SOME POLYNOMIAL PROBLEMS ARISING FROM PADÉ APPROXIMATION

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Abstract. In the convergence theory of Padé approximation, one needs to estimate the size of a set on which a suitably normalized polynomial q is small. For example, one needs to estimate the size of the set of $r \in [0,1]$ for which

$$\max_{|t|=1} |q(t)| \,/\, \min_{|t|=r} |q(t)|$$

is not "too large". We discuss some old and new problems of this type, and the methods used to solve them.

1. Introduction

Let f be a function analytic at 0, and hence possessing a Maclaurin series there. Recall that if $m, n \geq 0$, the (m,n) Padé approximant to f is a rational function

$$[m/n](z) = (p/q)(z),$$

where p, q are polynomials of degree $\leq m, n$ respectively, with q not identically zero, and

$$(fq - p)(z) = O\left(z^{m+n+1}\right).$$

The order relation indicates that the coefficients of $1, z, z^2, ..., z^{m+n}$ in the Maclaurin series of the left-hand side vanish. It may be reformulated as a system of homogeneous linear equations in the coefficients of q and p, with more unknowns than equations, and hence has a non-trivial solution. After the division by q, the solution becomes unique. For an introduction to the subject, see [2], [3].

An essential tool in studying the convergence of Padé approximants as m and or $n \to \infty$, is the contour integral error formula. Let us assume for simplicity that f is analytic in $\{z : |z| \leq 1\}$. Then if $[m/n] = p/q$, Cauchy's integral formula gives for $|z| < 1$,

$$\frac{(fq-p)(z)}{z^{m+n+1}} = \frac{1}{2\pi i}\int_{|t|=1} \frac{(fq-p)(t)}{t^{m+n+1}} \frac{dt}{t-z}, |z| < 1.$$

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Here for such a fixed z, $p(t) / \{t^{m+n+1}(t-z)\}$ is a rational function of t that is analytic outside the unit ball, and is $O(t^{-2})$ at ∞. It follows that the corresponding part of the integral is 0. On multiplying by $z^{m+n+1}/q(z)$, we obtain

$$(f-[m/n])(z)=\frac{1}{2\pi i}\int_{|t|=1}\frac{(fq)(t)}{q(z)(t-z)}\left(\frac{z}{t}\right)^{m+n+1}dt. \tag{1.1}$$

Estimation of the integral in a standard manner leads to

$$|f-[m/n]|(z)\leq\left\{|z|^{m+n}\frac{\max_{|t|=1}|q(t)|}{|q(z)|}\right\}\frac{C}{1-|z|} \tag{1.2}$$

and in particular,

$$\max_{|z|=r}|f-[m/n]|(z)\leq\left\{r^{m+n}\frac{\max_{|t|=1}|q(t)|}{\min_{|t|=r}|q(t)|}\right\}\frac{C}{1-r}, \tag{1.3}$$

for $0<r<1$, where C is independent of m,n,r,z.

How does one proceed from here? Unfortunately, in general one knows little about the zeros of q. It is only for special classes of functions that a great deal is known about the poles of $[m/n]$, that is, the zeros of q. So one may as well try to estimate the terms in $\{\}$ in (1.3) for an *arbitrary* polynomial q of degree $\leq n$. For how large a set of r can the term in $\{\}$ be small, and hence for how large a set of r can $[m/n]$ provide good approximation on $|z|=r$? Moreover, for how large a set of z can the term in $\{\}$ in (1.2) be not too large? This brings us to:

