Marcinkiewicz-Zygmund Inequalities: Methods and Results

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Abstract

The Gauss quadrature formula for a weight W^2 on the real line has the form

$$\sum_{j=1}^{n} \lambda_{jn} P(x_{jn}) = \int PW^2$$

for polynomials P of degree $\leq 2n-1$. In studying convergence of Lagrange interpolation in L_p norms, $p \neq 2$, one needs forward and converse quadrature sum estimates such as

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn}) \leq\geq C \int \mid PW \mid^p$$

with C independent of n and P. These are often called Marcinkiewicz-Zygmund inequalities after their founders. We survey methods to prove these and the results that can be achieved using them. Our focus is on weights on the whole real line, but we also refer to results for $(-1,1)$ and the plane. In particular, we present four methods to prove forward estimates and two to prove converse ones.

1. Introduction

There is an intimate connection between Gauss quadrature sums and mean convergence of Lagrange interpolation - hardly surprising, when both involve zeros of orthogonal polynomials.

Let $d\alpha$ be a non-negative measure on $\mathbb{R}$, and $\{p_n(x)\}_{n=0}^{\infty}$ be its orthonormal polynomials, so that

$$\int p_n p_m d\alpha = \delta_{mn}.$$

If $d\alpha(x) = w(x)dx$, and we need to indicate the dependence on w, we write $p_n(w, x)$, etc. Let

$$-\infty < x_{nn} < x_{n-1,n} < ... < x_{1n} < \infty$$

denote the zeros of $p_n(x)$ and let $\{\lambda_{jn}\}$ denote the Christoffel numbers. The Gauss quadrature formula is

$$\sum_{j=1}^{n} \lambda_{jn} P(x_{jn}) = \int P d\alpha, P \in \mathcal{P}_{2n-1}.$$

Here $\mathcal{P}_m$ denotes the polynomials of degree $\leq m$. Let $L_n[f] \in \mathcal{P}_{n-1}$ denote the Lagrange interpolation polynomial to f at the zeros of p_n, so that

$$L_n[f](x_{jn}) = f(x_{jn}), 1 \leq j \leq n.$$

The connection between convergence of Lagrange interpolation and convergence of Gauss quadrature is nowhere clearer than in the following result: Let f be in the $L_2(d\alpha)$ closure of the polynomials. Then

$$\lim_{n\to\infty} \int (f - L_n[f])^2 d\alpha = 0 \tag{1.1}$$

if and only if for every polynomial P

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn}(f - P)^2(x_{jn}) = \int (f - P)^2 d\alpha.$$

These relations are of course part of Shohat's extension to the infinite interval of the classic result of Erdös-Turan on L_2 convergence of Lagrange interpolation.

For $L_p, p \neq 2$, things are far more complicated and we need forward and converse quadrature sum estimates, often called Marcinkiewicz-Zygmund inequalities.

Zygmund's classic treatise contains a particularly elegant proof of both forward and converse estimates in the case of trigonometric polynomials [48,Ch.X,pp.28-29]. Let us illustrate the use of these in the context of weights of the form

$$d\alpha(x) = W^2(x)dx$$

where $W(x)$ is a non-negative function, the archetype being

$$W(x) = W_\beta(x) := \exp(- \mid x \mid^\beta), \beta > 1.$$

At first sight, the use of W^2 for the weight, rather than W, seems strange, but is standard for weights on the whole real line: it simplifies formulation of results.

1.1. Forward Quadrature sums in Lagrange Interpolation

Let us assume that we have a *forward quadrature sum estimate* of the form

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})\phi(x_{jn}) \mid PW \mid^p (x_{jn}) \leq C \int \mid PW \mid^p \phi, \tag{1.2}$$

$P \in \mathcal{P}_{n-1}$, where $C \neq C(n, P)$. Here ϕ is a slowly changing function. For our purposes, we can take $\phi(x) := (1 + x^2)^{-1/p}$.

The main part of proving mean convergence of Lagrange interpolation in a weighted L_p norm is showing uniform boundedness in n of the operator L_n. Let $1 < p < \infty$ and $q := p/(p-1)$. Then by duality

$$\| L_n[f]W \|_{L_p(\mathbb{R})} = \sup_g \int L_n[f]gW^2$$

where the sup is taken over all g with $\| gW \|_{L_q(\mathbb{R})} = 1$. To proceed, we use the partial sums

$$S_n[g](x) = \sum_{j=0}^{n-1} c_j p_j(x); c_j = \int g p_j W^2 \; \forall j.$$

of the orthonormal expansion of g. Since $g - S_n[g]$ is orthogonal to $\mathcal{P}_{n-1}$, we have

$$\int L_n[f]gW^2 = \int L_n[f]S_n[g]W^2 = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}) S_n[g](x_{jn})$$

by the Gauss quadrature formula. Let us now assume that

$$\mid fW \mid (x) \leq \phi(x) = (1 + x^2)^{-1/p}, x \in \mathbb{R}. \tag{1.3}$$

Then we obtain

$$| \int L_n[f]gW^2 | \leq \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})\phi(x_{jn}) \mid S_n[g]W \mid (x_{jn})$$

$$\leq C \int \mid S_n[g] \mid W\phi,$$

if we use (1.2) with $p = 1$. Setting $\sigma_n := sign(S_n[g])$ and then using the symmetry property of the operator S_n, we can continue this as

$$= C \int S_n[g](\sigma_n \phi W^{-1})W^2 = C \int g S_n[\sigma_n \phi W^{-1}]W^2$$

$$\leq C \parallel gW \parallel_{L_q(\mathbb{R})} \parallel S_n[\sigma_n \phi W^{-1}]W \parallel_{L_p(\mathbb{R})} .$$

Assuming a suitable mean boundedness of the operator S_n from L_p to L_p with suitable weights, we can continue this as

$$\leq C \parallel gW \parallel_{L_q(\mathbb{R})} C_1 \parallel [\sigma_n \phi W^{-1}]W \parallel_{L_p(\mathbb{R})} = CC_1 \parallel \phi \parallel_{L_p(\mathbb{R})} =: C_2 < \infty.$$

Thus we have shown that $\forall f$ satisfying (1.3),

$$\parallel L_n[f]W \parallel_{L_p(\mathbb{R})} \leq C_2, n \geq 1.$$

Here $C_2 \neq C_2(n, f)$. This and the reproducing property

$$L_n[P] = P, \; P \in \mathcal{P}_{n-1}$$

and the density of polynomials give convergence of $\{L_n[f]\}_{n=1}^{\infty}$ in weighted L_p norms.

We emphasise that this is just an illustration. The complete proofs are more complicated and require breaking up the L_p norm of $L_n[f]$ into several different pieces; the quadrature sum often includes x_{jn} only for those j satisfying $| x_{jn} | \leq (1 - \varepsilon)x_{1n}$ with fixed $0 < \varepsilon < 1$; and suitable factors are often inserted into the weighted L_p norms.

1.2. Converse Quadrature Sums in Lagrange Interpolation

Assume that we have a *converse quadrature sum inequality* of the form

$$\parallel PW \parallel_{L_p(\mathbb{R})} \leq C\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})\}^{1/p} \qquad (1.4)$$

for $P \in \mathcal{P}_{n-1}$ with $C \neq C(n, P)$. Then

$$\| L_n[f]W \|_{L_p(\mathbb{R})} \leq C\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid fW \mid^p (x_{jn})\}^{1/p}$$

$$\leq C\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})(1 + x_{jn}^2)^{-1}\}^{1/p}$$

provided (1.3) holds. This last quadrature sum converges as $n \to \infty$ to

$$C\{\int_{\mathbb{R}} (1 + x^2)^{-1} dx\}^{1/p}.$$

So again we have uniform boundedness in n for functions f satisfying (1.3), and hence convergence.

Clearly converse quadrature sum estimates yield boundedness of $\{L_n\}$ in a far simpler way than forward ones. However as we shall see, they are usually more difficult to prove and more restrictive in scope. There is also an almost incestuous duality between forward and converse estimates, as we shall see.

Historically, forward and converse quadrature sum estimates were first considered by Marcinkiewicz and Zygmund in the 1930's [19], [20]. As we have remarked, Zygmund's treatise contains a concise elegant treatment of both forward and converse estimates for trigonometric polynomials. R. Askey seems the first to have applied these estimates in studying Lagrange interpolation for Jacobi weights in the 1970's [1], and subsequently P. Nevai studied and applied these for the Hermite weight and Jacobi weights [34], [36], [37].

