leq \int_0^1 \left| P\left(e^{2\pi it}\right) \right|^p dt \leq \frac{B'_p}{n} \sum_{k=1}^{n} \left| P\left(e^{2\pi ik/n}\right) \right|^p. \tag{1.2}$$

Here too, A'_p and B'_p are independent of n and P, and the left-hand inequality is also true for $0 < p \leq 1$ [7]. The author [8] proved that the inequalities (1.1) and (1.2) are equivalent, in the sense that each implies the other. Moreover, the sharp constants are the same:

Theorem A
For $0 < p < \infty$, $A_p = A'_p$ *and for* $1 < p < \infty$, $B_p = B'_p$.

These inequalities are useful in studying convergence of Fourier series, Lagrange interpolation, in number theory, and weighted approximation. They have been extended to many settings, and there are a great many methods to prove them [2], [5], [7], [9], [10], [11], [15], [17], [18]. The sharp constants in (1.1) and (1.2) are unknown, except for the case $p = 2$, where of course we have equality rather than inequality, so that $A_2 = B_2 = A'_2 = B'_2 = 1$ [6, p. 150].

In this paper, we explore an analogous theme, where instead of roots of unity, we consider polynomial inequalities at zeros of Jacobi polynomials, and instead of the integers, we consider zeros of Bessel functions. We first need some notation. Let $\alpha, \beta > -1$ and

$$w^{\alpha,\beta}(x) = (1-x)^{\alpha} (1+x)^{\beta}, \ x \in (-1,1).$$

For $n \geq 1$, let $P_n^{\alpha,\beta}$ denote the standard Jacobi polynomial of degree n, so that it has degree n, satisfies the orthogonality conditions

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x)\, x^k w^{\alpha,\beta}(x)\, dx = 0, \ 0 \leq k < n,$$

and is normalized by

$$P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}.$$

Let

$$x_{nn} < x_{n-1,n} < \ldots < x_{1n}$$

denote the zeros of $P_n^{\alpha,\beta}$. Let $\{\lambda_{kn}\}$ denote the weights in the Gauss quadrature for $w^{\alpha,\beta}$, so that for all polynomials P of degree $\leq 2n-1$,

$$\int_{-1}^{1} P w^{\alpha,\beta} = \sum_{k=1}^{n} \lambda_{kn} P(x_{kn}).$$

There is a classical analogue of (1.2), established for special α, β by Richard Askey, and for all $\alpha, \beta > -1$ (and for more general "generalized Jacobi weights") by P. Nevai, and his collaborators [7], [9], [12], [13], with later work by König and Nielsen [5], and for doubling weights by Mastroianni and Totik [11]. The following special case follows from Theorem 5 in [9, eqn. (1.19), p. 534]:

Theorem B
Let $\alpha, \beta, \tau, \sigma$ *satisfy* $\alpha, \beta, \alpha+\sigma, \beta+\tau > -1$. *Let* $p > 0$. *For* $n \geq 1$, *let* $\{x_{kn}\}$

denote the zeros of the Jacobi polynomial $P_n^{\alpha,\beta}$ and $\{\lambda_{kn}\}$ denote the corresponding Gauss quadrature weights. There exists $A > 0$ such that for $n \geq 1$, and polynomials P of degree $\leq n-1$,

$$A\sum_{k=1}^{n}\lambda_{kn}\left|P\left(x_{kn}\right)\right|^{p}\left(1-x_{kn}\right)^{\sigma}\left(1+x_{kn}\right)^{\tau} \leq \int_{-1}^{1}\left|P\left(x\right)\right|^{p}\left(1-x\right)^{\alpha+\sigma}\left(1+x\right)^{\beta+\tau}dx. \tag{1.3}$$

We emphasize that this is not the most general form of this result, but one suitable for our purposes.

The converse inequality is much more delicate, and in particular holds only for $p > 1$, and even then only for special cases of the parameters. It too was investigated by P. Nevai, with later work by Yuan Xu [17], [18], König and Nielsen [5]. König and Nielsen gave the exact range of p for which

$$\int_{-1}^{1}\left|P\left(x\right)\right|^{p}\left(1-x\right)^{\alpha}\left(1+x\right)^{\beta}dx \leq B\sum_{k=1}^{n}\lambda_{kn}\left|P\left(x_{kn}\right)\right|^{p}, \tag{1.4}$$

holds with B independent of n and P. Let

$$\begin{aligned}
\mu\left(\alpha,\beta\right) &= \max\left\{1, 4\frac{\alpha+1}{2\alpha+5}, 4\frac{\beta+1}{2\beta+5}\right\};\\
m\left(\alpha,\beta\right) &= \max\left\{1, 4\frac{\alpha+1}{2\alpha+3}, 4\frac{\beta+1}{2\beta+3}\right\};\\
M\left(\alpha,\beta\right) &= \frac{m\left(\alpha,\beta\right)}{m\left(\alpha,\beta\right)-1}. \qquad (1.5)
\end{aligned}$$

Then (1.4) holds for all n and P iff

$$\mu\left(\alpha,\beta\right) < p < M\left(\alpha,\beta\right). \tag{1.6}$$

The most general sufficient condition for a converse quadrature inequality seems due to Yuan Xu [17, pp. 881-882]. When we restrict to Jacobi weights, with the same weight on both sides, Xu's inequality takes the following form:

Theorem C

Let $\alpha, \beta, \tau, \sigma$ satisfy $\alpha, \beta, \alpha+\sigma, \beta+\tau > -1$. Let $p > 1$, and assume that

$$\frac{p}{2}\left(\alpha+\frac{1}{2}\right)-\left(\alpha+1\right) < \sigma < \left(p-1\right)\left(\alpha+1\right)-\max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\}. \tag{1.7}$$

$$\frac{p}{2}\left(\beta+\frac{1}{2}\right)-\left(\beta+1\right) < \tau < \left(p-1\right)\left(\beta+1\right)-\max\left\{0, \frac{p}{2}\left(\beta+\frac{1}{2}\right)\right\}. \tag{1.8}$$

Then there exists $B > 0$ such that for $n \geq 1$, and polynomials P of degree $\leq n-1$,

$$\int_{-1}^{1}\left|P\left(x\right)\right|^{p}\left(1-x\right)^{\alpha+\sigma}\left(1+x\right)^{\beta+\tau}dx \leq B\sum_{k=1}^{n}\lambda_{kn}\left|P\left(x_{kn}\right)\right|^{p}\left(1-x_{kn}\right)^{\sigma}\left(1+x_{kn}\right)^{\tau}. \tag{1.9}$$

Remarks

For a given α, β, p, it is always possible to choose σ, τ satisfying (1.7), (1.8) and

$\alpha+\sigma, \beta+\tau>-1$. Indeed, the difference between the upper and lower bounds in (1.7) is

$$\begin{aligned} & p\left(\alpha+1\right)-\max\left\{\frac{p}{2}\left(\alpha+\frac{1}{2}\right), p\left(\alpha+\frac{1}{2}\right)\right\} \\ = \quad & p\min\left\{\frac{\alpha}{2}+\frac{3}{4},\frac{1}{2}\right\}>0. \end{aligned}$$

Inequalities of the type (1.9) for doubling weights have been established by Mastroianni and Totik [10], [11] under the additional condition that one needs to restrict the degree of P in (1.9) further, such as $\deg\left(P\right)\leq \eta n$ for some $\eta\in(0,1)$ depending on the particular doubling weight.

Now let $\alpha>-1$ and define the Bessel function of order α,

$$J_{\alpha}\left(z\right)=\left(\frac{z}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\left(-1\right)^{k}\frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma\left(k+\alpha+1\right)}. \tag{1.10}$$

We shall also use

$$J_{\alpha}^{*}\left(z\right)=J_{\alpha}\left(z\right)/z^{\alpha}, \tag{1.11}$$

which has the advantage of being an entire function for all $\alpha>-1$. J_{α}^{*} has real simple zeros, and we denote the positive zeros by

$$0<j_1<j_2<...$$

while for $k\geq 1$,

$$j_{-k}=-j_k.$$

It is unfortunate that the symbol j is used for this zero, but this is the standard notation, so we conform to it.