2. Cartan or Polya Lemmas

The first person to proceed for very general functions in a systematic way was John Nuttall [17], though Zinn-Justin's work [25] was at a similar time. Nuttall was the first to realize that one can take advantage of remarkable lemmas of Cartan and Polya [2], [3], [5] dealing with small values of polynomials. Let $\varepsilon>0$, m_2 denote planar Lebesgue measure, and $z_j\in\mathbb{C}, 1\leq j\leq n$. Polya's inequality says that

$$m_2\left\{z:\left|\prod_{j=1}^{n}(z-z_j)\right|\leq\varepsilon^n\right\}\leq\pi\varepsilon^2, \tag{2.1}$$

with strict inequality unless all z_j are equal. Cartan's lemma involves a parameter $\alpha>0$. It says that there exists $p\leq n$ and balls $B_j, 1\leq j\leq p$, with

$$\left\{z:\left|\prod_{j=1}^{n}(z-z_j)\right|\leq\varepsilon^n\right\}\subseteq\bigcup_{j=1}^{p}B_j, \tag{2.2}$$

and

$$\sum_{j=1}^{p}(diam\,(B_j))^{\alpha} \leq e\,(4\varepsilon)^{\alpha}\,. \tag{2.3}$$

(Of course $diam\,(B_j)$ denotes the diameter of B_j). Cartan's lemma may be viewed as an inequality involving α-dimensional Hausdorff outer measure, or even relating logarithmic capacity and Hausdorff content, for those familiar with those concepts. We remind the reader of an

Unsolved Problem
Find the sharp constant in (2.3) that should replace $e4^{\alpha}$.

Even for $\alpha = 1$, the sharp constant has not been established, though there it seems certain that it should be 4. Polya established sharpness of 4 when the set in (2.2) is restricted to a line in the plane. Of course the point about Cartan and Polya's lemmas is that the estimates do not depend on n. The naive approach of covering the set $\left\{z : \left|\prod_{j=1}^{n}(z-z_j)\right| \leq \varepsilon^n\right\}$ by balls of radius ε centred on each z_j leads to a factor of n in the right-hand sides of (2.1) and (2.3), and that would be fatal in most applications.

To apply Cartan or Polya's inequalities, we take our polynomial q, and decompose it into pieces with zeros close to, or far from, a circle centre 0, radius $r > 0$:

$$q\,(z) = c\left(\prod_{|z_j|<2r}(z-z_j)\right)\left(\prod_{|z_j|\geq 2r}\left(1-\frac{z}{z_j}\right)\right) =: cq_1\,(z)\,q_2\,(z)\,,$$

with $c \in \mathbb{C}$. Now for $|z| \leq r$,

$$\left(\frac{1}{2}\right)^{\deg(q_2)} \leq |q_2\,(z)| \leq \left(\frac{3}{2}\right)^{\deg(q_2)},$$

while for $|t| = r$,

$$|q_1\,(t)| \leq (3r)^{\deg(q_1)}\,.$$

Hence we obtain

$$\frac{\max_{|t|=r}|q\,(t)|}{|q\,(z)|} \leq \frac{(3\max\{1,r\})^{\deg(q)}}{|q_1\,(z)|},\, |z| < r. \tag{2.4}$$

Applying Polya's inequality to the monic polynomial q_1 with $r = 1$ and

$\varepsilon < 1$ gives

$$\frac{\max_{|t|=r}|q(t)|}{|q(z)|} \leq \frac{(3\max\{1,r\})^{\deg(q)}}{\varepsilon^{\deg(q_1)}} \leq \left(\frac{3}{\varepsilon}\right)^n, |z| < r, z \notin \mathcal{E},$$

where $m_2(\mathcal{E}) \leq \pi\varepsilon^2$. Now substituting the estimate in (1.2), we obtain

$$|f - [m/n]|(z) \leq 2C\left\{|z|^{m+n}\left(\frac{3}{\varepsilon}\right)^n\right\}, |z| \leq \frac{1}{2}, z \notin \mathcal{E}.$$

If we choose $\frac{1}{2} > \delta >> \varepsilon$ (that is, δ much larger than ε), and restrict $m = n$, and $|z| < \delta, z \notin \mathcal{E}$, we obtain

$$|f - [m/n]|(z) \leq 2C\left(\frac{3\delta^2}{\varepsilon}\right)^n. \tag{2.5}$$

This will be small for n large if $\delta^2 << \varepsilon/3$, which is consistent with $\delta >> \varepsilon$.