Indeed, it seems P. Nevai and his collaborators have been responsible for intensively studying and developing these inequalities [33-37], [15]. The author and his students have concentrated on the case of weights on the whole real line [3], [4], [8], [16-18] while Y.Xu has considered generalized Jacobi weights [42-45]. A particularly interesting method has been developed by H. König [9], [10] in the context of Banach spaces, but yields new results even in the scalar case. Complex methods such as Carleson measures and H_p space techniques have been developed by Zhong and Zhu [47], see also Peller [39].

In spirit, estimates for Lebesgue functions of Lagrange/ Hermite/ Hermite-Fejer interpolation are related to the quadrature sum estimates we consider here, but we shall not discuss them. See for example [23-26], [38], [41].

This paper is organised as follows: In Section 2, we outline four methods to prove forward quadrature sum estimates and discuss some of the results that can be proved using them. In Section 3, we outline two methods to prove converse quadrature sum estimates and results that they yield. In Section 4, we present some conclusions, and some open problems.

As a preparation for subsequent sections, we present more notation, and we also define classes of weights on the real line. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. Given real sequences $\{b_n\}, \{c_n\}$ we write

$$b_n \sim c_n$$

if there exist C_1, C_2 such that

$$C_1 \leq c_n/b_n \leq C_2$$

for the relevant range of n. Similar notation will be used for functions and sequences of functions.

Our weights on $\mathbb{R}$ always have the form $W^2(x) = e^{-2Q(x)}$ where Q is even and convex. Much as one distinguishes between entire functions of finite and infinite order, one distinguishes between Q of polynomial growth at ∞ (the so-called Freud weights) and of faster than polynomial growth at ∞ (the so-called Erdös weights). We define first a suitable class of the former:

Definition 1.1. *Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$, $Q' > 0$ in $(0, \infty)$, and for some $A, B > 1$,*

$$A \leq 1 + \frac{xQ''(x)}{Q'(x)} \leq B, x \in (0, \infty).$$

Then we write $W \in \mathcal{F}$.

The most important examples are $W(x) = W_\beta(x) = \exp(- \mid x \mid^\beta), \beta > 1$.

Definition 1.2. *Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$, $Q'', Q' > 0$ in $(0, \infty)$ and $T(x) := 1 + xQ''(x)/Q'(x)$ is increasing in $(0, \infty)$ with*

$$\lim_{x \to 0+} T(x) > 1; \lim_{x \to \infty} T(x) = \infty.$$

Moreover, assume that for some $C_j > 0, j = 1, 2, 3,$

$$C_1 \leq T(x)/(\frac{xQ'(x)}{Q(x)}) \leq C_2, x \geq C_3$$

and for every $\varepsilon > 0$,

$$T(x) = O(Q(x)^{\varepsilon}), x \to \infty.$$

Then we write $W \in \mathcal{E}$.

The most important examples are $W(x) = \exp(-\exp_k(\mid x \mid^{\beta}))$, where $\beta > 1, k \geq 1$ and $\exp_k = \exp(\exp(\ldots \exp()))$ denotes the kth iterated exponential.

For both Freud and Erdös weights, the *Mhaskar-Rahmanov-Saff number* a_n plays an important role. It is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}}. \tag{1.5}$$

One of its important properties is

$$\| PW \|_{L_\infty(\mathbb{R})} = \| PW \|_{L_\infty[-a_n, a_n]}, P \in \mathcal{P}_n \tag{1.6}$$

and for $0 < p < \infty$,

$$\| PW \|_{L_p(\mathbb{R})} \leq C \| PW \|_{L_p[-a_n, a_n]}, P \in \mathcal{P}_n \tag{1.7}$$

where $C \neq C(n, P)$ [27], [28], [11], [13]. Concerning its growth, we note that a_n is increasing in n, and grows roughly like $Q^{[-1]}(n)$, where $Q^{[-1]}$ is the inverse of Q on $(0, \infty)$. For those to whom it is new, a good example to think of is $W = W_\beta$, $Q(x) = \mid x \mid^{\beta}$, for which $a_n = Cn^{1/\beta}$, $n \geq 1$.

In presenting the various methods, we shall use the following estimates that hold for the class $\mathcal{F}$ of Freud weights (all of which can be found in [11], [2]). Define

$$\psi_n(x) := \max\{n^{-2/3}, 1 - \mid x \mid / a_n\}; \tag{1.8}$$

and

$$x_{0n} := x_{1n}(1 + n^{-2/3}); x_{n+1,n} := x_{nn}(1 + n^{-2/3}). \tag{1.9}$$

Then

$$\mid 1 - \frac{x_{1n}}{a_n} \mid \leq cn^{-2/3} \tag{1.10}$$

and uniformly in j, n

$$\lambda_{jn} W^{-2}(x_{jn}) \sim x_{j-1,n} - x_{jn} \sim \frac{a_n}{n} \psi_n^{-1/2}(x_{jn}). \tag{1.11}$$

The Christoffel numbers $\{\lambda_{jn}\}$ are special cases of the Christoffel functions

$$\lambda_n(W^2, x) := \inf_{P \in \mathcal{P}_{n-1}} \int (PW)^2(t)dt/P^2(x) \tag{1.12}$$

which admit the estimate

$$\lambda_n(W^2, x)W^{-2}(x) \geq C\frac{a_n}{n}\psi_n^{-1/2}(x), x \in \mathbb{R}. \tag{1.13}$$

The orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ for W^2 satisfy

$$| p_n(x) | W(x) \leq C\frac{a_n}{n}\psi_n^{-1/4}(x), x \in \mathbb{R}, n \geq 1. \tag{1.14}$$

2. Forward Quadrature Sum estimates

In illustrating the four methods to prove forward quadrature sum estimates, we shall assume that $W \in \mathcal{F}$, and that our weight is W^2. We shall also often use the estimates (1.10) - (1.14).

2.A Nevai's Method

This simple method requires an estimate like (1.11) and a suitable Markov-Bernstein inequality. The most influential papers (and possibly the first) papers in which it was used were those of P. Nevai for Jacobi and Hermite weights [34], [36], [37]. Given $u \in [x_{jn}, x_{j-1,n}]$, we have from the fundamental theorem of calculus,

$$| PW |^p (x_{jn}) \leq | PW |^p (u) + p\int_u^{x_{j-1,n}} | PW |^{p-1} (s) | (PW)'(s) | ds.$$

We can assume that u is the point in $[x_{jn}, x_{j-1,n}]$ where $| PW |^p$ attains its minimum. If we now use our estimate (1.11) for the Christoffel numbers, we obtain

$$\lambda_{jn}W^{-2}(x_{jn}) | PW |^p (x_{jn}) \leq C\int_{x_{jn}}^{x_{j-1,n}} | PW |^p (u)du$$

$$+C\frac{a_n}{n}\psi_n^{-1/2}(x_{jn})\int_{x_{jn}}^{x_{j-1,n}} | PW |^{p-1} (s) | (PW)'(s) | ds.$$

Summing over j, and using the fact that ψ_n does not change much in $[x_{jn}, x_{j-1,n}]$ (see [11] if you want a proof), we obtain

$$\sum_{j=1}^{n} \lambda_{jn}W^{-2}(x_{jn}) | PW |^p (x_{jn}) \leq C\int_{\mathbb{R}} | PW |^p (u)du$$

$$+C_1\frac{a_n}{n}\int_{\mathbb{R}} \mid PW \mid^{p-1}(s) \mid (PW)'(s) \mid \psi_n^{-1/2}(s)ds. \tag{2.1}$$

At this stage, we need a quite sophisticated Markov-Bernstein inequality of the form

$$\| (PW)'\psi_n^{-1/2} \|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n} \| PW \|_{L_p(\mathbb{R})}, P \in \mathcal{P}_n. \tag{2.2}$$