The connection between Jacobi polynomials and Bessel functions is given by the classical Mehler-Heine asymptotic, which holds uniformly for z in compact subsets of $\mathbb{C}$ [16, p. 192]:

$$\lim_{n\to\infty}n^{-\alpha}P_n^{\alpha,\beta}\left(1-\frac{1}{2}\left(\frac{z}{n}\right)^2\right)=\lim_{n\to\infty}n^{-\alpha}P_n^{\alpha,\beta}\left(\cos\frac{z}{n}\right)=\left(\frac{z}{2}\right)^{-\alpha}J_{\alpha}\left(z\right)=2^{\alpha}J_{\alpha}^{*}\left(z\right). \tag{1.12}$$

In this paper, we use this asymptotic to pass from inequalities such as (1.3) and (1.9) to analogous ones for entire functions involving zeros of Bessel functions.

There is an extensive literature dealing with quadrature sums and Lagrange interpolation at the $\{j_k\}$. In particular, there is the quadrature formula [3, p. 305], [4, p. 49]

$$\int_{-\infty}^{\infty}\left|x\right|^{2\alpha+1}f\left(x\right)dx=\frac{2}{\tau^{2\alpha+2}}\sum_{k=-\infty,k\neq 0}^{\infty}\frac{1}{\left|J_{\alpha}^{*\prime}\left(j_k\right)\right|^2}f\left(\frac{j_k}{\tau}\right),$$

valid for all entire functions f of exponential type at most 2τ, for which the integral on the left-hand side is finite. The paper of Grozev and Rahman also contains the following converse Marcinkiewicz-Zygmund type inequality: let $\alpha\geq-\frac{1}{2}$ and $p>1$; or $-1<\alpha<-\frac{1}{2}$ and $1<p<\frac{2}{|1+2\alpha|}$. Then for entire functions f of exponential type $\leq \tau$ for which $|x|^{\alpha+\frac{1}{2}} f(x) \in L_p(\mathbb{R}\backslash(-\delta,\delta))$, for some $\delta > 0$, [4, Lemma 14, p. 58; Lemma 13, p. 57]

$$\int_{-\infty}^{\infty} \left| |x|^{\alpha+\frac{1}{2}} f(x) \right|^p dx \leq \frac{B^*}{\tau} \sum_{k=-\infty, k\neq 0}^{\infty} \left| \frac{1}{\tau^{\alpha+\frac{1}{2}} J_\alpha^{*\prime}(j_k)} f\left(\frac{j_k}{\tau}\right) \right|^p. \tag{1.13}$$

Here B^* depends on α and p. In the other direction, since $j_{k+1} - j_k$ is bounded below by a positive constant for all k, classical inequalities from the theory of entire functions [6, p. 150] show that

$$\sum_{k=-\infty, k\neq 0}^{\infty} |f(j_k)|^p \leq C \int_{-\infty}^{\infty} |f(x)|^p \, dx$$

for entire functions of finite exponential type for which the right-hand side is finite.

While Grozev and Rahman note the analogous nature of Lagrange interpolation at zeros of Jacobi polynomials and Bessel functions, and also the Mehler-Heine formula, their proofs proceed purely from properties of Bessel functions. It is the purpose of this paper to show that one can pass from inequalities like (1.3) to analogues for Bessel functions using scaling limits, keeping the same constants, much as was done in [8].

For $k \geq 1$, let

$$\mathcal{L}_k(z) = 2j_k \frac{J_a^*(z)}{J_\alpha^{*\prime}(j_k)\left(z^2 - j_k^2\right)} \tag{1.14}$$

denote the fundamental "polynomial" of interpolation at $\{j_k\}$, so that

$$\mathcal{L}_k(j_m) = \delta_{jm}.$$

For functions $f : [0,\infty) \to \mathbb{R}$, define the Lagrange interpolation series

$$\mathcal{L}[f](x) = \sum_{k=1}^{\infty} f(j_k) \mathcal{L}_k(x). \tag{1.15}$$

For even entire functions f of exponential type, with appropriate growth restrictions, we have $f = \mathcal{L}[f]$, see Lemma 2.3 below, or [4]. Let $\mathcal{P}$ denote the set of all finite linear combinations of $\{\mathcal{L}_k\}_{k\geq 1}$, that is expressions of the form

$$\sum_{k=1}^{n} c_k \mathcal{L}_k(x)$$

with any $n \geq 1$ and arbitrary real $\{c_k\}$. Let $L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$ denote the space of all even entire functions f of exponential type ≤ 1 with

$$\int_0^{\infty} |f(t)|^p \, t^{2\alpha+2\sigma+1} dt < \infty.$$

It follows from estimates below that $\mathcal{P} \subset L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$ at least when (1.17) below holds. Also, let $\bar{\mathcal{P}}(p,\sigma)$ denote the closure of $\mathcal{P}$ in $L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$. This is the set of all functions $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$ such that for some sequence $\{P_m\}$ in $\mathcal{P}$, we have

$$\lim_{m\to\infty} \int_0^{\infty} |f(t) - P_m(t)|^p \, t^{2\alpha+2\sigma+1} dt = 0. \tag{1.16}$$

We prove:

Theorem 1.1
Assume that $p > 0$, $\alpha, \beta, \alpha + \sigma, \beta + \tau > -1$, and

$$-p\left(\frac{\alpha}{2}+\frac{5}{4}\right)+\alpha+\sigma+1<0. \tag{1.17}$$

Let A be as in Theorem B. Then

$$2A\sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \leq \int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt, \tag{1.18}$$

for all $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$.

Remarks
(a) In using Theorem B to prove Theorem 1.1, we only really need that for each fixed $M \geq 1$, and n exceeding some threshold depending on M, we have for all P of degree $\leq n-1$,

$$A\sum_{k=1}^{M} \lambda_{kn} |P(x_{kn})|^p (1-x_{kn})^\sigma (1+x_{kn})^\tau \leq \int_{-1}^{1} |P(x)|^p (1-x)^{\alpha+\sigma} (1+x)^{\beta+\tau} dx.$$

(b) It is not clear if one can pass from Theorem 1.1 back to Theorem B, in the way that we passed from the Plancherel-Polya inequalities back to the Marcinkiewicz-Zygmund inequalities in [8].
(c) The restriction (1.17) on the parameters is need to ensure convergence. It is implied by the restrictions in (1.7).
(d) We deduce a general result for not necessarily even functions:

Corollary 1.2
Assume that $p > 0$, $\alpha > -1, \alpha + \sigma > -1$, and

$$-p\left(\frac{\alpha}{2}+\frac{3}{4}\right)+\alpha+\sigma+1<0. \tag{1.19}$$

There exists $A_1 > 0$ such that

$$A_1 \sum_{k=-\infty, k\neq 0}^{\infty} |j_k|^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p |t|^{2\alpha+2\sigma+1} dt, \tag{1.20}$$

for all for all entire functions f of exponential type ≤ 1 for which the integral in the right-hand side converges.

Following is the converse quadrature sum inequality that we can deduce from Theorem C:

Theorem 1.3
Assume that $p > 1$, $\alpha, \beta, \alpha + \sigma, \beta + \tau > -1$, and that (1.7) and (1.8) hold. Let B be as in Theorem C. Then for $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$, we have

$$\int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B\sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p. \tag{1.21}$$

In particular this holds for $\sigma = \tau = 0$ if p satisfies (1.6) with $\beta = \alpha$. Moreover, for any α, β, p, it is possible to choose σ and τ satisfying (1.7), (1.8) so that (1.21) *holds.*

Remarks
(a) Note that Theorem 1.3 requires only the conclusion (1.9), not the hypotheses (1.7) and (1.8).
(b) By choosing $2\sigma + 2\alpha + 1 = p\left(\alpha + \frac{1}{2} - \Delta\right)$, we can also recast (1.21) as

$$\int_0^\infty \left| t^{\alpha+\frac{1}{2}-\Delta} f(t) \right|^p dt \le C \sum_{k=1}^{\infty} \left| j_k^{\alpha+\frac{1}{2}-\Delta} f(j_k) \right|^p, \tag{1.22}$$

some C independent of f.

