A MAXIMAL FUNCTION APPROACH TO CHRISTOFFEL FUNCTIONS AND NEVAI'S OPERATORS

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Abstract. Let μ be a compactly supported positive measure on the real line, with associated Christoffel functions $\lambda_n(d\mu,\cdot)$. Let g be a measurable function that is bounded above and below on supp$[\mu]$ by positive constants. We show that $\lambda_n(g\,d\mu,\cdot)/\lambda_n(d\mu,\cdot) \to g$ in measure in $\{x : \mu'(x) > 0\}$, and consequently in all L_p norms, $p < \infty$. The novelty is that there are no local or global restrictions on μ. The main idea is a new maximal function estimate for the "tail" in Nevai's operators.

Orthogonal Polynomials on the real line, Christoffel functions, ratio asymptotics, Nevai's operators. 42C05

1. Introduction[1]

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \ldots, \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

In analysis and applications of orthogonal polynomials, the reproducing kernel

$$K_n(d\mu, x, y) = \sum_{j=0}^{n-1} p_j(x) p_j(y).$$

plays a key role, as does its normalized cousin

$$\tilde{K}_n(d\mu, x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y).$$

If $\mu'(x)$ or $\mu'(y)$ is not finite, or does not exist, we set $\tilde{K}_n(d\mu, x, y) = 0$. When clear from the context that μ is the underlying measure, we omit $d\mu$. For $y = x$, K_n becomes the reciprocal of the Christoffel function

$$\lambda_n(d\mu, x) = \frac{1}{K_n(d\mu, x, x)}.$$

There is the classic extremum property

$$\lambda_n(d\mu, x) = \inf_{\deg(P) \le n-1} \frac{\int P^2 d\mu}{P^2(x)}.$$

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Ratio asymptotics for orthogonal polynomials associated with two different measures are a major topic in orthogonal polynomials. They were studied extensively by Maté, Nevai, and Totik [10], [11] as part of a program to extend Szegő's theory. Many others have taken up this topic - for example, Lopez [8] and Simon [17]. One of the essential limits within this topic is

$$\lim_{n\to\infty}\frac{\lambda_n\left(g\ d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}=g\left(x\right), \tag{1.1}$$

for appropriate functions g, and in an appropriate sense. For example, if μ is supported on $[-1,1]$, and $\mu'>0$ a.e. on $[-1,1]$, while $g^{\pm 1}$ is bounded on supp$[\mu]$, and g is continuous at x, then (1.1) holds at x. This follows from results of Nevai [9], [12].

Paul Nevai [12] introduced the operators

$$G_n\left[f\right]\left(x\right)=\int\frac{K_n^2\left(d\mu,x,t\right)}{K_n\left(d\mu,x,x\right)}f\left(t\right)d\mu\left(t\right),$$

as a means to establish (1.1). They are now called the *Nevai operators*, and have been studied for their own intrinsic interest, and have been widely generalized [3], [4], [7], [13]. They have turned out to be useful for orthogonal polynomials on the unit circle, and for questions in approximation theory. In all results to date, restrictions have had to be placed on the measure, often combined with bounds on the orthogonal polynomials.

In this paper, we shall use a maximal function approach to establish convergence in measure for general measures with compact support. For $r>0$, define the "tail" function

$$\Psi_n\left(x,r\right)=\frac{\int_{|t-x|\geq r/\tilde{K}_n(x,x)}K_n\left(x,t\right)^2d\mu\left(t\right)}{K_n\left(x,x\right)}. \tag{1.2}$$

Here if $\mu'\left(x\right)=0$, or does not exist, we set $\Psi_n\left(x,r\right)=0$. Also, let

$$A_n\left(x\right)=p_{n-1}^2\left(x\right)+p_n^2\left(x\right) \tag{1.3}$$

and define the maximal function

$$\mathcal{M}\left[d\nu\right]\left(x\right)=\sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h}d\nu \tag{1.4}$$

for positive measures ν on the real line. Our main new idea is a connection between maximal functions, and $\Psi_n\left(x,r\right)$, namely that for a.e. $x\in$supp$[\mu]$,

$$\Psi_n\left(x,r\right)\leq\frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\mathcal{M}\left[A_nd\mu\right]\left(x\right)\right)^2. \tag{1.5}$$

In the sequel, $\{\mu'>0\}$ denotes the set $\{x:\mu'\left(x\right)>0\}$. Recall that a sequence of functions $\{f_n\}$ on the real line is said to converge in measure to a function f on the set $\mathcal{A}$ if for every $\varepsilon>0$,

$$meas\left\{x\in\mathcal{A}:\left|f_n-f\right|\left(x\right)>\varepsilon\right\}\to 0,$$

as $n\to\infty$. Here meas denotes linear Lebesgue measure.