This was proved in [12] for the class $\mathcal{F}$, using Carleson measures. Applying Hölder's inequality with parameters $q := p/(p-1)$ and p to the second term in (2.1) and then this Markov-Bernstein inequality give

$$\frac{a_n}{n}\int_{\mathbb{R}} \mid PW \mid^{p-1}(s) \mid (PW)'(s) \mid \psi_n^{-1/2}(s)ds$$

$$\leq \frac{a_n}{n} \| PW \|_{L_p(\mathbb{R})}^{p-1} \| (PW)'\psi_n^{-1/2} \|_{L_p(\mathbb{R})} \leq C \| PW \|_{L_p(\mathbb{R})}^p .$$

So we have shown

$$\{\sum_{j=1}^n \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})\}^{1/p} \leq C \| PW \|_{L_p(\mathbb{R})} .$$

The real bugbear of this method, at least for full quadrature sums, is the sophisticated Markov-Bernstein inequality (2.2). In his treatment of the Hermite weight, P. Nevai used a somewhat weaker inequality, namely

$$\| (PW)' \|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n} \| PW \|_{L_p(\mathbb{R})}, P \in \mathcal{P}_n. \tag{2.3}$$

Later authors [8], [9] did likewise. Since for fixed $0 < \varepsilon < 1$, and $\mid x_{jn} \mid \leq (1-\varepsilon)a_n$,

$$\lambda_{jn} W^{-2}(x_{jn}) \leq C(x_{j-1,n} - x_{jn}) \leq C_1\frac{a_n}{n}$$

the same arguments as above yield (assuming (2.3))

$$\{\sum_{|x_{jn}| \leq (1-\varepsilon)a_n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})\}^{1/p} \leq C \| PW \|_{L_p(\mathbb{R})} .$$

This last inequality is typically enough for mean convergence of Lagrange interpolation. In fact, in it one may allow for fixed $k \geq 1$, $P \in \mathcal{P}_{kn}$, rather than just $P \in \mathcal{P}_n$.

The following result is what D. Matjila and the author [17] could prove using this method:

Theorem 2.1
Let $W := e^{-Q} \in \mathcal{F}$.
(a) Let $1 \le p < \infty, r > 0$ and $-\infty < b \le 2$. Then

$$\sum_{j=1}^{n} \lambda_{jn} W^{-b}(x_{jn}) \mid PW \mid^p (x_{jn}) \le C \int_R \mid PW \mid^p (t) W^{2-b}(t) dt, \qquad (2.4)$$

for all P of degree at most $n + rn^{1/3}$.

(b) More generally, let $\phi : \mathbb{R} \to (0, \infty)$ be even, continuous with ϕ'' continuous for large x, and

$$\limsup_{x\to\infty} \frac{x \mid \phi'(x) \mid}{\phi(x) Q(x)^{1/3}} < \infty; \lim_{x\to\infty} \frac{1}{Q'(x)} \frac{d}{dx} \left(\frac{x\phi'(x)}{\phi(x)}\right) = 0.$$

Then for every P of degree at most $n + rn^{1/3}$

$$\sum_{j=1}^{n} \lambda_{jn} W^{-b}(x_{jn}) \phi(x_{jn}) \mid PW \mid^p (x_{jn}) \le C \int_{\mathbb{R}} \mid PW \mid^p (t) \phi(t) W^{2-b}(t) dt. \qquad (2.5)$$

For example, we could choose $\phi(x) := (1+ \mid x \mid)^b$ where $b \in \mathbb{R}$, or $\phi(x) := \exp(\mid x \mid^a)$ where $a > 0$ is small enough. The upper bound of $n + rn^{1/3}$ on the degree of P is curious, but essential. If $m = n + \xi_n n^{1/3}, \xi_n \to \infty$, one can choose $P \in \mathcal{P}_m$ for which (2.4) fails with $b = p = 2$ as $n \to \infty$. See [17].

This method has also been used by H. König [9], [10] in the context of Banach spaces with the Hermite weight and Jacobi weights, where instead of scalar polynomials P, one has polynomials P with vector values or values in a Banach space. The inequalities take the form

$$\{ \sum_{|x_{jn}| \le (1-\varepsilon)a_n} \lambda_{jn} W^{-2}(x_{jn}) \parallel PW \parallel^p (x_{jn})\}^{1/p} \le C\{\int_{\mathbb{R}} \parallel PW \parallel^p (t) dt\}^{1/p}$$

where $\|\cdot\|$ is the norm of the Banach space in which P takes values.

2.B The Large Sieve Method

This method is in spirit closely related to the large sieve of number theory, and was already used by Marcinkiewicz and Zygmund [48, Ch.X,pp.28-29]. Let us illustrate this for Freud weights. Our starting point is the estimate (1.13) for the Christoffel function, which gives

$$\lambda_{m+1}^{-1}(W^2, x) W^2(x) \le C \frac{m}{a_m}, x \in \mathbb{R}.$$

The definition of λ_{m+1} and the infinite-finite range inequality (1.7) lead to

$$| PW |^2 (x) \leq C\frac{m}{a_m}\int_{-a_m}^{a_m}(PW)^2(t)dt, P \in \mathcal{P}_m, x \in \mathbb{R}.$$

In order to deal with L_p norms other than $p = 2$, we fix a large positive integer l, and we replace P by P^l and W by W^l. Since the Mhaskar-Rahmanov-Saff number of order ml for W^l is just the Mhaskar-Rahmanov-Saff number of order m for W, we obtain

$$| PW |^{2l} (x) \leq C\frac{m}{a_m}\int_{-a_m}^{a_m}(PW)^{2l}(t)dt, P \in \mathcal{P}_m, x \in \mathbb{R}.$$

Hence if $0 < p < 2l$,

$$\| PW \|_{L_\infty(\mathbb{R})}^{2l} \leq C\frac{m}{a_m}\int_{-a_m}^{a_m} | PW |^p (t) \| PW \|_{L_\infty(\mathbb{R})}^{2l-p} dt$$

so for $x \in \mathbb{R}$,

$$| PW |^p (x) \leq \| PW \|_{L_\infty(\mathbb{R})}^{p} \leq C\frac{m}{a_m}\int_{-a_m}^{a_m} | PW |^p (t)dt. \tag{2.6}$$

It is now that the idea of the large sieve enters: It is largely P. Nevai and his collaborators that have been responsible for developing the method in this form [15], [22], [35], [37]; R. Askey's [1] variant of this depends on having a suitable non-negative kernel for the Jacobi weight to replace $K_n(x, t)$. We need the reproducing kernel $K_n(x, t)$ for the Chebyshev weight

$$K_n(x,t) = \frac{1}{\pi}(1 + 2\sum_{j=1}^{n-1} T_j(x)T_j(t))$$

(as usual $T_j(\cos\theta) = \cos(j\theta)$). It is well known [35,p.108] that

$$K_n(x,x) \sim n, x \in [-1,1];$$

$$| K_n(x,t) | \leq n \min\{1, \frac{1}{n | x-t |}\}, x,t \in [-1,1].$$

We now apply (2.6) to the polynomial $P(t)K_n(\frac{x}{a_{2kn}}, \frac{t}{a_{2kn}})^k$ where k and x are fixed and $P \in \mathcal{P}_{kn}$. This polynomial has degree $\leq 2kn$ in t, so we can apply (2.6) with $m = 2kn$:

$$| PW |^p (x) | K_n(\frac{x}{a_{2kn}}, \frac{x}{a_{2kn}}) |^{kp} \leq C\frac{2kn}{a_{2kn}}\int_{-a_{2kn}}^{a_{2kn}} | PW |^p (t) | K_n(\frac{x}{a_{2kn}}, \frac{t}{a_{2kn}}) |^{kp} dt$$

and hence if $kp > 2$, our bounds for K_n give

$$| PW |^p (x) \leq \frac{C_1}{na_n} \int_{-a_{2kn}}^{a_{2kn}} | PW |^p (t) K_n^2(\frac{x}{a_{2kn}}, \frac{t}{a_{2kn}}) dt.$$

This holds uniformly for $| x | \leq a_{2kn}$ with $C_1 \neq C_1(n, P, x)$. Choosing $x = x_{jn}$ in the above inequality gives