Corollary 1.4
Assume that $p > 1$, $\alpha \ge -\frac{1}{2}$, $\alpha + \sigma > -1$, and that

$$\sigma > D := \frac{p}{2}\left(\alpha + \frac{1}{2}\right) - (\alpha + 1) \tag{1.23}$$

but

$$\sigma \ne D + \ell \frac{p}{2} \text{ for some non-negative integer } \ell. \tag{1.24}$$

There exists $B_1 > 0$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p |t|^{2\alpha+2\sigma+1} dt \le B_1 \sum_{k=-\infty, k\ne 0}^{\infty} |j_k|^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p, \tag{1.25}$$

for all entire functions f of exponential type ≤ 1, for which the left-hand side converges.
We expect that the restriction (1.24) can be dropped.

In the sequel, $C, C_1, C_2, \ldots$ denote constants independent of n, z, x, t, and polynomials of degree $\le n$. The same symbol does not necessarily denote the same constant in different occurrences. $[x]$ denotes the greatest integer $\le x$. Given two sequences $\{x_n\}, \{y_n\}$ of non-zeros real numbers, we write

$$x_n \sim y_n$$

if there exist constants C_1 and C_2 such that

$$C_1 \le x_n / y_n \le C_2$$

for $n \ge 1$. Similar notation is used for functions and sequences of functions. We let

$$S(z) = \frac{\sin z}{z}.$$

We prove Theorem 1.1, and Corollary 1.2 in Section 2; and Theorem 1.3 and Corollary 1.4 in Section 3.

2. Proof of Theorem 1.1 and Corollary 1.2

Let us fix α, β and as in Section 1, let $\{x_{jn}\}$ denote the zeros of $P_n^{\alpha,\beta}$, and $\{\ell_{jn}\}$ denote the fundamental polynomials of Lagrange interpolation at $\{x_{jn}\}$. Thus

$$\ell_{jn}(x) = \frac{P_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta\prime}(x_{jn})(x - x_{jn})} \tag{2.1}$$

and

$$\ell_{jn}\left(x_{kn}\right)=\delta_{jk}. \tag{2.2}$$

We shall also need the orthonormal polynomials $p_n^{\alpha,\beta}(x)=\gamma_n x^n+...$ satisfying

$$\int_{-1}^{1} p_n^{\alpha,\beta} p_m^{\alpha,\beta} w^{\alpha,\beta}=\delta_{mn}; \tag{2.3}$$

the nth reproducing kernel

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n-1} p_k^{\alpha,\beta}(x)\, p_k^{\alpha,\beta}(y) \\ &= \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n^{\alpha,\beta}(x)\, p_{n-1}^{\alpha,\beta}(y)-p_n^{\alpha,\beta}(y)\, p_{n-1}^{\alpha,\beta}(x)}{x-y}; \end{aligned} \tag{2.4}$$

and nth Christoffel function

$$\lambda_n(x)=1/K_n(x,x). \tag{2.5}$$

Lemma 2.1
(a) For each $k\geq 1$,

$$\lim_{n\to\infty} n^2\left(1-x_{kn}\right)=\frac{j_k^2}{2}. \tag{2.6}$$

(b) Uniformly for $n\geq 2$, $1\leq k\leq n$,

$$n^2\left(1-x_{kn}\right)\sim k^2. \tag{2.7}$$

(c) Uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty}\ell_{kn}\left(1-\frac{1}{2}\left(\frac{z}{n}\right)^2\right)=\lim_{n\to\infty}\ell_{kn}\left(\cos\frac{z}{n}\right)=\mathcal{L}_k(z). \tag{2.8}$$

(d)

$$\mathcal{L}_k\left(j_m\right)=\delta_{km}. \tag{2.9}$$

(e) For $n\geq 1, 1\leq k\leq n$, $\eta>0$ and $-1\leq x\leq x_{kn}-\eta n^{-2}$,

$$\left|\ell_{kn}(x)\right|\leq Ck^{\alpha+\frac{3}{2}}\left(n^2(1-x)\right)^{-\frac{\alpha}{2}-\frac{5}{4}}\left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}. \tag{2.10}$$

(f) For $k\geq 1$, and $x\in[0,\infty)$,

$$\left|\mathcal{L}_k(x)\right|\leq C\frac{j_k^{\alpha+\frac{3}{2}}(1+x)^{-\alpha-\frac{1}{2}}}{\left|x^2-j_k^2\right|}. \tag{2.11}$$

(g) For $k\geq 1$, and $|x-j_k|\leq j_k^{-1}$,

$$\left|\mathcal{L}_k(x)\right|\leq C. \tag{2.12}$$

Proof
(a) This is a classical limit [16, p, 193, eqn. (8.1.3)].
(b) It is shown in [16, p. 238, eqn. (8.9.1)] that

$$\theta_{kn}:=\arccos x_{kn}=\frac{1}{n}\left(k\pi+O(1)\right),$$

uniformly for $1 \leq k \leq n, n \geq 1$. Then for $k = o(n)$,

$$n^2 (1 - x_{kn}) = 2n^2 \left(\sin \frac{\theta_{kn}}{2} \right)^2 = \frac{1}{2} (k\pi + O(1))^2 . \tag{2.13}$$

Together with (2.6), this implies that for all $k \leq n$,

$$n^2 (1 - x_{kn}) \geq Ck^2.$$

This easily implies the result.
(c) Since the Mehler-Heine formula (1.12) holds uniformly for z in compact subsets of the plane, we may differentiate it. Thus uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty} n^{-\alpha-2} P_n^{\alpha,\beta\prime} \left(1 - \frac{1}{2} \left(\frac{z}{n} \right)^2 \right) z = -2^\alpha J_\alpha^{*\prime}(z) . \tag{2.14}$$

Then

$$\begin{aligned} & \lim_{n\to\infty} n^{-\alpha-2} P_n^{\alpha,\beta\prime}(x_{kn}) \\ = & \lim_{n\to\infty} n^{-\alpha-2} P_n^{\alpha,\beta\prime} \left(1 - \frac{1}{2n^2} \left[j_k^2 + o(1) \right] \right) = -2^\alpha J_\alpha^{*\prime}(j_k) / j_k, \end{aligned}$$

and consequently,

$$\begin{aligned} & \lim_{n\to\infty} \ell_{kn} \left(1 - \frac{1}{2} \left(\frac{z}{n} \right)^2 \right) \\ = & \lim_{n\to\infty} \frac{n^{-\alpha} P_n^{\alpha,\beta} \left(1 - \frac{1}{2} \left(\frac{z}{n} \right)^2 \right)}{\left[n^{-\alpha-2} P_n^{\alpha,\beta\prime}(x_{kn}) \right] \left[n^2 [1 - x_{kn}] - \frac{z^2}{2} \right]} \\ = & \frac{J_\alpha^*(z)\, 2 j_k}{J_\alpha^{*\prime}(j_k) \left[z^2 - j_k^2 \right]} = \mathcal{L}_k(z) . \end{aligned}$$

(d) This is an immediate consequence of (a), (c), and (2.2).
(e) We use the alternate representation

$$\begin{aligned} \ell_{kn}(x) &= \lambda_n(x_{kn}) K_n(x, x_{kn}) \\ &= \lambda_n(x_{kn}) \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n^{\alpha,\beta}(x)\, p_{n-1}^{\alpha,\beta}(x_{kn})}{x - x_{kn}}, \end{aligned}$$

and standard estimates, that are conveniently summarized in [14, p. 36]: uniformly in k and n,

$$\left| p_{n-1}^{\alpha,\beta}(x_{kn}) \right| \sim w^{\alpha,\beta}(x_{kn})^{-1/2} \left(1 - x_{kn}^2 \right)^{1/4} ; \tag{2.15}$$

$$\lambda_n(x_{kn}) \sim \frac{1}{n} w^{\alpha,\beta}(x_{kn}) \left(1 - x_{kn}^2 \right)^{1/2} ; \tag{2.16}$$

uniformly in $x \in [-1, 1]$ and n,

$$\left| p_n^{\alpha,\beta}(x) \right| \leq C \left(1 - x + \frac{1}{n^2} \right)^{-\frac{\alpha}{2} - \frac{1}{4}} \left(1 + x + \frac{1}{n^2} \right)^{-\frac{\beta}{2} - \frac{1}{4}} ; \tag{2.17}$$