Our result for Christoffel functions is:

Theorem 1.1
Let μ be a compactly supported measure on the real line with infinitely many points in its support. Let $g : \mathbb{R} \to (0, \infty)$ be a $d\mu$ measurable function such that $g^{\pm 1}$ are bounded on $\operatorname{supp}[\mu]$. *Then*

$$\frac{\lambda_n (g\ d\mu, \cdot)}{\lambda_n (d\mu, \cdot)} \to g \text{ in measure in } \{\mu' > 0\}. \tag{1.6}$$

Moreover, for every $p > 0$,

$$\lim_{n\to\infty} \int_{\{\mu'>0\}} \left| \frac{\lambda_n (g\ d\mu, x)}{\lambda_n (d\mu, x)} - g(x) \right|^p dx = 0. \tag{1.7}$$

The novelty is the lack of restrictions on μ, especially the lack of a global condition.

Theorem 1.1 follows from a convergence result for the Nevai operators. As mentioned above, their convergence has been studied by many authors. One of the very first results, due to Nevai [12, Thm. 2, p. 74], remains the most relevant:

Theorem A
Assume that μ is a measure supported on $[-1, 1]$ with $\mu' > 0$ a.e. there. Let $f : [-1, 1] \to \mathbb{R}$ be $d\mu$ measurable and bounded on $[-1, 1]$. Then

$$\lim_{n\to\infty} G_n [f] (x) = f (x) \tag{1.8}$$

at every point of continuity of f in $(-1, 1)$.

Nevai actually proved this in a larger class, now called Nevai's class, or the Nevai-Blumenthal class, which is defined in terms of recurrence coefficients. Another very interesting recent result is due to Breuer, Last, and Simon [2], though stated there in a different form:

Theorem B
Let μ be a measure on the real line with

$$0 < \inf_n \frac{\gamma_{n-1}}{\gamma_n} \leq \sup_n \frac{\gamma_{n-1}}{\gamma_n} < \infty. \tag{1.9}$$

Then (1.8) holds at a given x for every continuous compactly supported function $f : \mathbb{R} \to \mathbb{R}$, iff

$$\lim_{n\to\infty} \frac{p_n^2 (x)}{\sum_{k=0}^{n-1} p_k^2 (x)} = 0. \tag{1.10}$$

Thus convergence of Nevai's operators at a point x is equivalent to subexponential growth of the orthonormal polynomials. The condition (1.9) ensures that $\operatorname{supp}[\mu]$ is compact. Breuer, Last and Simon, also constructed examples of measures that are regular in the sense of Stahl and Totik on $[-2, 2]$, but for which (1.10), and hence (1.8) fails, in $[-2, 2] \setminus [-1, 1]$.

We prove:

Theorem 1.2
Let μ be a compactly supported measure on the real line with infinitely many points

in its support. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded $d\mu$ measurable function. Then $\{G_n[f]\}$ converges in measure to f in $\{\mu'>0\}$. Moreover, for every $p>0$,

$$\lim_{n\to\infty}\int_{\{\mu'>0\}}|G_n[f]-f|^p=0. \tag{1.11}$$

Remarks

(a) One can prove analogues of Theorems 1.1 and 1.2 for measures on the unit circle.

(b) One can weaken the restriction that $g^{\pm 1}$ be bounded to the restriction adopted by Nevai and others: there is some polynomial R such that Rg and Rg^{-1} are bounded, while R is positive in supp$[\mu]$.

2. Proof of Theorem 1.2

Recall our notation (1.2) to (1.4). The main idea for estimating the "tail integral" in Nevai's operator is:

Theorem 2.1

Let μ be a measure on the real line with infinitely many points in its support. Let $r>0$.