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \psi_n^{1/2}(x_{jn}) | PW |^p (x_{jn}) \leq C \int_{-a_{2kn}}^{a_{2kn}} | PW |^p (t) \sum(t) dt$$

where

$$\sum(t) := \frac{1}{na_n} \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \psi_n^{1/2}(x_{jn}) K_n^2(\frac{x_{jn}}{a_{2kn}}, \frac{t}{a_{2kn}})$$

$$\leq C \sum_{j=1}^{n} \min\{1, \frac{1}{n \mid \frac{x_{jn}}{a_{2kn}} - \frac{t}{a_{2kn}} \mid}\}^2$$

by our bounds (1.11) on the Christoffel numbers λ_{jn} and the bounds on K_n. Since uniformly in j and n, the zeros of $p_n(W^2, x)$ satisfy

$$x_{j-1,n} - x_{jn} \geq C \frac{a_n}{n}$$

it is not difficult to estimate the sum by an integral, so that

$$\sum(t) \leq C_1 \frac{n}{a_n} \int_{\mathbb{R}} \min\{1, \frac{1}{n \mid \frac{x}{a_{2kn}} - \frac{t}{a_{2kn}} \mid}\}^2 dx$$

$$= C_1 \frac{a_{2kn}}{a_n} \int_{\mathbb{R}} \min\{1, \frac{1}{| u |}\}^2 du \leq C_2 < \infty.$$

Thus we have for $P \in \mathcal{P}_{kn}$,

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \psi_n^{1/2}(x_{jn}) | PW |^p (x_{jn}) \leq C \int_{\mathbb{R}} | PW |^p (t) dt \qquad (2.7)$$

with $C \neq C(n, P)$. As $\psi_n(t) \geq C(\varepsilon), | t | \leq (1 - \varepsilon) a_n$, we also deduce that

$$\sum_{|x_{jn}| \leq (1-\varepsilon)a_n} \lambda_{jn} W^{-2}(x_{jn}) | PW |^p (x_{jn}) \leq C \int_{\mathbb{R}} | PW |^p (t) dt. \qquad (2.8)$$

One of the powerful features of the estimates (2.7) and (2.8) is the freedom to allow polynomials of degree kn and not just n. Thus if $\phi : \mathbb{R} \to (0, \infty)$ is such that one can find polynomials R_n of degree $O(n)$ such that

$$R_n(x) \sim \phi(x), \mid x \mid \leq a_n \tag{2.9}$$

and $\phi^{\pm 1}$ is "small" relative to W^{-1}, one can insert ϕ in (2.7) to obtain

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \psi_n^{1/2}(x_{jn}) \phi(x_{jn}) \mid PW \mid^p (x_{jn}) \leq C \int_{\mathbb{R}} \mid PW \mid^p (t) \phi(t) dt, \tag{2.10}$$

with $C \neq C(P, n)$. In particular $\phi(x) := (1+ \mid x \mid)^b, b \in \mathbb{R}$ will do.

The first time the full quadrature sums (2.7) or (2.10) have been considered, with the factor $\psi_n^{1/2}$ in the left, is in the recent Ph.D thesis of Haewon Joung [7], a student of P. Nevai. There not just ordinary polynomials, but *generalized non-negative polynomials* P were considered. These have the form

$$P(x) = \mid c \mid \prod_{j=1}^{m} \mid x - z_j \mid^{r_j}$$

where $c, z_j \in \mathbb{C}$, and

$$d = \deg(P) = \sum_{j=1}^{m} r_j.$$

Of course in the polynomial case, all r_j are positive integers. Joung's estimate depends on first finding estimates for Christoffel functions that involve generalized non-negative polynomials.

The "large sieve" method has many advantages over the method that we called Nevai's method. It works for all $p > 0$, not just $p \geq 1$; it requires only estimates on spacing of Christoffel functions and spacing of zeros, not the deeper Bernstein inequality. Nevertheless, it does not seem to be able to yield the full quadrature sum estimates (2.4), (2.5) in Theorem 2.1; the latter do not involve the factor $\psi_n^{1/2}$.

We note that in both the large sieve method and Nevai's method, we are not really using intrinsic properties of the zeros $\{x_{jn}\}$, only estimates on their *spacing*. Thus if

$$t_{j+1,n} - t_{j,n} \geq C \frac{a_n}{n} \; \forall j, n; \; \max_j \mid t_{jn} \mid \leq C a_n$$

then the same methods yield

$$\sum_{j=1}^{n} | PW |^p (t_{jn})\phi(t_{jn}) \leq C \int_{\mathbb{R}} | PW |^p \phi, P \in \mathcal{P}_{kn}.$$

We remark that both (2.7), (2.8) and (2.10) hold for $W \in \mathcal{F}$ and $P \in \mathcal{P}_{kn}$ and more generally probably for generalized non-negative polynomials of degree $\leq kn$, via Joung's method of proof. For $W \in \mathcal{E}$, the function ψ_n has to be replaced by another more complicated function in (2.7) and (2.10); (2.8) is still true, but is not sufficient for mean convergence of Lagrange interpolation. S.Damelin and the author [3] found it necessary to prove (using the large sieve method) that given $0 < \eta < 1$,

$$\sum_{|x_{jn}| \leq (1-\varepsilon)a_n} \lambda_{jn} W^{-2}(x_{jn})\phi(x_{jn}) \mid PW \mid^p (x_{jn}) \leq C \int_{\mathbb{R}} \mid PW \mid^p (t)\phi(t)dt,$$

where ϕ is any function for which (2.9) is possible. This is sharper than (2.8), since for Erdös weights,

$$a_{\eta n}/a_n \to 1, n \to \infty.$$

The reader may find further applications and developments of this method in [15], [30], [31], [37].

2.C The Duality Method

This method is based on applying duality to a suitable converse quadrature sum estimate, and is in a way indicative of the almost incestuous relationship between forward and converse estimates. It was apparently first used by H. König [9], [10].

Let n be fixed and let μ_n be the discrete (pure jump) measure having mass $\lambda_{jn} W^{-2}(x_{jn})$ at x_{jn}. Then

$$\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})\}^{1/p} = \{\int \mid PW \mid^p d\mu_n\}^{1/p}$$

$$= \sup_g \int PW^2 g \; d\mu_n$$

where the sup is taken over all g with

$$\{\int \mid gW \mid^q d\mu_n\}^{1/q} = 1.$$

Here of course $q = p/(p-1)$. Since g needs to be defined only at the n points $\{x_{jn}\}_{j=1}^{n}$, we can assume that $g \in \mathcal{P}_{n-1}$. So

$$\int PW^2 g \; d\mu_n = \sum_{j=1}^{n} \lambda_{jn} (Pg)(x_{jn})$$

$$= \int_{\mathbb{R}} PgW^2 \leq \| PW \|_{L_p(\mathbb{R})} \| gW \|_{L_q(\mathbb{R})} .$$

Now we make our major assumption: There is a converse quadrature sum estimate of the form

$$\| SW \|_{L_q(\mathbb{R})} \leq C \{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid SW \mid^q (x_{jn}) \}^{1/q}, S \in \mathcal{P}_{n-1}. \tag{2.11}$$

Then

$$\| gW \|_{L_q(\mathbb{R})} \leq C \{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid gW \mid^q (x_{jn}) \}^{1/q} = C \int \mid gW \mid^q d\mu_n = C.$$

Thus we obtain

$$\{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn}) \}^{1/p} \leq C \| PW \|_{L_p(\mathbb{R})} .$$

The attraction of this method is that it comes "for free". After spending a lot of effort proving a converse quadrature sum inequality involving the L_q norm, we immediately obtain a forward quadrature sum estimate for the dual L_p norm, and one that holds for full quadrature sums. The disadvantage of this method is that usually the range of q for which we can prove (2.11) is quite restricted. For example, in König's work, he showed that (2.11) is true for the Hermite weight only for $1 < q < 4$, and so one deduces the forward estimate for $\frac{4}{3} < p < \infty$, whereas it should hold in some form for all $1 \leq p < \infty$. Another disadvantage is that it works only for $P \in \mathcal{P}_{n-1}$.

2.D Complex Methods and Carleson Measures

Complex methods have been used primarily by Zhong and Zhu [47] for forward and converse quadrature sum estimates in the plane. A principal ingredient are *Carleson measures.* The latter also underlie the Markov-Bernstein inequality (2.2).