The reader may well ask, so what have we gained? We have restricted z to lie in a small ball, centre 0, radius δ. True, since $\delta >> \varepsilon$, the set $\mathcal{E}$, which has area at most $\pi\varepsilon^2$, is a negligible proportion of that ball. But what of the rest of the unit ball? Now we come up with one of the main requirements of the Nuttall-Pommerenke theorem: suppose that f is not just analytic in the unit ball, but is entire. Then we can apply the above not to $f(z)$ but to the scaled function $f(z/\eta)$ for some small η. Moreover, it is easy to show that the (m,n) Padé approximant to $f(z/\eta)$ is just the (m,n) Padé approximant to $f(z)$, but evaluated at z/η. Then a little thought shows that (2.5) gives:

Theorem 1 (Weak form of Nuttall-Pommerenke Theorem)
Let f be entire, and let $r, \varepsilon > 0$. Then

$$m_2\{z : |z| \leq r \text{ and } |f - [n/n]|(z) \geq \varepsilon^n\} \to 0, n \to \infty. \tag{2.6}$$

Thus $\{[n/n]\}_{n=1}^{\infty}$ converges in planar measure to f in each ball centre 0, even with geometric rate. In fact, the same result holds if we allow f to be meromorphic in the plane, or if f has singularities of logarithmic capacity 0 (in particular essential singularities), or if f is rapidly approximable by rational functions, that is, lies in the Gonchar-Walsh class [6], [11]. Moreover, one may replace planar measure by α-dimensional Hausdorff content or logarithmic capacity, and the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ by more general "ray" sequences, for which m/n, the ratio of numerator and denominator degrees, is bounded above and below by positive constants [18]. There are also deeper analogues for functions with branchpoints, due to Stahl [21], [22], and many generalisations [6], [7], [24].

To cover any of these important developments, would take us too far from the main purpose of this paper, so we refer to [12], [22], [24].

3. Cartan's Lemma Revisited

We saw above how useful it was to have f entire, so that we could apply (2.5) to $f(z/\eta)$ with η small, rather than to $f(z)$. What happens when, for example, f is analytic only in the unit ball, and has a natural boundary on the unit circle? Then, unfortunately, there is no analogue of the Nuttall-Pommerenke Theorem. E.A. Rakhmanov and the author showed independently in the 1980's [9], [10], [19], that $\{[n/n]\}_{n=1}^{\infty}$ need not converge in measure in any ball contained in the unit circle. In fact, for each such ball, no matter how small, within the unit ball, there is a corresponding subsequence of $\{[n/n]\}_{n=1}^{\infty}$ having very bad divergence properties.

This seemed to indicate that one cannot say anything positive about the full diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ for functions known to be analytic (or meromorphic) in only a finite ball centre 0. Perhaps for that reason there has been more effort devoted to the 1961 Baker-Gammel-Wills Conjecture for *subsequences.* Following is one form of the conjecture:

Baker-Gammel-Wills Conjecture
Let f be analytic at 0 and meromorphic in the unit ball. Then there is an infinite sequence S of positive integers with

$$\lim_{n\to\infty, n\in S} [n/n](z) = f(z),$$

uniformly in compact subsets of the unit ball omitting poles of f.

The conjecture is widely believed to be false in the above form, though possibly true for functions meromorphic in the plane. The author has convincing evidence, but not yet a full proof, that a continued fraction of Rogers and Ramanujan provides a counterexample - see [12].

But let us return to full sequences $\{[n/n]\}_{n=1}^{\infty}$. OK, they cannot converge in measure, but is there any positive convergence property? After some thought, one realizes that there are positive things one can say, but this requires solutions of some problems involving polynomials. I believe that these problems have intrinsic interest, and I shall discuss them one by one.