Since also $\frac{\gamma_{n-1}}{\gamma_n} \leq 1$ (of course they have limit $\frac{1}{2}$), these estimates give uniformly in $x \in [-1,1]$ and $n \geq 1$,

$$|\ell_{kn}(x)| \leq \frac{C}{n} \frac{\left(1-x+\frac{1}{n^2}\right)^{-\frac{\alpha}{2}-\frac{1}{4}} \left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}} w^{\alpha,\beta}(x_{kn})^{1/2} \left(1-x_{kn}^2\right)^{3/4}}{x-x_{kn}}.$$

Then for $1 \leq k \leq n$ and $x \leq x_{kn} - \eta n^{-2}$, so that $x_{kn} - x \geq C(k,\eta)(1-x)$, we have for $n \geq n_0(k)$, from (2.7),

$$\begin{aligned} |\ell_{kn}(x)| &\leq \frac{C}{n}(1-x)^{-\frac{\alpha}{2}-\frac{5}{4}} \left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}} (1-x_{kn})^{\frac{\alpha}{2}+\frac{3}{4}} \\ &\leq Ck^{\alpha+\frac{3}{2}} \left(n^2(1-x)\right)^{-\frac{\alpha}{2}-\frac{5}{4}} \left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}. \end{aligned}$$

(f) A convenient summary of what is needed is given in [4, pp. 49-50]. We have [4, eqn. (11), (19)]

$$|J_\alpha^*(x)| \leq C(1+x)^{-\alpha-\frac{1}{2}},\ x \in [0,\infty);$$

$$|J_\alpha^{*\prime}(j_k)| > Cj_k^{-\alpha-\frac{1}{2}}, k \geq 1. \tag{2.18}$$

Then (2.11) follows.
(g) We use that for some ξ between x and j_k,

$$\mathcal{L}_k(x) = 2j_k \frac{J_a^{*\prime}(\xi)}{J_\alpha^{*\prime}(j_k)(x+j_k)}.$$

Next, we need [4, pp. 49-50]

$$\begin{aligned} J_\alpha(x) &= \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(x^{-1}\right)\right]; \\ J_\alpha'(x) &= -\sqrt{\frac{2}{\pi x}} \left[\sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(x^{-1}\right)\right]; \\ j_k &= \left(k + \frac{\alpha}{2} - \frac{1}{4}\right)\pi + O\left(k^{-1}\right). \end{aligned}$$

From these we see that for large enough k and $|\xi - j_k| \leq j_k^{-1}$,

$$\begin{aligned} J_\alpha^{*\prime}(\xi) &= -\alpha\xi^{-\alpha-1}J_\alpha(\xi) + \xi^{-\alpha}J_\alpha'(\xi) \\ &= O\left(j_k^{-\alpha-\frac{3}{2}}k^{-1}\right) + O\left(j_k^{-\alpha-\frac{1}{2}}\right) = O\left(|J_\alpha^{*\prime}(j_k)|\right), \end{aligned}$$

by (2.18). See [4, p. 50, eqn. (18)]. Thus for large enough k, and $|x - j_k| \leq j_k^{-1}$,

$$|\mathcal{L}_k(x)| \leq C.$$

Since each $\mathcal{L}_k$ is continuous, this is also trivially true for small k. ■

Next, we need the Lagrange interpolation series of a function $f : [0,\infty) \to \mathbb{R}$, defined at (1.15). We also need its mth partial sum,

$$\mathcal{S}_m[f](x) = \sum_{k=1}^{m} f(j_k)\mathcal{L}_k(x). \tag{2.19}$$

Following is the main part of the proof of Theorem 1.1:

Lemma 2.2
Assume the hypotheses of Theorem 1.1. For $P \in \mathcal{P}$, we have

$$2A \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P(j_k)|^p \leq \int_0^\infty |P(t)|^p t^{2\alpha+2\sigma+1} dt. \tag{2.20}$$

Proof
We shall need a limit for $\lambda_n(x_{kn})$: for each fixed k, as $n \to \infty$, [16, p. 353, eqn. (15.3.11)]

$$\begin{aligned} \lambda_n(x_{kn}) &= 2^{\alpha+\beta+1} \left[\left(\frac{j_k}{2} \right)^\alpha J'_\alpha(j_k)^{-1} \right]^2 n^{-2\alpha-2} (1 + o(1)) \\ &= 2^{\beta-\alpha+1} J_\alpha^{*\prime}(j_k)^{-2} n^{-2\alpha-2} (1 + o(1)). \end{aligned} \tag{2.21}$$

We shall assume that

$$P(x) = \sum_{k=1}^{M} c_k \mathcal{L}_k(x).$$

Fix a positive integer L (it will be chosen large enough later), $\varepsilon \in \left(0, \frac{1}{2}\right)$, and let

$$R_n(x) = \left(\sum_{k=1}^{M} c_k \ell_{k,[n(1-\varepsilon)]}(x) \right) U_{\left[n \frac{\varepsilon}{L}\right]}(x)^L, \tag{2.22}$$

where

$$U_m(x) = \frac{T_m(x) - T_m(1)}{T'_m(1)(x-1)} = \frac{T_m(x) - T_m(1)}{m^2(x-1)}, \tag{2.23}$$

and T_m is the usual Chebyshev polynomial of degree m. We shall apply the Marcinkiewicz-Zygmund inequality in Theorem B to R_n. The factor involving U_m is included to ensure convergence of integrals below. Note that R_n has degree $\leq [n(1-\varepsilon)] - 1 + L\left(\left[n \frac{\varepsilon}{L}\right] - 1\right) \leq n - 1$. We see that

$$\begin{aligned} U_m\left(1 - \frac{1}{2}\left(\frac{z}{m}\right)^2\right) &= 2\frac{T_m\left(\cos\frac{z}{m} + o\left(\frac{1}{m^2}\right)\right) - 1}{-z^2} \\ &= 2\frac{\cos z - 1}{-z^2} + o(1) \\ &= S\left(\frac{z}{2}\right)^2 + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane. (Recall that $S(z) = \frac{\sin z}{z}$; we are also using that U_m is a polynomial). Then using this last limit and (2.8), we see that

$$\begin{aligned} R_n\left(1 - \frac{1}{2}\left(\frac{z}{n}\right)^2\right) &= \left(\sum_{k=1}^{M} c_k \ell_{k,[n(1-\varepsilon)]}\left(1 - \frac{1}{2}\left(\frac{z}{n}\right)^2\right) \right) U_{\left[n\frac{\varepsilon}{L}\right]}\left(1 - \frac{1}{2}\left(\frac{z}{n}\right)^2\right)^L \\ &= \left(\sum_{k=1}^{M} c_k \mathcal{L}_k((1-\varepsilon)z) \right) S\left(\frac{\varepsilon z}{2L}\right)^{2L} + o(1) \\ &= P((1-\varepsilon)z) S\left(\frac{\varepsilon z}{2L}\right)^{2L} + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane. Thus, given fixed $r>0$, we have as $n\to\infty$,

$$\begin{aligned}&\int_{1-\frac{1}{2}\left(\frac{r}{n}\right)^2}^{1}\left|R_n(x)\right|^p(1-x)^{\alpha+\sigma}(1+x)^{\beta+\tau}dx\\ =\;& 2^{\beta+\tau-(\alpha+\sigma)}n^{-2(\alpha+\sigma)-2}\int_0^r\left|P\left((1-\varepsilon)t\right)S\left(\frac{\varepsilon t}{2L}\right)^{2L}+o(1)\right|^p t^{2\alpha+2\sigma+1}dt.\end{aligned}\tag{2.24}$$

Also, for each fixed k, we see that as $n\to\infty$,

$$\begin{aligned}&\lambda_n(x_{kn})\left|R_n(x_{kn})\right|^p(1-x_{kn})^{\sigma}(1+x_{kn})^{\tau}\\ =\;& 2^{\beta+\tau-(\alpha+\sigma)+1}n^{-2(\alpha+\sigma)-2}j_k^{2\sigma}J_\alpha^{*\prime}(j_k)^{-2}\left|P\left((1-\varepsilon)j_k\right)S\left(\frac{\varepsilon j_k}{2L}\right)^{2L}+o(1)\right|^p.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{k=1}^{M}\lambda_n(x_{kn})\left|R_n(x_{kn})\right|^p(1-x_{kn})^{\sigma}(1+x_{kn})^{\tau}\\ =\;&(1+o(1))\,2^{\beta+\tau-(\alpha+\sigma)+1}n^{-2(\alpha+\sigma)-2}\sum_{k=1}^{M}j_k^{2\sigma}J_\alpha^{*\prime}(j_k)^{-2}\\ &\times\left|P\left((1-\varepsilon)j_k\right)S\left(\frac{\varepsilon j_k}{2L}\right)^{2L}+o(1)\right|^p.\end{aligned}\tag{2.25}$$