Recall that a Carleson measure is a positive measure $d\sigma$ on the upper half plane, that satisfies

$$\sigma([a-\frac{1}{2}h, a+\frac{1}{2}h] \times [0,h]) \leq Ch \tag{2.12}$$

for all $a \in \mathbb{R}$, $h > 0$. Thus the σ-measure of any square S in the upper-half plane with base on the real line should be bounded by a constant times the side of S. The smallest C in (2.12) is called the *Carleson norm* $N(\sigma)$ of σ.

The point about Carleson measures is the following: Let $0 < p < \infty$, and H^p be the Hardy space of the upper-half plane, that is, the set of all functions f analytic in the upper-half plane with boundary values $f(x)$ satisfying

$$\| f \|_{H^p} := \| f \|_{L_p(\mathbb{R})} < \infty.$$

Then

$$\int | f |^p \, d\sigma \leq N(\sigma) \int_{\mathbb{R}} | f |^p (x) dx. \tag{2.13}$$

Thus Carleson measures can be used to pass from the upper-half plane back to the real line. To illustrate how this idea can be used in the context of Freud weights; we follow closely the proofs given in [12] for (2.2).

Our first step is to pass from an estimate for $| PW |^p (x)$, $x \in \mathbb{R}$, to one over an arc in the upper-half plane, via Cauchy's integral formula. The problem is that W is not analytic! So for a given x, define

$$H_x(z) := e^{-[Q(x)+Q'(x)(z-x)]}.$$

Let us assume P has real coefficients. Cauchy's integral formula and the reflection principle give

$$| PW | (x) \leq \frac{1}{\pi} \int_0^{\pi} | PH_x | (x + \varepsilon e^{i\theta}) d\theta.$$

If we set $W(z) := W(| z |)$ and choose

$$\varepsilon := \varepsilon_n(x) := \frac{a_n}{n} \psi_n^{-1/2}(x),$$

it can be shown [12,Lemma 2.1,p.234] that for $| x | \leq x_{1n}$,

$$| H_x(x + \varepsilon_n(x) e^{i\theta}) | \leq CW(x + \varepsilon_n(x) e^{i\theta})$$

with $C \neq C(n, x, P)$. Hölder's inequality gives for $p \geq 1$,

$$| PW |^p (x) \leq C_1 \int_0^{\pi} | PW |^p (x + \varepsilon_n(x) e^{i\theta}) d\theta.$$

We deduce that

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn}) \leq C_1 \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \int_0^{\pi} \mid PW \mid^p (x_{jn}+\varepsilon_n(x_{jn})e^{i\theta}) d\theta$$

$$=: C_1 \int \mid PW \mid^p d\sigma_n.$$

We see that the measure σ_n is supported on the union of semicircular arcs, centred on the points $\{x_{jn}\}$. If we can show that the Carleson norms $N(\sigma_n)$ of σ_n satisfy

$$N(\sigma_n) \leq C_2, n \geq 1, \tag{2.14}$$

and if PW belongs to the Hardy space of the upper half plane, we could use (2.13) to deduce that

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn}) \leq C_1 C_2 \int_{\mathbb{R}} \mid PW \mid^p (x) dx.$$

As W is not in general analytic, we have to use a function $G_n(z)$ that is, in essence, derived from solving the Dirichlet problem for the domain $\mathbb{C}\backslash[-a_n, a_n]$, with suitable boundary values on $[-a_n, a_n]$. It was used by H.N. Mhaskar and E.B. Saff in proving (1.5) and in a different form by E.A. Rahmanov [27], [28], [40]. The properties of G_n that we need are that G_n is analytic in $\overline{\mathbb{C}}\backslash[-a_n, a_n]$ with a simple zero at ∞ and for $x \in \mathbb{R}$, [12,pp.234-235],

$$\mid G^n(x+i0) \mid := \lim_{y \to 0+} \mid G^n(x+iy) \mid = W(x), x \in [-a_n, a_n]\backslash\{0\};$$

$$W(\mid x+\varepsilon_n(x)e^{i\theta} \mid) \leq C \mid G^n(x+\varepsilon_n(x)e^{i\theta}) \mid, \mid x \mid \leq x_{1n}, \theta \in [0, \pi].$$

Then

$$\int \mid PW \mid^p d\sigma_n \leq C \int \mid PG^n \mid^p d\sigma_n$$

$$\leq CN(\sigma_n) \int_{\mathbb{R}} \mid PG^n \mid^p (x) dx$$

$$= CN(\sigma_n) \{ \int_{-a_n}^{a_n} \mid PW \mid^p dx + \int_{\mathbb{R}\backslash[-a_n, a_n]} \mid PG^n \mid^p dx \}$$

$$\leq CN(\sigma_n) \int_{-a_n}^{a_n} \mid PW \mid^p dx.$$

In this last step, one uses a representation of PG^n as a Hilbert transform of a function supported on $[-a_n, a_n]$, and boundedness of the Hilbert transform from L_p to L_p, $p > 1$.

What about (2.14)? Our estimate (1.11) for the Christoffel numbers gives

$$\int | PW |^p \, d\sigma_n \leq C \sum_{j=1}^{n} (x_{j-1,n} - x_{jn}) \int_0^{\pi} | PW |^p \, (x_{jn} + \varepsilon_n(x_{jn}) e^{i\theta}) d\theta$$

$$\leq C \int_{x_{nn}}^{x_{1n}} \int_0^{\pi} | PW |^p \, (x + \varepsilon_n(x) e^{i\theta}) d\theta \, dx$$

$$=: C \int | PW |^p \, d\tilde{\sigma}_n.$$

Of course the second last step requires proof, but is intuitively reasonable. In [12, Lemma 2.4], it is shown that

$$N(\tilde{\sigma}_n) \leq C_3, n \geq 1$$

and the same proof shows that (2.14) holds.

This method of proof is attractive, but as already remarked, it involves essentially the same tools as to prove the Markov-Bernstein inequality (2.2).

Perhaps the only published paper where this method has been used to prove quadrature sum estimates is that of Zhong and Zhu [47]. They proved:

Theorem 2.2

Let Γ be a $C^{2+\delta}$ smooth simple arc in $\mathbb{C}$, that is $\Gamma = \{\gamma(t) : t \in [a, b]\}$ where γ'' satisfies a Lipschitz condition of order $\delta > 0$. There exist $\{z_{k,n}\}_{k=0}^{n-1} \subset \Gamma$, $n \geq 1$, such that for $1 < p < \infty$ and $P \in \mathcal{P}_{n-1}$,

$$\{\sum_{k=0}^{n-1} | P(z_{k,n}) |^p | z_{k+1,n} - z_{k,n} |\}^{1/p} \leq C \| P \|_{L_p(\Gamma)} .$$

Here $z_{n,n} := z_{0,n}$.

Essentially the authors use a conformal map Ψ of the exterior of the unit ball onto $\mathbb{C} \backslash \Gamma$, and form the *Fejer points*

$$z_{k,n}^* := \Psi(e^{2\pi i k/n}), 0 \leq k \leq n-1.$$

As some of these may be too close, they modify these to obtain $\{z_{k,n}\}_{k=0}^{n-1}$. Instead of estimating $P(z_{k,n})$ in terms of values of P on a semi-circle centre $z_{k,n}$, the authors estimate $P(z_{k,n})$ in terms of values of P on the "level curve"

$$\Gamma_n := \{\Psi((1+\frac{1}{n})e^{it}) : t \in [0, 2\pi]\}$$

which encircles Γ. A suitable Carleson measure is formed, and moreover it is shown that for all f in a suitable Smirnov space of functions analytic inside Γ_n,

$$\| f \|_{L_p(\Gamma)} \leq C \| f \|_{L_p(\Gamma_n)}, C \neq C(n, f).$$

Other ingredients are Lagrange interpolation and careful estimation of the spacing $| z_{k+1,n} - z_{k,n} |$, and of

$$\Omega_n(z) := \prod_{k=0}^{n-1} (z - z_{k,n}).$$

A related paper of Zhong and Shen is [46]. Unfortunately this paper is not available in South Africa, and the British Lending Library could not provide a copy to the author. So the reader should please take note that [46] is excluded from this survey.

3. Converse Quadrature Sum Estimates

We shall present two methods for these, illustrated in the case of Freud weights.