Let us start from (1.3). An old inequality of Bernstein provides the bound

$$\max_{|t|=1} |q(t)| \leq r^{-n} \max_{|t|=r} |q(t)|, r < 1. \tag{3.1}$$

(In fact, this is an easy consequence of the maximum-modulus principle.) Then (1.3) becomes the more symmetric inequality

$$\max_{|z|=r} |f - [m/n]|(z) \leq \left\{ r^m \frac{\max_{|t|=r} |q(t)|}{\min_{|t|=r} |q(t)|} \right\} \frac{C}{1-r}, r < 1. \tag{3.2}$$

This suggests:

Problem 1
Let $\lambda > 1$, and q be a polynomial of degree $\leq n$. Let m_1 denote linear Lebesgue measure. Estimate below

$$m_1 \left\{ r \in [0,1] : \frac{\max_{|t|=r} |q(t)|}{\min_{|t|=r} |q(t)|} < \lambda^n \right\}.$$

More generally, let R be a rational function with numerator degree $\leq m$, denominator degree $\leq n$. Estimate below

$$m_1 \left\{ r \in [0,1] : \frac{\max_{|t|=r} |R(t)|}{\min_{|t|=r} |R(t)|} < \lambda^{m+n} \right\}. \tag{3.3}$$

The idea is that, having solved this, one can use the power r^m in (3.2) to "kill" the growing factor λ^m. In [13], I proved that

$$m_1 \left\{ r \in [0,1] : \frac{\max_{|t|=r} |R(t)|}{\min_{|t|=r} |R(t)|} < \lambda^{m+n} \right\} \geq \frac{1}{4} \exp\left(-\frac{13}{\log \lambda} \right), \tag{3.4}$$

but may be no larger than $\exp\left(-\frac{2-\varepsilon}{\log \lambda}\right)$, for any $\varepsilon > 0$. The method involves the:

Theorem 2 ("Abstract" Cartan's Lemma)
Let (X,d) be a metric space, $0 < r_1 < r_2 < ... < r_n$, and $a_1, a_2, ..., a_n \in X$. There exist positive integers $p, \lambda_1, \lambda_2, ..., \lambda_p$ and balls $B_1, B_2, ..., B_p$ in X, such that
(i) $\lambda_1 + \lambda_2 + ... + \lambda_p = n$;
(ii) Each B_j has diameter $4r_{\lambda_j}$;

$$\prod_{j=1}^{n} d(x, a_j) > \prod_{j=1}^{n} r_j, x \in X \backslash \bigcup_{j=1}^{p} B_j.$$

The proof of this is almost identical to the usual Cartan's Lemma. By applying this to $X = (0, \infty)$, equipped with the metric

$$d(x,t) := \left| \frac{x-t}{x+t} \right|,$$

one may prove [13, Theorem 2], that if $a_1, a_2, ..., a_n \in \mathbb{C}$; $\varepsilon \in \left(0, \frac{1}{6}\right)$, and

$$\mathcal{E} := \left\{ x \in [0, \infty) : \left| \prod_{j=1}^{n} \left(\frac{x-a_j}{x+a_j} \right) \right| \leq \varepsilon^n \right\},$$

then

$$\int_{\mathcal{E}} \frac{dx}{x} \leq 37\varepsilon. \tag{3.5}$$

How does this relate to Problem 1? Well observe that for any complex number a,

$$\frac{\max_{|t|=r} |t-a|}{\min_{|t|=r} |t-a|} \leq \left| \frac{r+|a|}{r-|a|} \right|, \tag{3.6}$$

so the "worst case" in (3.3) comes from R with real zeros and poles: replacing R in (3.3) by a rational function with real zeros and poles only decreases the size of the set in (3.3). Then after some work, and applying (3.5), we obtain (3.4). That inequality enables us to prove:

Theorem 3
Let f be analytic at 0 *and meromorphic in the unit ball. Let $0 < \delta < 1$. There exists n_0 and for $n \geq n_0$, $\mathcal{S}_n \subset \left(0, \frac{1}{2}\right)$, with $m_1(\mathcal{S}_n) \geq \exp\left(-\frac{40}{\delta}\right)$ and for $r \in \mathcal{S}_n$,*

$$\max_{|z|=r} \frac{|f - [n/n]|(z)}{|z|^n} \leq (1+\delta)^n. \tag{3.7}$$

Thus on a set $\mathcal{S}_n$ whose linear measure is bounded below independently of n, $[n/n]$ provides good approximation. That is, $\{[n/n]\}_{n=1}^{\infty}$ approximates well on "sets of positive proportion".