Next, for some C_P depending on P,ε, we see from (2.10) and (2.22-2.23) that for large enough r and $-1\le x\le 1-r/n^2$,

$$\left|R_n(x)\right|\le C_P\left(n^2(1-x)\right)^{-\frac{\alpha}{2}-\frac{5}{4}-L}\left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}},\tag{2.26}$$

so that

$$\begin{aligned}&\int_{-1}^{1-\frac{1}{2}\left(\frac{r}{n}\right)^2}\left|R_n(x)\right|^p(1-x)^{\alpha+\sigma}(1+x)^{\beta+\tau}dx\\ \le\;& CC_P^p\left\{\begin{array}{c}\left(n^{-\alpha-\frac{5}{2}-2L}\right)^p\int_{-1}^0\left(1+x+\frac{1}{n^2}\right)^{-\left(\frac{\beta}{2}+\frac{1}{4}\right)p}(1+x)^{\beta+\tau}dx\\ +\int_0^{1-\frac{1}{2}\left(\frac{r}{n}\right)^2}\left(n^2(1-x)\right)^{-\left(\frac{\alpha}{2}+\frac{5}{4}+L\right)p}(1-x)^{\alpha+\sigma}dx\end{array}\right\}\\ \le\;& CC_P^p\left\{\begin{array}{c}\left(n^{-\alpha-\frac{5}{2}-2L+\max\left\{\beta+\frac{1}{2},0\right\}}\right)^p\int_{-1}^0(1+x)^{\beta+\tau}dx\\ +n^{-2\alpha-2\sigma-2}\int_{\frac{1}{2}r^2}^{\infty}s^{-\left(\frac{\alpha}{2}+\frac{5}{4}+L\right)p+\alpha+\sigma}ds\end{array}\right\}.\end{aligned}$$

We assume that L is so large that

$$-p\left(\alpha+\frac{5}{2}+2L\right)+p\max\left\{\beta+\frac{1}{2},0\right\}<-2\alpha-2\sigma-2\tag{2.27}$$

and

$$-\left(\frac{\alpha}{2}+\frac{5}{4}+L\right)p+\alpha+\sigma<-2. \tag{2.28}$$

Then

(2.29)

$$\int_{-1}^{1-\frac{1}{2}\left(\frac{r}{n}\right)^2}\left|R_n\left(x\right)\right|^p\left(1-x\right)^{\alpha+\sigma}\left(1+x\right)^{\beta+\tau}dx\leq Cn^{-2\alpha-2\sigma-2}\left(o\left(1\right)+r^{-1}\right),$$

where C is independent of n and r. Combining this, (1.3), (2.24), and (2.25), yields, as $n\to\infty$,

$$\begin{aligned}&2A\sum_{k=1}^{M}j_k^{2\sigma}J_\alpha^{*\prime}\left(j_k\right)^{-2}\left[P\left(\left(1-\varepsilon\right)j_k\right)S\left(\frac{\varepsilon j_k}{2L}\right)^{2L}\right]^p\\ \leq\ &\int_0^r\left|P\left(\left(1-\varepsilon\right)t\right)S\left(\frac{\varepsilon t}{2L}\right)^{2L}\right|^p t^{2\alpha+2\sigma+1}dt+C/r.\end{aligned}$$

Letting $r\to\infty$ gives,

$$\begin{aligned}&2A\sum_{k=1}^{M}j_k^{2\sigma}J_\alpha^{*\prime}\left(j_k\right)^{-2}\left[P\left(\left(1-\varepsilon\right)j_k\right)S\left(\frac{\varepsilon j_k}{2L}\right)^{2L}\right]^p\\ \leq\ &\int_0^\infty\left|P\left(\left(1-\varepsilon\right)t\right)S\left(\frac{\varepsilon t}{2L}\right)^{2L}\right|^p t^{2\alpha+2\sigma+1}dt\\ \leq\ &\int_0^\infty\left|P\left(\left(1-\varepsilon\right)t\right)\right|^p t^{2\alpha+2\sigma+1}dt, \qquad (2.30)\end{aligned}$$

as $|S(t)|\leq 1$. Next, we want to let $\varepsilon\to 0+$. Observe that $P((1-\varepsilon)t)\to P(t)$ uniformly for t in compact subsets of $[0,\infty)$. Moreover, (2.11) shows that for $\varepsilon\in\left(0,\frac{1}{2}\right)$, and for some C_2 depending only on P, we have for $t\geq C_2$,

$$\left|P\left(\left(1-\varepsilon\right)t\right)\right|\leq Ct^{-\alpha-\frac{5}{2}} \tag{2.31}$$

Then uniformly in $\varepsilon\in\left(0,\frac{1}{2}\right), t\geq C_2$,

$$\left|P\left(\left(1-\varepsilon\right)t\right)\right|^p t^{2\alpha+2\sigma+1}\leq Ct^{\alpha(2-p)-\frac{5}{2}p+2\sigma+1}\leq Ct^{-1-\eta},$$

for some $\eta>0$, by (1.17). Lebesgue's Dominated Convergence Theorem gives, from (2.30)

$$2A\sum_{k=1}^{M}j_k^{2\sigma}J_\alpha^{*\prime}\left(j_k\right)^{-2}\left|P\left(j_k\right)\right|^p\leq\int_0^\infty\left|P\left(t\right)\right|^p t^{2\alpha+2\sigma+1}dt.$$

Since $P(j_k)=0$ for $k>M$, we obtain the result. ■

Next, we show that $\bar{\mathcal{P}}(p,\sigma)$ contains $L_1^p\left((0,\infty),t^{2\alpha+2\sigma+1}\right)$:

Lemma 2.3

Suppose that $f\in L_1^p\left((0,\infty),t^{2\alpha+2\sigma+1}\right)$. *Then* $f\in\bar{\mathcal{P}}(p,\sigma)$.

Proof

Suppose first that for some

$$E>\alpha+5\frac{1}{2}, \tag{2.32}$$

$$|f(x)| \leq C(1+x)^{-E}, \ x \in (0,\infty). \tag{2.33}$$

Note that then $|x|^{\alpha+\frac{1}{2}} f(x) \in L_2(\mathbb{R}\backslash(-1,1))$, so $f = \mathcal{L}[f]$ [4, p. 57, Lemma 13]. Note too that we use the evenness of f to simplify the Lagrange interpolation series in [4]. We shall show that $\mathcal{S}_m[f] \to f$ in the norm of $\bar{\mathcal{P}}(p,\sigma)$, that is

$$\lim_{m\to\infty} \int_0^\infty |f(x) - \mathcal{S}_m[f](x)|^p x^{2\sigma+2\alpha+1} dx = 0. \tag{2.34}$$

Let

$$I_k = \left[j_k - j_k^{-2}, j_k + j_k^{-2}\right] \text{ for } k \geq 1.$$

Observe that if $x \leq j_k/2$

$$|x - j_k| \geq j_k/2 \geq x$$

and if $x \geq 2j_k$, then

$$|x - j_k| \geq x/2.$$

If $\frac{j_k}{2} \leq x \leq 2j_k$ and $x \notin I_k$, then

$$|x - j_k| \geq j_k^{-2} \geq \frac{1}{2} x j_k^{-3}.$$

Thus using (2.11), if $x \notin I_k$, then

$$|\mathcal{L}_k(x)| \leq C j_k^{\alpha+4\frac{1}{2}} (1+x)^{-\alpha-2\frac{1}{2}}. \tag{2.35}$$