3.A The Duality Method

This method already appears, in the setting of trigonometric polynomials, in the treatise of Zygmund [48,ChX, pp.28-29]. It is based on duality and "deep" results on mean boundedness of orthogonal expansions. Let $1 \leq p < \infty$ and $q = p/(p-1)$. Let $P \in \mathcal{P}_{n-1}$. We have

$$\| PW \|_{L_p(\mathbb{R})} = \sup_g \int gPW^2$$

where the sup is taken over all g with $\| gW \|_{L_q(\mathbb{R})} = 1$. By orthogonality of $g - S_n[g]$ to $\mathcal{P}_{n-1}$, and then by the Gauss quadrature formula,

$$\int gPW^2 = \int S_n[g]PW^2 = \sum_{j=1}^{n} \lambda_{jn}(PS_n[g])(x_{jn})$$

$$\leq \{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^{p} (x_{jn})\}^{1/p} \{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid S_n[g] W \mid^{q} (x_{jn})\}^{1/q}$$
$$=: T_1 \times T_2.$$

Let us suppose that we have a suitable forward quadrature sum estimate like (2.4) and that the partial sum operators $\{S_n\}$ are bounded uniformly in n in a suitable weighted setting. Then

$$T_2 \leq C_1 \parallel S_n[g]W \parallel_{L_q(\mathbb{R})} \leq C_2 \parallel gW \parallel_{L_q(\mathbb{R})} = C_2.$$

So we have shown that

$$\parallel PW \parallel_{L_p(\mathbb{R})} \leq C\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^{p} (x_{jn})\}^{1/p}.$$

This duality method is elegant but it depends on having a forward quadrature sum estimate, and, much deeper, results on mean boundedness of orthogonal expansions. It is the difficulty of proving the latter that severely restricts this method. Chiefly it is a tool to pass from results on mean convergence of orthogonal expansions to corresponding results for Lagrange interpolation.

Typically, the mean boundedness required above is valid only for $\frac{4}{3} < q < 4$; to ensure its validity for other values of q, one needs to insert suitable powers of $1+ \mid x \mid$ as weights on $S_n[g]$ and/ or g. Moreover, in proving even these, one needs bounds on functions of the second kind or on $p_{n+1} - p_{n-1}$. For Jacobi weights, the requisite bounds are classical, but these bounds are not generally available in the setting of Freud weights. This explains the severe restrictions of the following result [14]:

Theorem 3.1

Let $W(x) := \exp(- \mid x \mid^{\beta}), \beta = 2, 4, 6, , \ldots$. *Let* $4 < p < \infty$ *and* $r, R \in \mathbb{R}$ *satisfy*

$$R > -\frac{1}{p}; \tag{3.1}$$

and

$$r - \min\{R, 1 - \frac{1}{p}\} + \frac{\beta}{6}(1 - \frac{4}{p}) \begin{cases} \leq 0, \text{ if } R \neq 1 - \frac{1}{p} \\ < 0, \text{ if } R = 1 - \frac{1}{p} \end{cases}. \tag{3.2}$$

Then for $P \in \mathcal{P}_{n-1}$,

$$\parallel (PW)(x)(1+ \mid x \mid)^{r} \parallel_{L_p(\mathbb{R})} \leq C\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^{p} (x_{jn})(1+ \mid x_{jn} \mid)^{Rp}\}^{1/p}. \tag{3.3}$$

For $p = 4$, (3.3) holds if (3.1) holds and

$$r - \min\{R, 1 - \frac{1}{p}\} < 0. \quad (3.4)$$

The conditions on r, R are disconcerting, but it was shown in [14] that (3.2) is necessary for (3.3). (It is not clear if (3.1) is also necessary). In particular, (3.2) requires $r < R$, so that for $p > 4$, we can never have $r = R$ in (3.3). However, there are always r, R that satisfy (3.2), (3.1), one just needs to choose r small enough. More generally, we proved:

Theorem 3.2
Let $W \in \mathcal{F}$, with the additional condition that the orthonormal polynomials $\{p_n\}$ for W^2 satisfy

$$\sup_{x\in\mathbb{R}} \mid p_{n+1} - p_{n-1} \mid (x)W(x)\psi_n(x)^{-1/2} \leq Ca_n^{-1/2}, n \geq 1. \quad (3.5)$$

Then if (3.1) holds and

$$a_n^{r-\min\{R,1-\frac{1}{p}\}} n^{\frac{1}{6}(1-\frac{4}{p})} = \begin{cases} O(1),\ R \neq 1 - \frac{1}{p} \\ O((\log n)^{-R}),\ R = 1 - \frac{1}{p} \end{cases} \quad (3.6)$$

we have (3.3). For $p = 4$, if (3.1) and (3.4) hold, then we have (3.3).

It was also shown in [14] that a slightly weaker form of (3.6) is necessary for the converse estimate. In both the above results, we restricted ourselves to $p \geq 4$; For $p < 4$, the next method will give better results.

Since the theory of mean convergence of orthogonal expansions for weights on $(-1, 1)$ is far more developed than that for weights on $\mathbb{R}$, it is hardly surprising that more impressive results can be achieved by this duality method for weights on $(-1, 1)$. Here is a result of Yuan Xu [45,p.82] for *generalized Jacobi weights* that extends earlier results of P. Nevai. Recall that a generalized Jacobi weight has the form

$$w(x) = \prod_{j=1}^{M} \mid x - t_j \mid^{\beta_j}, x \in (-1, 1)$$

where

$$-1 = t_1 < t_2 < ... < t_M = 1; \beta_j \in \mathbb{R}, 1 \leq j \leq M.$$

(It is possible that some $\beta_j < -1$). We call $\tilde{w}$ a *generalized Jacobi distribution* if it is integrable and has the form ψw, where w is a generalized Jacobi weight, and ψ is a continuous function on $[-1,1]$ with ψ^{-1} bounded on $[-1,1]$, and whose modulus of continuity ω satisfies

$$\int_0^1 \frac{\omega(t)}{t}dt < \infty.$$

Corresponding to $\tilde{w}$, we define

$$\tilde{w}_n(x) := \psi(x)(\sqrt{1+x}+\frac{1}{n})^{2\beta_1+1}\prod_{j=2}^{M-1}(\mid x-t_j\mid+\frac{1}{n})^{\beta_j}(\sqrt{1-x}+\frac{1}{n})^{2\beta_M+1}.$$

Theorem 3.3 *Let u, v be generalized Jacobi distributions on* $(-1,1)$ and w be a generalized Jacobi weight on $(-1,1)$. *Let $1 < p < \infty$ and $q := p/(p-1)$. Then if $\{x_{jn}\}, \{\lambda_{jn}\}$ are the Gauss points and weights for u,*

$$\{\int_{-1}^{1}\mid P(t)\mid^p v(t)dt\}^{1/p} \le C\{\sum_{j=1}^{n}\lambda_{jn}\tilde{w}_n(x_{jn})\mid P(x_{jn})\mid^p\}^{1/p}, P \in \mathcal{P}_{n-1}$$

provided the following four conditions hold:

$$w^{1-q}u \in L_1[-1,1];$$
$$w^{1-q}(x)u(x)\{u(x)\sqrt{1-x^2}\}^{-q/2} \in L_1[-1,1];$$
$$wu \ge Cv \text{ in } (-1,1);$$
$$v(x)\{u(x)\sqrt{1-x^2}\}^{-p/2} \in L_1[-1,1].$$

Xu's paper also contains a converse quadrature sum inequality that involves not just the values of P, but also of its derivatives [45,p.83]. These are useful in studying mean convergence of Hermite interpolation.

3.B König's Method

König's method is based on Lagrange interpolation, a clever estimate for Hilbert transforms of characteristic functions of intervals, and bounds on the norms of linear operators derived via Hölder's inequality. It is technically the most difficult amongst those we have presented, but is extremely powerful, and relatively direct - it does not depend on deep results such as mean convergence of orthogonal expansions or Markov-Bernstein inequalities.