4. Potential Theory and Green Potentials

While the abstract Cartan lemma may have some appeal, it is with hindsight the wrong approach. There is this tool called potential theory that has been sweeping across orthogonal polynomials, and polynomial and rational approximation (especially in the complex plane) in the last twenty years. When it is applicable, it is simply unbeatable. It turns out here too that the sharp estimates for (3.3) may be obtained this way.

The connection between polynomials

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

and potentials is contained in the identity

$$\frac{1}{n} \log \frac{1}{|P(z)|} = \int \log \frac{1}{|z-t|} d\mu(t) =: U^{\mu}(z), \tag{4.1}$$

where μ denotes a probability measure with mass $\frac{1}{n}$ at each z_j. Potentials U^{μ} are useful because we can apply theorems like monotone convergence, Fatou's Lemma, weak convergence of sequences of probability measures, and so on, to sequences of potentials. In this particular problem, one writes

$$\frac{1}{n} \log \left| \frac{P(-z)}{P(z)} \right| = \int \log \left| \frac{z+t}{z-t} \right| d\mu(t) = \text{a Green potential,}$$

since one notes that when z is restricted to $(0, \infty)$, $\log \left| \frac{z+t}{z-t} \right|$ is the Green's function for the right-half plane, with pole at z. The reason for the term $P(-z)/P(z)$, where P has all real zeros, comes from (3.6), which shows that the "worst case" comes from such a P: for a particular rational function R, one replaces each of its zeros α by $|\alpha|$, and each of its poles β by $|\beta|$. Then one constructs a P with zeros at each $|\alpha|, |\beta|$.

Green's functions and Green potentials are especially useful in bounding the growth of rational functions when one is working in a domain such as a half-plane or a ball. See [20] for a comprehensive discussion of their properties and uses.

The statement of the result involves the complete elliptic integral

$$K(b) := \int_0^1 \frac{dx}{\sqrt{(1-b^2x^2)(1-x^2)}}, b \in (0,1),$$

and the complementary modulus

$$b' = \sqrt{1-b^2}.$$

It turns out that the function

$$F(b) := \frac{K(b')}{\pi K(b)}, b \in (0,1),$$

is a strictly decreasing function of b, mapping $(0,1)$ onto $(0,\infty)$ and hence has an inverse $F^{[-1]} : (0,\infty) \to (0,1)$. Thus,

$$F\left(F^{[-1]}(x)\right) = x, x \in (0,\infty).$$

The sharp estimate is:

Theorem 4
Let $\lambda > 1$ and $m, n \geq 0$.
(a) For rational functions R with numerator, denominator degrees$\leq$ m, n respectively,

$$m_1 \left\{ r \in [0,1] : \frac{\max_{|t|=r} |R(t)|}{\min_{|t|=r} |R(t)|} < \lambda^{m+n} \right\} \geq F^{[-1]}\left(\frac{1}{\log \lambda}\right). \tag{4.2}$$

(b) This is sharp in the following sense. Given $\varepsilon > 0$, there exists for large enough m, a polynomial R of degree $\leq m$, such that with $n = 0$, the set above has linear measure $\leq F^{[-1]}\left(\frac{1}{\log \lambda}\right) + \varepsilon$.

We note that

$$F^{[-1]}\left(\frac{1}{\log \lambda}\right) = \exp\left(-\frac{\pi^2}{2\log\lambda}(1 + o(1))\right), \lambda \to 1+.$$

Thus $\pi^2/2$ replaces 13 in (3.4) for $\lambda \to 1+$. For intrinsic interest, we suggest:

Problem 2
Let $a \geq 0, \lambda > 0$. For rational functions R with numerator, denominator degrees$\leq$ m, n respectively, estimate below

$$m_1 \left\{ r \in [0,1] : r^{a(m+n)} \frac{\max_{|t|=r} |R(t)|}{\min_{|t|=r} |R(t)|} < \lambda^{m+n} \right\}. \tag{4.3}$$

We shall not sketch the application of (4.2) to Padé approximation, since the next approach yields sharper results. As it turns out (and perhaps we should not be surprised), the use of the Bernstein inequality at (3.1), is not such a good idea, at least as regards Padé approximation.