(a) *Then for all $x\in$supp$[\mu]$,*

$$\Psi_n(x,r)\le\frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 A_n(x)\,\mu'(x)\,\mathcal{M}[A_n d\mu](x). \tag{2.1}$$

Here both sides are interpreted as 0 if $\mu'(x)$ does not exist.

(b) For a.e. $x\in$supp$[\mu]$,

$$\Psi_n(x,r)\le\frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\mathcal{M}[A_n d\mu](x)\right)^2. \tag{2.2}$$

(c) Let $\varepsilon>0$. Then

$$meas\{x\in\text{supp}[\mu]:\Psi_n(x,r)\ge\varepsilon\}\le\frac{\gamma_{n-1}}{\gamma_n}\frac{17}{\sqrt{r\varepsilon}}. \tag{2.3}$$

Proof

(a) Assume $\mu'(x)$ exists and is non-zero, otherwise there is nothing to prove. Observe that

$$\begin{aligned}|K_n(x,t)| &= \frac{\gamma_{n-1}}{\gamma_n}\left|\frac{p_n(x)p_{n-1}(t)-p_{n-1}(x)p_n(t)}{x-t}\right|\\ &\le \frac{\gamma_{n-1}}{\gamma_n}\frac{A_n(x)^{1/2}A_n^{1/2}(t)}{|x-t|},\end{aligned}$$

by Cauchy-Schwarz. Let

$$\beta=\frac{r}{\tilde{K}_n(x,x)}.$$

Then we see that

$$\begin{aligned}
\Psi_n(x,r) &= \frac{\int_{|t-x|\geq\beta} K_n(x,t)^2\, d\mu(t)}{K_n(x,x)} \\
&\leq \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \sum_{j=0}^{\infty} \int_{2^j\beta\leq|x-t|\leq 2^{j+1}\beta} \frac{1}{|x-t|^2} A_n(t)\, d\mu(t) \\
&\leq \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \sum_{j=0}^{\infty} 2^{-2j}\beta^{-2} \int_{2^j\beta\leq|x-t|\leq 2^{j+1}\beta} A_n(t)\, d\mu(t).
\end{aligned}$$

Here

$$\begin{aligned}
&\frac{1}{2\left(2^{j+1}\beta\right)} \int_{|x-t|\leq 2^{j+1}\beta} A_n(t)\, d\mu(t) \\
\leq\ & \mathcal{M}\left[A_n d\mu\right](x),
\end{aligned}$$

so we can continue this as

$$\begin{aligned}
\Psi_n(x,r) &\leq \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \beta^{-1} 8\mathcal{M}\left[A_n d\mu\right](x) \\
&= \frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 A_n(x)\,\mu'(x)\,\mathcal{M}\left[A_n d\mu\right](x).
\end{aligned}$$

(b) Now a.e. $x \in$supp$[\mu]$ is a Lebesgue point of the measure $A_n d\mu$, so for such x, we have

$$\begin{aligned}
&\mathcal{M}\left[A_n d\mu\right](x) \\
\geq\ & \lim_{h\to 0+} \frac{1}{2h}\int_{x-h}^{x+h} A_n(t)\, d\mu(t) \\
=\ & A_n(x)\,\mu'(x).
\end{aligned}$$

(c) By the classical weak (1,1) inequality for maximal functions [15, p.137], for $\lambda > 0$,

$$\begin{aligned}
&meas\left\{x\in\mathbb{R} : \mathcal{M}\left[A_n d\mu\right](x) \geq \lambda\right\} \\
\leq\ & \frac{3}{\lambda}\int A_n d\mu = \frac{6}{\lambda}.
\end{aligned}$$

Then

$$meas\left\{x \in \text{supp}\,[\mu] : \Psi_n(x,r) \geq \frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\lambda\right)^2\right\} \leq \frac{6}{\lambda}.$$

Now choose λ such that

$$\frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\lambda\right)^2 = \varepsilon \Leftrightarrow \frac{1}{\lambda} = \frac{\gamma_{n-1}}{\gamma_n}\frac{2\sqrt{2}}{\sqrt{r\varepsilon}}.$$