We shall need some extra notation. Let

$$I_{jn} := [x_{jn}, x_{j-1,n}); \mid I_{jn} \mid = x_{j-1,n} - x_{jn}, 1 \leq j \leq n;$$

(recall $x_{0n} = x_{1n}(1 + n^{-2/3})$) and the characteristic function of I_{jn} is denoted by

$$\chi_{jn}(x) := \chi_{I_{jn}}(x).$$

The fundamental polynomials of Lagrange interpolation are

$$l_{jn}(x) := \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})}, 1 \leq j \leq n;$$

The Hilbert transform $H[f]$ of $f \in L_1(\mathbb{R})$ is

$$H[f](x) := \lim_{\varepsilon \to 0+} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{t - x} dt, \text{ a.e. } x \in \mathbb{R}.$$

We write, for fixed $P \in \mathcal{P}_{n-1}$,

$$y_{jn} := a_n^{-1/2} \frac{P(x_{jn})}{p_n'(x_{jn})}$$

so that

$$P(x) = L_n[P](x) = a_n^{1/2} p_n(x) \sum_{j=1}^{n} \frac{y_{jn}}{x - x_{jn}}$$

$$= a_n^{1/2} p_n(x) \sum_{j=1}^{n} y_{jn} \{ \frac{1}{x - x_{jn}} - \frac{1}{\mid I_{jn} \mid} H[\chi_{jn}](x) \}$$

$$+ a_n^{1/2} p_n(x) \sum_{j=1}^{n} \frac{y_{jn}}{\mid I_{jn} \mid} H[\chi_{jn}](x) =: J_1(x) + J_2(x). \tag{3.7}$$

We first deal with the easier term J_2. We shall use the bound (1.14) in the form

$$\mid p_n W \mid (x) \leq C a_n^{-1/2} \mid 1 - \frac{\mid x \mid}{a_n} \mid^{-1/4}, x \in \mathbb{R},$$

and also the following bound on the Hilbert transform: For all functions g with support in $[-2a_n, 2a_n]$,

$$\| H[g](x) \mid 1-\frac{|x|}{a_n} \mid^{-1/4} \|_{L_p(|x|\leq 2a_n)} \leq C \| g(x) \mid 1-\frac{|x|}{a_n} \mid^{-1/4} \|_{L_p(|x|\leq 2a_n)} \quad (3.8)$$

with $C \neq C(n,g)$ provided $1 < p < 4$. This bound is a variant of M. Riesz' theorem that the Hilbert transform is a bounded operator from L_p to L_p, $1 < p < \infty$. For the case $a_n = \sqrt{2n}$, this lemma already appears in a 1970 paper of B. Muckenhoupt [32]. It was in this paper that the modern study of weighted inequalities for the Hilbert transform began, leading ultimately to Muckenhoupt's A_p condition. Then

$$\begin{aligned}
\| J_2 W \|_{L_p[-2a_n,2a_n]} &= a_n^{1/2} \| (p_n W)(x) H[\sum_{j=1}^{n} \frac{y_{jn}}{|I_{jn}|}\chi_{jn}](x) \|_{L_p(|x|\leq 2a_n)} \\
&\leq C \|| 1-\frac{|x|}{a_n} |^{-1/4} \sum_{j=1}^{n} \frac{y_{jn}}{|I_{jn}|}\chi_{jn}(x) \|_{L_p(|x|\leq 2a_n)} \\
&= C\{\sum_{j=1}^{n}(\frac{|y_{jn}|}{|I_{jn}|})^p \int_{I_{jn}} | 1-\frac{|x|}{a_n} |^{-p/4} dx\}^{1/p} \\
&\leq C_1\{\sum_{j=1}^{n}(\frac{|y_{jn}|}{|I_{jn}|})^p | I_{jn} | \psi_n^{-p/4}(x_{jn})\}^{1/p}.
\end{aligned}$$

It can be shown that uniformly in j, n,

$$a_n^{1/2} | p_n' W | (x_{jn}) \sim | I_{jn} |^{-1} \psi_n^{-1/4}(x_{jn}); | I_{jn} | \sim \lambda_{jn} W^{-2}(x_{jn}) \quad (3.9)$$

and hence

$$\frac{|y_{jn}|}{|I_{jn}|} \sim | PW | (x_{jn})\psi_n^{1/4}(x_{jn}). \quad (3.10)$$

Hence we deduce that

$$\| J_2 W \|_{L_p[-2a_n,2a_n]} \leq C_2\{\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})\}^{1/p}. \quad (3.11)$$

The estimation of J_1(defined by (3.7)) is more difficult. If we set

$$f_{jn}(x) := \psi_n^{-1/4}(x) \times \begin{cases} \frac{1}{|I_{jn}|}, | x - x_{jn} | \leq 2 | I_{jn} | \\ \frac{|I_{jn}|}{|x-x_{jn}|}\{\frac{1}{|x-x_{jn}|} + \frac{1}{1+|x_{jn}|}\}, | x - x_{jn} | > 2 | I_{jn} | \end{cases}$$

then it can be shown that uniformly in j, n and $x \in [x_{nn}, x_{1n}]$,

$$\mid \frac{1}{x - x_{jn}} - \frac{1}{\mid I_{jn} \mid} H[\chi_{jn}](x) \mid a_n^{1/2} \mid p_n W \mid (x) \le C f_{jn}(x)$$

[14,p.542, Lemma 5.2] so that

$$\mid J_1 W \mid (x) \le C \sum_{j=1}^{n} \mid y_{jn} \mid f_{jn}(x), x \in [x_{nn}, x_{1n}].$$

Then as each f_{jn} doesn't change much in each I_{kn},

$$\parallel J_1 W \parallel_{L_p[x_{nn}, x_{1n}]} \le C \{ \sum_{k=2}^{n} \mid I_{kn} \mid [\sum_{j=1}^{n} \mid y_{jn} \mid f_{jn}(x_{kn})]^p \}^{1/p}.$$

Taking account of the form of f_{jn}, we see that

$$\parallel J_1 W \parallel_{L_p[x_{nn}, x_{1n}]} \le C \{ S_1 + S_2 + S_3 \} \tag{3.12}$$

where

$$S_1 := \{ \sum_{k=2}^{n} \mid I_{kn} \mid [\sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\mid y_{jn} \mid\mid I_{jn} \mid}{(x_{jn} - x_{kn})^2} \psi_n^{-1/4}(x_{kn})]^p \}^{1/p};$$

$$S_2 := \{ \sum_{k=2}^{n} \mid I_{kn} \mid [\sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\mid y_{jn} \mid\mid I_{jn} \mid}{\mid x_{jn} - x_{kn} \mid (1 + \mid x_{jn} \mid)} \psi_n^{-1/4}(x_{kn})]^p \}^{1/p};$$

and

$$S_3 := \{ \sum_{j=2}^{n} \mid I_{jn} \mid [\frac{\mid y_{jn} \mid}{\mid I_{jn} \mid} \psi_n^{-1/4}(x_{jn})]^p \}^{1/p}.$$

Exactly as for J_2, we deduce that

$$S_3 \le C_3 \{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn}) \}^{1/p}.$$

If we use

$$\psi_n^{-1/4}(x_{jn}) \sim [\frac{n}{a_n} \mid I_{jn} \mid]^{1/2}$$

and our estimate (3.10) for y_{jn}, then we obtain

$$S_1 \le C \{ \sum_{k=1}^{n} [\sum_{j=1}^{n} b_{kj} \mid I_{jn} \mid^{1/p} \mid PW \mid (x_{jn})]\}^{1/p}$$

where $b_{kk} = b_{1k} = 0 \ \forall k$, and otherwise

$$b_{kj} :=\mid I_{kn} \mid^{\frac{1}{p}+\frac{1}{2}}\mid I_{jn} \mid^{\frac{3}{2}-\frac{1}{p}} (x_{kn} - x_{jn})^{-2}.$$

Defining the $n \times n$ matrix

$$B := (b_{kj})_{k,j=1}^{n}$$

(note the reversed order of our indices), we obtain

$$S_1 \leq C_4 \parallel B \parallel_{l_p^n \to l_p^n} [\sum_{j=1}^{n} \mid I_{jn} \mid\mid PW \mid^p (x_{jn})]^{1/p}$$

$$\leq C_5 \parallel B \parallel_{l_p^n \to l_p^n} [\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})]^{1/p}.$$

If we can show that

$$\parallel B \parallel_{l_p^n \to l_p^n} \leq C_6 \tag{3.13}$$

then we obtain the desired estimate for S_1. Similarly

$$S_2 \leq C_6 \parallel D \parallel_{l_p^n \to l_p^n} [\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})]^{1/p}$$

where

$$D := (d_{kj})_{k,j=1}^{n}$$

and

$$d_{kj} := b_{kj} \frac{\mid x_{kn} - x_{jn} \mid}{1+ \mid x_{jn} \mid} \forall j, k.$$