5. Weighted Potential Theory

Let us recall (1.3):

$$\max_{|z|=r} |f - [m/n]|(z) \leq \left\{ r^{m+n} \frac{\max_{|t|=1} |q(t)|}{\min_{|t|=r} |q(t)|} \right\} \frac{C}{1-r}, 0 < r < 1.$$

This suggests:

Problem 3
Let $a \geq 0, \lambda > 0$. For polynomials q of degree $\leq n$, estimate below

$$m_1 \left\{ r \in [0,1] : r^{an} \frac{\max_{|t|=1} |q(t)|}{\min_{|t|=r} |q(t)|} < \lambda^n \right\}. \tag{5.1}$$

This problem was solved in [15], using the sort of weighted potential theory (or potential theory with external fields) that has had such an impact on orthogonal polynomials and rational approximation in the last twenty years. An introduction to this appears in [8] and [16], and a comprehensive treatment is given in [20].

Let us sketch the steps in the solution:

Step 1: q with only real zeros
First we note that the set in (5.1) is smallest when q has only real zeros, all lying in $[0,1]$ - compare (3.6). Thus we need only consider

$$\mathcal{E} := \left\{ r \in [0,1] : r^{an} \prod_{j=1}^{n} \left| \frac{1+\alpha_j}{r-\alpha_j} \right| < \lambda^n \right\},$$

where $\alpha_j \in [0,1], 1 \leq j \leq n$. Rather than finding a lower bound for $m_1(\mathcal{E})$, it is easier to find an upper bound for the linear measure of the complementary set

$$\mathcal{F} := \left\{ r \in [0,1] : r^{an} \prod_{j=1}^{n} \left| \frac{1+\alpha_j}{r-\alpha_j} \right| \geq \lambda^n \right\}.$$

Of course,

$$m_1(\mathcal{E}) = 1 - m_1(\mathcal{F}).$$

Step 2: Reformulate using potentials
One can show that in the worst case, that is when $m_1(\mathcal{F})$ is as large as possible,

$$\mathcal{F} = [c, 1].$$

Thus $\mathcal{F}$ is an interval with 1 as a right endpoint. The underlying reason is that r^{an} is an increasing function of r, and by shifting some of the α_j to the right, we may ensure that $\mathcal{F}$ becomes a single interval, without reducing its linear measure.

If ν denotes a probability measure with mass $\frac{1}{n}$ at each α_j, then we see that we may reformulate the inequality defining $\mathcal{F}$ in the following way:

$$r \in \mathcal{F} = [c, 1] \Leftrightarrow a \log r + U^{\nu}(r) - U^{\nu}(-1) \geq \log \lambda. \tag{5.2}$$

(Recall our notation for the potential U^{ν} from (4.1)).

Step 3: Use weighted potential theory
It is at this stage that one needs some potential theory. It turns out that for the given a and for $c \in (0, 1)$, there exists a unique probablility measure $\mu_{a,c}$, that is $\mu_{a,c}$ is a non-negative Borel measure with total mass 1, supported on some interval $[c, d] \subseteq [c, 1]$, such that

$$U^{\mu_{a,c}}(x) + a \log x \begin{cases} = F_{a,c}, & x \in [c, d]; \\ \geq F_{a,c}, & x \in [c, 1]. \end{cases} \tag{5.3}$$

Here $F_{a,c}$ is a constant. The measure $\mu_{a,c}$ is called the *equilibrium measure for the external field* $a \log r$ on $[c, 1]$. Then (5.2) gives

$$U^{\nu}(r) - U^{\mu_{a,c}}(r) + F_{a,c} - U^{\nu}(-1) \geq \log \lambda, r \in [c, d]. \tag{5.4}$$