If $x \in I_k$, we instead use (2.12):

$$|\mathcal{L}_k(x)| \leq C.$$

Then if

$$I = \bigcup_{k\geq 1} I_k,$$

we have for $x \in [0,\infty)\backslash I$,

$$\begin{aligned}
&|f - \mathcal{S}_m[f]|(x) = |\mathcal{L}[f] - \mathcal{S}_m[f]|(x) \\
\leq\ & C(1+x)^{-\alpha-2\frac{1}{2}} \sum_{k=m+1}^{\infty} |f(j_k)| j_k^{\alpha+4\frac{1}{2}} \\
\leq\ & C(1+x)^{-\alpha-2\frac{1}{2}} \sum_{k=m+1}^{\infty} k^{-E+\alpha+4\frac{1}{2}} \\
\leq\ & C(1+x)^{-\alpha-2\frac{1}{2}} m^{-E+\alpha+5\frac{1}{2}},
\end{aligned}$$

by (2.32) and the fact that $j_k \sim k$. If $x \in I_{k_1}$ for some k_1, then since all the $\{I_k\}$ are disjoint, we have instead

$$\begin{aligned}
|f - \mathcal{S}_m[f]|(x) &\leq C(1+x)^{-\alpha-2\frac{1}{2}} m^{-E+\alpha+5\frac{1}{2}} + C \begin{cases} k_1^{-E}, & \text{if } k_1 \geq m \\ 0, & \text{if } k_1 < m \end{cases} \\
&\leq C \begin{cases} m^{-E}, & \text{if } k_1 \geq m \\ (1+x)^{-\alpha-2\frac{1}{2}} m^{-E+\alpha+5\frac{1}{2}}, & \text{if } k_1 < m \end{cases}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^\infty |f-\mathcal{S}_m[f]|^p(x)\,x^{2\sigma+2\alpha+1}dx\\
\le\ & C\left(m^{-E+\alpha+5\frac{1}{2}}\right)^p\int_{[0,\infty)\backslash\bigcup\limits_{k\ge m}I_k}(1+x)^{\left[-\alpha-2\frac{1}{2}\right]p}(x)\,x^{2\sigma+2\alpha+1}dx+Cm^{-Ep}\int_{\bigcup\limits_{k\ge m}I_k}1\\
\le\ & C\left(m^{-E+\alpha+5\frac{1}{2}}\right)^p+Cm^{-Ep}\sum_{k=m}^\infty j_k^{-2}\\
\to\ & 0 \text{ as } m\to\infty.
\end{aligned}$$

Here we also use (1.17) to ensure the convergence of the integral over $[0,\infty)\backslash\bigcup\limits_{k\ge m}I_k$.

Finally, we drop the restrictions (2.32) and (2.33). Suppose that f is an even entire function of exponential type satisfying

$$\int_0^\infty |f(t)|^p\,t^{2\sigma+2\alpha+1}dt<\infty. \tag{2.36}$$

Let $\varepsilon\in\left(0,\frac{1}{2}\right)$ and

$$g_\varepsilon(z)=f((1-\varepsilon)z)\,S\left(\frac{\varepsilon}{M}z\right)^M, \tag{2.37}$$

where $S(z)=\frac{\sin z}{z}$ and M is some large positive number. Note that g_ε is entire of exponential type ≤ 1 as $S\left(\frac{\varepsilon}{M}z\right)$ has type $\frac{\varepsilon}{M}$. Moreover, we claim that for large enough M, g_ε satisfies the bound (2.33). Indeed, the Hadamard factorization theorem shows that $f((1-\varepsilon)z)$ will have infinitely many zeros, as it is entire of type ≤ 1 and satisfies (2.36). Thus if we choose sufficiently many zeros, say $\{z_j\}_{j=1}^N$, we will have from (2.36) that

$$\int_{-\infty}^\infty\left|\frac{f((1-\varepsilon)t)}{\prod\limits_{j=1}^N\left(t^2-z_j^2\right)}\right|^p dt<\infty.$$

Then the function $\frac{f((1-\varepsilon)t)}{\prod\limits_{j=1}^N\left(t^2-z_j^2\right)}$ belongs to the Paley-Wiener space L_1^p and so is bounded on the real line [6, p.149]. Then for $t\ge 1$,

$$|g_\varepsilon(t)|\le C\,|t|^{2N-M},$$

so choosing M large enough, g_ε satisfies (2.33). Consequently $g_\varepsilon\in\bar{\mathcal{P}}(p,\sigma)$. Next, we show that as $\varepsilon\to 0+$, $g_\varepsilon\to f$ in the appropriate norm. Let $R>0$. Using that

$|S| \leq 1$, we see that

$$\begin{aligned}
&\int_0^\infty |f(t) - g_\varepsilon(t)|^p t^{2\sigma+2\alpha+1} dt \\
\leq\ & \int_0^R |f(t) - g_\varepsilon(t)|^p t^{2\sigma+2\alpha+1} dt + 2^p \int_R^\infty \left(|f(t)|^p + |g_\varepsilon(t)|\right) t^{2\sigma+2\alpha+1} dt \\
\leq\ & \int_0^R |f(t) - g_\varepsilon(t)|^p t^{2\sigma+2\alpha+1} dt + 2^p \int_R^\infty |f(t)|^p t^{2\sigma+2\alpha+1} dt \\
& + 2^p (1-\varepsilon)^{-(2\alpha+2\sigma+2)} \int_{R(1-\varepsilon)}^\infty |f(t)|^p t^{2\sigma+2\alpha+1} dt \\
\leq\ & \int_0^R |f(t) - g_\varepsilon(t)|^p t^{2\sigma+2\alpha+1} dt + 2^{p+1+(2\alpha+2\sigma+2)} \int_{R/2}^\infty |f(t)|^p t^{2\sigma+2\alpha+1} dt,
\end{aligned}$$

as $\varepsilon \in \left(0, \frac{1}{2}\right)$. We can first choose $R > 0$ so large that the second integral in the last right-hand side is smaller than a given $\delta > 0$, and then use that $g_\varepsilon(t) \to f(t)$ uniformly in $[0, R]$ as $\varepsilon \to 0+$. Then as $g_\varepsilon \in \bar{\mathcal{P}}(p, \sigma)$, so also $f \in \bar{\mathcal{P}}(p, \sigma)$. ■

The Proof of Theorem 1.1

Suppose first that $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$ and in addition (2.32) and (2.33) hold. Then we know that (2.34) is true. By Lemma 2.2, for $m \geq 1$,

$$\begin{aligned}
& 2A \sum_{k=1}^m j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \\
=\ & 2A \sum_{k=1}^m j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |\mathcal{S}_m[f](j_k)|^p \\
\leq\ & \int_0^\infty |\mathcal{S}_m[f](t)|^p t^{2\alpha+2\sigma+1} dt.
\end{aligned}$$

Letting $m \to \infty$, and using (2.34), we obtain for each fixed $N \geq 1$,

$$2A \sum_{k=1}^N j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \leq \int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt.$$

Now letting $N \to \infty$ gives (1.18). Now consider the general case, so that we drop (2.32-33). Let $\varepsilon > 0$ and define g_ε by (2.37), with M large enough. Since $|S| \leq 1$, we have for $N \geq 1$,

$$\begin{aligned}
2A \sum_{k=1}^N j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |g_\varepsilon(j_k)|^p &\leq \int_0^\infty |f((1-\varepsilon)t)|^p t^{2\alpha+2\sigma+1} dt \\
&= (1-\varepsilon)^{-p} \int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt.
\end{aligned}$$