If we can show that

$$\parallel D \parallel_{l_p^n \to l_p^n} \leq C_7 \tag{3.14}$$

we obtain the desired estimate for S_1. Then (3.12) and (3.11) yield

$$\parallel PW \parallel_{L_p(\mathbb{R})} \leq C_8 [\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})]^{1/p}. \tag{3.15}$$

We proceed to prove (3.13) and (3.14). König's method to bound these depends on the following:

Proposition

Let (Ω, μ) be a measure space, and $s, r : \Omega \times \Omega \to \mathbb{R}$. Define the operator.

$$T_k[f](x) := \int_\Omega s(x,y) f(y) d\mu(y).$$

Let $M > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that

$$\sup_x \int_\Omega | s(x,y) || r(x,y) |^q \, d\mu(y) \leq M;$$

$$\sup_y \int_\Omega | s(x,y) || r(x,y) |^{-p} \, d\mu(x) \leq M.$$

Then

$$\| T_k \|_{L_p(\Omega) \to L_p(\Omega)} \leq M,$$

that is, $\forall f \in L_p(d\mu)$,

$$\{ \int_\Omega | T_k[f] |^p \, (y) d\mu(y) \}^{1/p} \leq M \{ \int_\Omega | f(x) |^p \, d\mu(x) \}^{1/p}.$$

This proposition is easily proved using Hölder's inequality - see [4], [9]. To prove (3.1) one chooses $\Omega := \{1, 2, ..., n\}$, $\mu(\{j\}) = 1, 1 \leq j \leq n$, and

$$s(k,j) := b_{kj}; r(k,j) := \left(\frac{| I_{jn} |}{| I_{kn} |} \frac{1+ | x_{kn} |}{1+ | x_{jn} |}\right)^{1/pq}.$$

One can show that [14]

$$\sup_k \sum_{j=1}^{n} | s(k,j) || r(k,j) |^q \leq M;$$

$$\sup_j \sum_{k=1}^{n} | s(k,j) || r(k,j) |^{-p} \leq M;$$

with $M \neq M(n)$. The actual proof of these involves re-expressing certain sums in terms of integrals and then careful estimation of the integrals. Similarly to prove (3.14), one chooses the same Ω, μ and chooses

$$s(k,j) := d_{kj}; r(k,j) := \left(\frac{| I_{jn} |}{| I_{kn} |} \frac{1+ | x_{kn} |}{1+ | x_{jn} |}\right)^{1/pq} (1+ | x_{jn} |)^{1/q} \psi_n^{1/4q}(x_{jn}) \psi_n^{1/4q}(x_{kn}).$$

This is what we could prove using König's method:

Theorem 3.4
Let $1 < p < 4$ *and* $W \in \mathcal{F}$. *Let*

$$r < 1 - \frac{1}{p}; r \leq R; R > -\frac{1}{p}. \tag{3.16}$$

Then

$$\| (PW)(x)(1+ \mid x \mid)^r \|_{L_p(\mathbb{R})} \leq C[\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \mid PW \mid^p (x_{jn})(1+ \mid x_{jn} \mid)^{Rp}]^{1/p}. \tag{3.17}$$

In [14] it is also shown that the first two conditions in (3.16) are necessary. For Erdös weights, we proved [4,Thm. 3.1]:

Theorem 3.5 *Let* $1 < p < 4$ *and* $W \in \mathcal{E}$. *Then (3.17) holds with* $r = R = 0$.

For Jacobi weights, König and Nielsen [10] proved the following elegant theorem using this method. In fact they worked in the more general setting of Banach spaces that admit a Hilbert transform bound. For simplicity, we quote this result for the case of polynomials:

Theorem 3.6
Let $u(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$ *be a Jacobi weight and* $\{\lambda_{jn}\}, \{x_{jn}\}$ *be the corresponding Gauss weights and points. Let*

$$\mu(\alpha, \beta) := \max\{1, \frac{4(\alpha+1)}{2\alpha+5}, \frac{4(\beta+1)}{2\beta+5}\};$$

$$m(\alpha, \beta) := \max\{1, \frac{4(\alpha+1)}{2\alpha+3}, \frac{4(\beta+1)}{2\beta+3}\};$$

and

$$M(\alpha, \beta) := \frac{m(\alpha, \beta)}{m(\alpha, \beta) - 1}.$$

The following are equivalent:

(i) For $n \geq 1$ *and* $P \in \mathcal{P}_n$,

$$(\int_{-1}^{1} \mid P(t) \mid^p u(t) dt)^{1/p} \leq C[\sum_{j=1}^{n} \lambda_{jn} \mid P \mid^p (x_{jn})]^{1/p}.$$

(ii)

$$\mu(\alpha,\beta) < p < M(\alpha,\beta).$$

In comparing to Xu's Theorem 3.3, we note that in the "unweighted case" of that theorem, for which there $v = u; w = 1$, and u is a Jacobi weight, König's result is more extensive.

3.C Complex Methods

In Section 2D, we discussed the results of Zhong and Zhu for forward quadrature sum estimates. The same framework of ideas of Carleson measures and Smirnov spaces, contour integral error formulas for Lagrange interpolation, and conformal maps, enabled them to prove:

Theorem 3.7

Let $1 < p < \infty$. Under the hypotheses of Theorem 2.2, the points $\{z_{k,n}\}_{k=0}^{n-1}$ there satisfy

$$\| P \|_{L_p(\Gamma)} \leq C\{\sum_{k=0}^{n-1} | P(z_{k,n}) |^p | z_{k+1,n} - z_{k,n} |\}^{1/p}, P \in \mathcal{P}_{n-1}.$$

It is notable that all the methods we have presented for converse quadrature sum estimates work only for $p > 1$. Using operator theoretic methods, and complex ones, Peller [39,p.480] proved a converse quadrature sum for $(4n-1)$st roots of unity that works even for $p < 1$, but involving polynomials of degree at most $n-1$:

Theorem 3.8

Let $0 < p < \infty$. For polynomials P of degree $\leq n-1$,

$$\frac{1}{2\pi}\int_0^{2\pi} | P(e^{i\theta}) |^p \, d\theta \leq \frac{C}{n}\sum_{j=0}^{4n-1} | P(e^{2\pi i j/(4n)}) |^p .$$

It seems likely that the same methods should allow one to replace 4 by $1+\varepsilon$.

4. Conclusions

We have seen four methods for proving forward quadrature sum estimates. For purposes of weighted approximation, I believe that the "large sieve" method is the most versatile, and the most generally applicable, yielding adequate results most generally. However when full quadrature sums need to be estimated, without damping factors, Nevai's method is the most appropriate. The duality and complex methods seem to yield less in weighted approximation, though are powerful in some circumstances.

For converse quadrature sum estimates, I believe that König's method is the most direct and powerful, though at present it works only for $p < 4$. The method based on duality and mean convergence of orthogonal expansions is elegant but the mean convergence results required are very deep. Perhaps chiefly it can be used to pass from mean convergence of orthogonal expansions to mean convergence of Lagrange interpolation.

There are several worthwhile open problems:

(I) Make König's method for converse estimates work for all $1 \leq p < \infty$. One of the main sticking points seems to be to extend the Hilbert transform inequality (3.8) in some form to $p \geq 4$, by inserting suitable damping factors on both sides.

(II) There seems to be little on converse quadrature sum estimates in L_p for $p \leq 1$. As far as the author could determine, Peller's Theorem 3.7 is about the only one. Surely $4n$ there can be replaced by n? And what about weights on $(-1, 1)$ or $\mathbb{R}$?

(III) There are gaps between the necessary and sufficient conditions for converse quadrature sum estimates in [14]. The gaps arise because the necessary conditions are derived from results on mean convergence of Lagrange interpolation, while the sufficient ones are derived via König's and the duality method. Close these gaps!

(IV) Yuan Xu's extensive result for generalized Jacobi weights Theorem 3.3 involves sufficient conditions. Find the necessary and sufficient ones, thereby extending the scope of König's Theorem 3.6. Most probably, König's methods will have to be used.

(V) Explore the implications of Peller's methods for converse quadrature sum estimates.

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