A fundamental principle in potential theory, the Principle of Domination, then shows that (5.4) holds for all $r \in \mathbb{C}$. In particular, taking $r = -1$ gives

$$-U^{\mu_{a,c}}(-1) + F_{a,c} \geq \log \lambda.$$

This is the basic inequality we need. For a given a and λ, to get the largest $\mathcal{F}$, we solve the equation

$$-U^{\mu_{a,c}}(-1) + F_{a,c} = \log \lambda$$

for c. The function

$$G_a(c) := U^{\mu_{a,c}}(-1) - F_{a,c}, c \in (0, 1)$$

turns out to be a strictly decreasing function of c that maps $(0, 1)$ onto $\mathbb{R}$ if $a > 1$, and maps $(0, 1)$ onto $\left(-\infty, -(1-a)\log\left(3+\sqrt{8}\right)\right)$ if $a \leq 1$. In [15], we derive an explicit (but lengthy) representation for G_a in terms of a, c and the right endpoint $d = d(a, c)$ of the support interval

$[c,d]$ of $\mu_{a,c}$:

$$\begin{aligned}
G_a(c) &= -a\log\left(\frac{d+c}{d-c}+\sqrt{\left(\frac{d+c}{d-c}\right)^2-1}\right) \\
&\quad -\log\left(\frac{d+c+2}{d-c}+\sqrt{\left(\frac{d+c+2}{d-c}\right)^2-1}\right)-a\log\frac{d-c}{4} \\
&\quad -2a\log\left(\begin{array}{c} 1-\left[\frac{d+c}{d-c}-\sqrt{\left(\frac{d+c}{d-c}\right)^2-1}\right]\times \\ \times\left[\frac{d+c+2}{d-c}-\sqrt{\left(\frac{d+c+2}{d-c}\right)^2-1}\right]\end{array}\right),
\end{aligned}$$

where

$$d:=d(a,c):=\begin{cases} 1, & a\le 1; \\ \min\left\{1,c\left(\frac{a}{a-1}\right)^2\right\}, & a>1.\end{cases}$$

Let $G_a^{[-1]}$ denote the inverse function of G_a, and define

$$H_a(\varepsilon):=G_a^{[-1]}(-\log\varepsilon),\begin{cases}\varepsilon\in(0,\infty), & a>1 \\ \varepsilon\in\left(\left(3+\sqrt{8}\right)^{1-a},\infty\right), & a\le 1\end{cases}.$$

We may now state the main result of [15]:

Theorem 5
Let $a\ge 0$, and let

$$\begin{array}{ll}\varepsilon\in(0,\infty), & \text{if } a>1; \\ \varepsilon\in\left(\left(3+\sqrt{8}\right)^{1-a},\infty\right), & \text{if } a\le 1.\end{array}$$

(a) If $n\ge 1$ and P is a polynomial of degree $\le n$, then

$$m_1\left\{r\in[0,1]:r^{an}\frac{\max_{|t|=1}|P(t)|}{\min_{|t|=r}|P(t)|}<\varepsilon^n\right\}\ge H_a(\varepsilon). \tag{5.5}$$

(b) This is sharp in the sense that we may find for large enough n a polynomial P for which the left hand side in (5.5) is as close to $H_a(\varepsilon)$ as we please.

Two special cases ($a=2,3$ and $\varepsilon=1$) are of particular interest:

$$\begin{aligned}
m_1\left\{r\in[0,1]:r^{2n}\frac{\max_{|t|=1}|P(t)|}{\min_{|t|=r}|P(t)|}<1\right\} &\ge \frac{1}{8}; \\
m_1\left\{r\in[0,1]:r^{3n}\frac{\max_{|t|=1}|P(t)|}{\min_{|t|=r}|P(t)|}<1\right\} &\ge \frac{1}{4}.
\end{aligned}$$

We emphasise that $\frac{1}{4}$ and $\frac{1}{8}$ are sharp.

We now turn to results on Padé approximation. Note that if λ is fixed and $m = \lambda n$, then $m + n = (\lambda + 1) n$, so the $[\lambda n/n]$ Padé approximant corresponds to $a = \lambda + 1$ in (5.5).