Letting $\varepsilon \to 0$, and then $N \to \infty$ gives (1.18). ■

Proof of Corollary 1.2

Let

$$f_e(t) = f(t) + f(-t) \text{ and } f_o(t) = f(t) - f(-t) \tag{2.38}$$

denote the even and odd parts of f, and let

$$g(t) = f_o(t)/t. \tag{2.39}$$

Since $f(t) = f_e(t) + tg(t)$, and both f_e and g are even,

$$\begin{aligned}
& \sum_{k=-\infty,k\neq 0}^{\infty} |j_k|^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p \\
\leq\ & 2^p \sum_{k=-\infty,k\neq 0}^{\infty} |j_k|^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} \{|f_e(j_k)|^p + |j_k|^p |g(j_k)|^p\} \\
=\ & 2^{p+1} \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} \{|f_e(j_k)|^p + |j_k|^p |g(j_k)|^p\} \\
\leq\ & 2^{p+1} \left\{ A_1^{-1} \int_0^\infty |f_e(t)|^p t^{2\sigma+2\alpha+1} dt + A_2^{-1} \int_0^\infty |g(t)|^p t^{2\left(\sigma+\frac{p}{2}\right)+2\alpha+1} dt \right\},
\end{aligned}$$

by Theorem 1.1 with A_1 and A_2 corresponding to the constants for σ and $\sigma + \frac{p}{2}$ respectively. We continue this as

$$\begin{aligned}
\leq\ & 2^{2p+1} \left\{ A_1^{-1} \int_0^\infty (|f(t)|^p + |f(-t)|^p) t^{2\sigma+2\alpha+1} dt + A_2^{-1} \int_0^\infty (|f(t)|^p + |f(-t)|^p) t^{2\sigma+2\alpha+1} dt \right\} \\
\leq\ & 2^{2p+1} \left(A_1^{-1} + A_2^{-1}\right) \int_{-\infty}^{\infty} |f(t)|^p |t|^{2s+2\alpha+1} dt.
\end{aligned}$$

Note that Theorem 1.1 is applicable to the function g with σ replaced by $\sigma + \frac{p}{2}$, since (1.17) becomes (1.19) with this adjusted choice of σ. ∎

3. Proof of Theorem 1.3 and Corollary 1.4

Lemma 3.1
Assume the conclusion (1.9) of Theorem C.
(a) For each $P \in \mathcal{P}$, we have

$$\int_0^\infty |P(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P(j_k)|^p. \tag{3.1}$$

(b) Assume that $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$. Then

$$\int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p. \tag{3.2}$$

Proof
(a) Let P be as in the proof of Lemma 2.2 and define R_n by (2.22). We know that by (2.24) and (2.29),

$$\begin{aligned}
& \int_{-1}^{1} |R_n(x)|^p (1-x)^{\alpha+\sigma} (1+x)^{\beta+\tau} dx \\
=\ & 2^{\beta+\tau-(\alpha+\sigma)} n^{-2(\alpha+\sigma)-2} \int_0^r \left| P((1-\varepsilon)t) S\left(\frac{\varepsilon t}{2L}\right)^{2L} + o(1) \right|^p t^{2\alpha+2\sigma+1} dt \\
& + n^{-2\alpha-2\sigma-2} \left(o(1) + O\left(r^{-1}\right)\right).
\end{aligned} \tag{3.3}$$

Next by (2.16) and (2.26), for $x_{kn} \leq 1 - \frac{r}{n^2}$,

$$\begin{aligned} & \lambda_n(x_{kn})\left|R_n(x_{kn})\right|^p (1-x_{kn})^\sigma (1+x_{kn})^\tau \\ \leq \; & \frac{C}{n}(1-x_{kn})^{\alpha+\sigma+\frac{1}{2}}(1+x_{kn})^{\beta+\tau+\frac{1}{2}}\left|\left(n^2(1-x_{kn})\right)^{-\frac{\alpha}{2}-\frac{5}{4}-L}\left(1+x_{kn}+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}\right|^p. \end{aligned}$$

Since uniformly in k and n, we have

$$x_{kn} - x_{k+1,n} \sim \frac{1}{n}\sqrt{1-x_{kn}^2},$$

we see that given $N = N(n) < n$,

$$\begin{aligned} & \sum_{k=N+1}^{n} \lambda_n(x_{kn})\left|R_n(x_{kn})\right|^p (1-x_{kn})^\sigma (1+x_{kn})^\tau \\ \leq \; & C\sum_{k=N+1}^{n}(x_{kn}-x_{k+1,n})(1-x_{kn})^{\alpha+\sigma}(1+x_{kn})^{\beta+\tau}\left|\left(n^2(1-x_{kn})\right)^{-\frac{\alpha}{2}-\frac{5}{4}-L}\left(1+x_{kn}+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}\right|^p \\ \leq \; & C\int_{-1}^{x_{Nn}}(1-x)^{\alpha+\sigma}(1+x)^{\beta+\tau}\left|\left(n^2(1-x)\right)^{-\frac{\alpha}{2}-\frac{5}{4}-L}\left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}\right|^p dx \\ \leq \; & C\int_{-1}^{0}(1+x)^{\beta+\tau}\left|\left(n^2\right)^{-\frac{\alpha}{2}-\frac{5}{4}-L}\left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}}\right|^p dx \\ & +C\int_{0}^{x_{Nn}}(1-x)^{\alpha+\sigma}\left|\left(n^2(1-x)\right)^{-\frac{\alpha}{2}-\frac{5}{4}-L}\right|^p dx \\ \leq \; & Cn^{-\left(\alpha+\frac{5}{2}+2L\right)p+p\max\left\{\beta+\frac{1}{2},0\right\}} + Cn^{-2(\alpha+\sigma)-2}\int_{n^2(1-x_{Nn})}^{\infty} t^{\alpha+\sigma-\left(\frac{\alpha}{2}+\frac{5}{4}+L\right)p}dt \\ = \; & n^{-2\alpha-2\sigma-2}\left(o(1)+O\left(N^{-1}\right)\right), \end{aligned}$$

(3.4)

by (2.7) and provided L is so large that (2.27) and (2.28) hold. Combining (1.9), (2.25), (3.3) and (3.4), gives as $n \to \infty$,

$$\begin{aligned} & \left[\int_0^r \left|P((1-\varepsilon)t)\,S\left(\frac{\varepsilon t}{2L}\right)^{2L}\right|^p t^{2\alpha+2\sigma+1}dt + O\left(r^{-1}\right)\right] \\ \leq \; & 2B\sum_{k=1}^{N} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2}\left[P((1-\varepsilon)j_k)\,S\left(\frac{\varepsilon j_k}{2L}\right)^{2L}\right]^p + O\left(N^{-1}\right). \end{aligned}$$

Since $|S(t)| \leq 1$, we obtain, letting $N \to \infty$, and then $r \to \infty$,

$$\int_0^\infty \left|P((1-\varepsilon)t)\,S\left(\frac{\varepsilon t}{2L}\right)^{2L}\right|^p t^{2\alpha+2\sigma+1}dt \leq 2B\sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2}\left|P((1-\varepsilon)j_k)\right|^p.$$

We now want to let $\varepsilon \to 0+$. We use (2.31) and (2.18) above, which give uniformly in $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$
\begin{aligned}
& \sum_{k=N}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} \left|P\left((1-\varepsilon) j_k\right)\right|^p \\
\leq \; & C \sum_{k=N}^{\infty} j_k^{2\sigma} \left(j_k^{\alpha+\frac{1}{2}}\right)^2 \left(j_k^{-\alpha-\frac{5}{2}}\right)^p = C \sum_{k=N}^{\infty} j_k^{-\left(\alpha+\frac{5}{2}\right)p+2\sigma+2\alpha+1} \to 0,
\end{aligned}
$$

as $N \to \infty$, since $j_k \geq Ck$, and by (1.17). Then we can let $\varepsilon \to 0+$, so that for each $R > 0$,

$$
\int_0^R |P(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P(j_k)|^p .
$$

Now we let $R \to \infty$ in the left-hand side.
(b) By Lemma 2.3, $f \in \bar{\mathcal{P}}(p, \sigma)$. Let $\{P_m\} \subset \mathcal{P}$ satisfy (1.16). Recall that we are assuming $p > 1$. We use the inequality

$$
\left||b|^p - |a|^p\right| \leq |b-a| \, p \left(|a|^{p-1} + |b|^{p-1}\right).
$$