Thus

$$meas\left\{x \in \text{supp}\,[\mu] : \Psi_n(x,r) \geq \varepsilon\right\} \leq 12\sqrt{2}\frac{\gamma_{n-1}}{\gamma_n}\frac{1}{\sqrt{r\varepsilon}}$$

■

Our second main idea is to bound the ratio $\frac{K_n^2(x,t)}{K_n^2(x,x)}$ for t close to x and most $x \in \{\mu' > 0\}$. In the proof, we use elementary potential theory, and the maximal Hilbert transform $\mathcal{H}^*$, defined for $f \in L_1(\mathbb{R})$ and a.e. x by

$$\mathcal{H}^*[f](x) = \sup_{\varepsilon>0} \left| \int_{|x-t|>\varepsilon} \frac{f(t)}{t-x} dt \right|.$$

Lemma 2.2

Assume the hypotheses of Theorem 1.1. Let $\varepsilon > 0$. There exist $C_1, C_2 > 0$ and for $n \geq 1$, sets $\mathcal{E}_n$ of measure $\leq \varepsilon$, with the following property: let $r > 0$. For $n \geq 1$ and $x \in \{\mu' > 0\} \backslash \mathcal{E}_n$,

$$\sup_{|t-x| \leq \frac{r}{K_n(x,x)}} \frac{K_n^2(x,t)}{K_n^2(x,x)} \leq C_1 e^{C_2 r}. \tag{2.4}$$

Here C_1, C_2 depend on ε, but are independent of r, n and x.

Proof

Step 1: The set E_n on which K_n is not too large

First we show that there is $\Lambda > 1$ independent of n, such that for $n \geq 1$,

$$E_n = \{t : K_n(t,t) \leq \Lambda n\} \tag{2.5}$$

has

$$meas(\{\mu' > 0\} \backslash E_n) < \frac{\varepsilon}{3}. \tag{2.6}$$

Indeed,

$$\int K_n(t,t)\, \mu'(t)\, dt \leq n,$$

so

$$meas\left\{t : K_n(t,t)\, \mu'(t) \geq \sqrt{\Lambda} n\right\} \leq \frac{1}{\sqrt{\Lambda}}.$$

Moreover, for sufficiently large Λ,

$$meas\left\{t : t \in \{\mu' > 0\} \text{ and } \mu'(t) \leq \frac{1}{\sqrt{\Lambda}}\right\} \leq \frac{\varepsilon}{6}.$$

Combining the last two inequalities, with appropriately large Λ, gives (2.6).

Step 2: Using the Bernstein-Walsh Inequality

Next, E_n consists of at most finitely many intervals, some of which may reduce to a point (as K_n is a polynomial). Let ν_n denote the equilibrium measure of E_n, and $g_{\mathbb{C}\backslash E_n}$ denote the Green's function for $\mathbb{C}\backslash E_n$, with pole at ∞. Since E_n has finitely many intervals, ν_n is absolutely continuous, and ν_n' is positive and continuous in the interior of E_n [16, p. 412]. Moreover, $g_{\mathbb{C}\backslash E_n}(x) = 0$ at every point x of E_n (cf. [14, p. 111, Thm. 4.4.9]). We use the Bernstein-Walsh inequality for $\mathbb{R}\backslash E_n$ [14, p. 156, Thm. 5.5.7]:

$$K_n(t,t) \leq \Lambda n e^{(2n-2) g_{\mathbb{C}\backslash E_n}(t)},\ t \in \mathbb{R}\backslash E_n. \tag{2.7}$$

For $\xi \in E_n$, and $u \in \mathbb{R}$, by the potential theoretic representation for the Green's function [14, p. 107]

$$\begin{aligned} g_{\mathbb{C}\backslash E_n}(\xi+u) &= g_{\mathbb{C}\backslash E_n}(\xi+u) - g_{\mathbb{C}\backslash E_n}(\xi) \\ &= \left[\int_{|\xi-t|\le|u|} + \int_{|\xi-t|>|u|}\right] \log\left|1+\frac{u}{\xi-t}\right| \nu_n'(t)\,dt \\ &= I_1 + I_2. \end{aligned}$$