Theorem 6

Let f be analytic at 0, and let f be meromorphic in $\{z : |z| < 1\}$.
(i) Let $0 < \delta < \frac{1}{8}$. Then there exists $\varepsilon = \varepsilon(\delta) \in (0, 1)$ and $n_0 > 0$ such that for $n \geq n_0$,

$$m_1 \left\{ r \in [0, 1] : \max_{|z|=r} |f - [n/n]| (z) \leq \varepsilon^n \right\} \geq \frac{1}{8} - \delta.$$

(ii) Let $0 < \delta < \frac{1}{4}$. Then there exists $\varepsilon = \varepsilon(\delta) \in (0, 1)$ and $n_0 > 0$ such that for $n \geq n_0$,

$$m_1 \left\{ r \in [0, 1] : \max_{|z|=r} |f - [2n/n]| (z) \leq \varepsilon^n \right\} \geq \frac{1}{4} - \delta.$$

Thus $[n/n]$ provides good approximation to f on almost $\frac{1}{8}$ of the circles centre 0 in the unit ball, and $[2n/n]$ on almost $\frac{1}{4}$ of the circles centre 0. One may consider more general ray sequences $\{[m_k/n_k]\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} m_k/n_k = \lambda.$$

As λ increases, the proportion of circles on which we get good approximation increases. Because of duality properties of Padé approximants, the same conclusions apply when λ is replace by $1/\lambda$.

We believe that the above theorem already shows that even for functions with finite radius of meromorphy, diagonal or ray sequences of Padé approximants can have reasonable approximation properties. Of course this immediately suggests the question: is $\frac{1}{8}$ sharp? is $\frac{1}{4}$ sharp? Unfortunately, it seems not, as we are not using the full power of the contour integral error formula.

6. Some Unsolved Problems

If we go back to the error formula (1.1), we recall that we are not using the full power of the contour integral error formula. Indeed, if S is an arbitrary polynomial of degree $\leq m$, we may insert it into (1.1), giving for $|z| < 1$,

$$(f - [m/n])(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{(fSq)(t)}{(Sq)(z)(t - z)} \left(\frac{z}{t} \right)^{m+n+1} dt.$$

This leads to the estimate

$$\max_{|z|=r} |f - [m/n]|(z) \leq \left\{ r^{m+n} \frac{\max_{|t|=1} |Sq(t)|}{\min_{|t|=r} |Sq(t)|} \right\} \frac{C}{1-r},$$

for $0 < r < 1$, with C independent of m, n, r, z, S. Since S is arbitrary, this suggests:

Problem 4
Let $a, b \geq 0, \lambda > 0$. For polynomials q of degree $\leq n$, estimate below

$$m_1 \left\{ r \in [0,1] : r^{an} \min_{\deg(S) \leq bn} \frac{\max_{|t|=1} |Sq(t)|}{\min_{|t|=r} |Sq(t)|} \leq \lambda^n \right\}.$$

Notice that we may choose a different polynomial S for each $r \in (0,1)$. This may be reformulated as a problem involving potentials. However, the solution is not as simple as that of Problem 3. What is certain that one obtains better results for the proportion of the "good" circles for Padé approximants. In fact, even choosing

$$S(z) := q(-z),$$

already leads to an improvement, for example for $a = 3$.

Another important direction that we have hardly covered, is estimates involving planar measure m_2, rather than linear Lebesgue measure m_1. Here one does not take maxima over circles centre 0, radius r. Thus, we may consider:

Problem 5
Let $a, b \geq 0, \lambda > 0$. For polynomials q of degree $\leq n$, estimate below

$$m_2 \left\{ z : |z| < 1 \text{ and } |z|^{an} \min_{\deg(S) \leq bn} \frac{\max_{|t|=1} |Sq(t)|}{|Sq(z)|} \leq \lambda^n \right\}.$$

This leads to an estimate of the planar measure of the set on which $[m/n]$ provides good approximation. In fact, this type of inequality, like most of those in this paper, may be reformulated as a weighted Remez inequality. Curiously enough, despite the apparent similarity to Zolotarev numbers, there is a closer link to Remez inequalities. See [1], [4] for a discussion of Remez inequalities and [20] for a discussion of Zolotarev numbers.

There is a lesson to learn from all this: even in old subjects like Padé approximation, there are new twists that can be explored, especially when along comes a powerful new tool like potential theory for external fields.

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