Then using Hőlder's inequality,

$$
\begin{aligned}
& \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} \left||P_m(j_k)|^p - |f(j_k)|^p\right| \\
\leq \; & p \sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P_m(j_k) - f(j_k)| \left(|P_m(j_k)|^{p-1} + |f(j_k)|^{p-1}\right) \\
\leq \; & p \left(\sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P_m(j_k) - f(j_k)|^p\right)^{1/p} \\
& \times \left\{ \left(\sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P_m(j_k)|^p\right)^{\frac{p-1}{p}} + \left(\sum_{k=1}^{\infty} j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p\right)^{\frac{p-1}{p}} \right\} \\
\leq \; & pA^{-1} \left(\int_0^\infty |P_m - f|(t)^p t^{2\alpha+2\sigma+1} dt\right)^{1/p} \\
& \times \left\{ \left(\int_0^\infty |P_m|(t)^p t^{2\alpha+2\sigma+1} dt\right)^{(p-1)p} + \left(\int_0^\infty |f|(t)^p t^{2\alpha+2\sigma+1} dt\right)^{(p-1)p} \right\},
\end{aligned}
$$

by Theorem 1.1. Note that this is applicable as each $\mathcal{L}_k \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$. Using (1.16), we obtain from the above

$$
\lim_{m\to\infty} \sum_{k=1}^{\infty} j_k J_\alpha^{*\prime}(j_k)^{-2} |P_m(j_k)|^p = \sum_{k=1}^{\infty} j_k J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p .
$$

Hence, using (a) of this lemma,

$$\begin{aligned}\int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt &= \lim_{m\to\infty} \int_0^\infty |P_m(t)|^p t^{2\alpha+2\sigma+1} dt \\ &\leq \lim_{m\to\infty} 2B \sum_{k=1}^\infty j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |P_m(j_k)|^p \\ &= 2B \sum_{k=1}^\infty j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p .\end{aligned}$$

■

Next, we reformulate Xu's sufficient conditions for converse quadrature sum inequalities into the form given in Theorem C:

Lemma 3.2
Assume that (1.7) and (1.8) hold. Then the converse quadrature inequality (1.9) holds.
Proof
Let $q = \frac{p}{p-1}$. By taking $u = w^{\sigma,\tau}$ and $\beta' = uw^{\alpha,\beta}$ in Xu's Theorem 2.1 [17, pp. 881-882], we see that his conditions for (1.9) become:

$$\begin{aligned}\int_{-1}^1 (w^{\sigma,\tau})^{1-q} w^{\alpha,\beta} &< \infty; \\ \int_{-1}^1 (w^{\sigma,\tau})^{1-q} \left(w^{\alpha,\beta}\right)^{1-q/2} \left(w^{1/2,1/2}\right)^{-q/2} &< \infty; \\ \int_{-1}^1 w^{\sigma,\tau} \left(w^{\alpha,\beta}\right)^{1-p/2} \left(w^{1/2,1/2}\right)^{-p/2} &< \infty.\end{aligned}$$

In terms of the parameters α, σ, p, q, these require

$$\sigma(1-q) + \alpha > -1;$$

$$\sigma(1-q) + \alpha(1-q/2) - q/4 > -1;$$

$$\sigma + \alpha(1-p/2) - p/4 > -1. \tag{3.5}$$

Since $(q-1)(p-1) = 1$, we can reformulate the first two as

$$\sigma - (\alpha+1)(p-1) < 0;$$

$$\sigma - (p-1)\left[\alpha\left(1 - \frac{p}{2(p-1)}\right) - \frac{p}{4(p-1)} + 1\right] < 0,$$

or

$$\sigma - (\alpha+1)(p-1) + \frac{p}{2}\left(\alpha + \frac{1}{2}\right) < 0.$$

Together, these first two conditions give the upper bound

$$\sigma < (\alpha+1)(p-1) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\}.$$

The third condition (3.5) gives

$$\sigma > \frac{p}{2}\left(\alpha + \frac{1}{2}\right) - (\alpha+1).$$

Combining the last two conditions gives (1.7). The inequality (1.8) involving β, τ, p, q follows similarly. Then Xu's Theorem 2.1 gives (1.9). ■

Proof of Theorem 1.3
This follows directly from Lemmas 3.1 and 3.2 above. Note that the upper bound on σ in (1.7) implies that on σ in (1.17). Indeed the former upper bound minus the latter is

$$\begin{aligned} & (p-1)(\alpha+1) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\} - \left[p\left(\frac{\alpha}{2}+\frac{5}{4}\right) - (\alpha+1)\right] \\ = \quad & p\left(\frac{\alpha}{2}-\frac{1}{4}\right) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\} = \min\left\{p\left(\frac{\alpha}{2}-\frac{1}{4}\right), -\frac{p}{2}\right\} \\ \leq \quad & -\frac{p}{2} < 0, \end{aligned}$$

so (1.7) is indeed more restrictive. ■

The Proof of Corollary 1.4
Step 1: We first prove this for even $f \in L_1^p\left((0,\infty), t^{2\sigma+2\alpha+1}\right)$.
Choose a non-negative integer ℓ such that

$$\ell\frac{p}{2} < \sigma - D < (\ell+1)\frac{p}{2}.$$

Here D is defined by (1.23). Then

$$\sigma = D + \ell\frac{p}{2} + \rho,$$

for some $\rho \in \left(0, \frac{p}{2}\right)$. Let

$$\hat{\sigma} = D + \rho,$$

so that

$$2\sigma = 2\hat{\sigma} + \ell p. \tag{3.6}$$

We next show that $\hat{\sigma}$ satisfies (1.7). Firstly $\hat{\sigma} > D$, so satisfies the lower bound in (1.7). Next, the difference of the upper bound in (1.7) and $\hat{\sigma}$ is

$$\begin{aligned} & (p-1)(\alpha+1) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\} - D - \rho \\ > \quad & (p-1)(\alpha+1) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\} \\ & -\frac{p}{2}\left(\alpha+\frac{1}{2}\right) + (\alpha+1) - \frac{p}{2} \\ = \quad & \frac{p}{2}\left(\alpha+\frac{1}{2}\right) - \max\left\{0, \frac{p}{2}\left(\alpha+\frac{1}{2}\right)\right\} = 0, \end{aligned}$$

as $\alpha \geq -\frac{1}{2}$. So indeed, the right inequality in (1.7) holds for $\hat{\sigma}$. Also, (1.23) shows that the left inequality in (1.7) holds. Thus the conclusion of Theorem 1.3 holds for the class $L_1^p\left((0,\infty), t^{2\alpha+2\hat{\sigma}+1}\right)$. Here we also note that $\alpha + \hat{\sigma} > -1$ follows from (1.23). Now let $f \in L_1^p\left((0,\infty), t^{2\alpha+2\sigma+1}\right)$, and then set

$$h(t) = t^{\ell} f(t),$$

which is entire of exponential type ≤ 1. Note that by (3.6),

$$\int_0^\infty |h(t)|^p t^{2\alpha+2\hat{\sigma}+1} dt = \int_0^\infty |f(t)|^p t^{2\alpha+2\hat{\sigma}+1} dt < \infty.$$

Then Theorem 1.3 applied to h gives

$$\int_0^\infty |h(t)|^p t^{2\alpha+2\hat{\sigma}+1} dt \leq 2B(\hat{\sigma}) \sum_{k=1}^\infty j_k^{2\hat{\sigma}} J_\alpha^{*\prime}(j_k)^{-2} |h(j_k)|^p.$$

In view of (3.6), and our choice of h, this becomes

$$\int_0^\infty |f(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B(\hat{\sigma}) \sum_{k=1}^\infty j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f(j_k)|^p,$$

which implies (1.25) since f is even.

Step 2: Now consider non-even f

As in the proof of Corollary 1.2, we define f_e, f_o, g by (2.38) and (2.39). We have both

$$\int_0^\infty |f_e(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B_3 \sum_{k=1}^\infty j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f_e(j_k)|^p, \tag{3.7}$$

and since the restrictions (1.23) and (1.24) hold for g with σ replaced by $\sigma + \frac{p}{2}$,

$$\int_0^\infty |g(t)|^p t^{2\alpha+2\left(\sigma+\frac{p}{2}\right)+1} dt \leq 2B_3 \sum_{k=1}^\infty j_k^{2\left(\sigma+\frac{p}{2}\right)} J_\alpha^{*\prime}(j_k)^{-2} |g(j_k)|^p,$$

which gives

$$\int_0^\infty |f_0(t)|^p t^{2\alpha+2\sigma+1} dt \leq 2B_3 \sum_{k=1}^\infty j_k^{2\sigma} J_\alpha^{*\prime}(j_k)^{-2} |f_0(j_k)|^p.$$

This and (3.7) and some elementary inequalities give the result. ■

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