Here

$$\begin{aligned} I_1 &\le \int_{|\xi-t|\le|u|} \log\left(1+\left|\frac{u}{\xi-t}\right|\right)\nu_n'(t)\,dt \\ &\le \sum_{k=0}^{\infty} \log\left(1+2^{k+1}\right) \int_{2^{-k-1}|u|\le|\xi-t|\le 2^{-k}|u|} \nu_n'(t)\,dt \\ &\le \sum_{k=0}^{\infty} \log\left(1+2^{k+1}\right) 2^{-k+1}|u|\,\mathcal{M}[\nu_n'](\xi) = C_0|u|\,\mathcal{M}[\nu_n'](\xi). \end{aligned}$$

Next, in I_2, we have $\left|\frac{u}{\xi-t}\right| < 1$, so the inequality $\log(1+t) \le t, t > -1$, gives

$$I_2 \le \int_{|\xi-t|>|u|} \frac{u}{\xi-t}\nu_n'(t)\,dt \le |u|\,\mathcal{H}^*[\nu_n'](\xi),$$

where $\mathcal{H}^*$ is the maximal Hilbert transform, defined above. Thus for all real u,

$$g_{\mathbb{C}\backslash E_n}(\xi+u) \le |u|\left\{C_0\mathcal{M}[\nu_n'](\xi) + \mathcal{H}^*[\nu_n'](\xi)\right\}. \tag{2.8}$$

As both the maximal function and maximal Hilbert transform are weak type (1,1), [1, p. 139], [6, p. 129], we have for all $\lambda > 0$,

$$meas\left\{\xi : C_0\mathcal{M}[\nu_n'](\xi) + \mathcal{H}^*[\nu_n'](\xi) > \lambda\right\} \le \frac{C_1}{\lambda}\int \nu_n' = \frac{C_1}{\lambda}.$$

Here C_1 is a constant independent of λ, n, ν_n. Choose $\lambda = \frac{6C_1}{\varepsilon}$, and let

$$\mathcal{F}_n = \left\{\xi : C_0\mathcal{M}[\nu_n'](\xi) + \mathcal{H}^*[\nu_n'](\xi) > \frac{6C_1}{\varepsilon}\right\}, \tag{2.9}$$

which has

$$meas(\mathcal{F}_n) \le \frac{\varepsilon}{6}. \tag{2.10}$$

From (2.7), (2.8), and (2.9), we obtain for $\xi \in E_n\backslash\mathcal{F}_n$, and all $r > 0$,

$$\sup_{|s|\le r} K_n\left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right) \le \Lambda n e^{(2n-2)\frac{r}{\tilde{K}_n(\xi,\xi)}\frac{6C_1}{\varepsilon}}. \tag{2.11}$$

Step 3 A Lower Bound for K_n

Take an interval J containing supp$[\mu]$ and let

$$d\rho = dx_{|J} + d\mu.$$

Then

$$\int_J \log\rho' > -\infty,$$

so a theorem of Maté, Nevai and Totik [9] shows that for a.e. $x \in J$,

$$\lim_{n\to\infty} \frac{K_n(d\rho, x, x)}{n\nu'_J(x)/\rho'(x)} = 1,$$

where ν_J denotes the equilibrium measure of the interval J. As $d\mu \leq d\rho$, $K_n(d\mu, t, t) \geq K_n(d\rho, t, t)$, so we deduce that for a.e. $x \in$supp$[\mu]$,

$$\liminf_{n\to\infty} \frac{K_n(d\mu, x, x)}{n\nu'_J(x)/\rho'(x)} \geq 1. \tag{2.12}$$

By applying Egorov's theorem to the sequence of functions

$$\min\left\{\frac{K_n(d\mu, x, x)}{n\nu'_J(x)/\rho'(x)}, 1\right\}, n \geq 1,$$

we obtain $C_2 > 0$ and a set $\mathcal{G}$ of measure $\leq \frac{\varepsilon}{4}$ such that

$$K_n(x, x) \geq C_2 n \text{ for } n \geq 1 \text{ and } x \in \operatorname{supp}[\mu] \backslash \mathcal{G}.$$

In turn, this leads to a set $\mathcal{H}$ of measure $\leq \frac{\varepsilon}{3}$ and $C_3 > 0$ such that

$$K_n(x, x) \geq C_3 n \text{ and } \tilde{K}_n(x, x) \geq C_3 n \text{ for } n \geq 1 \text{ and } x \in \operatorname{supp}[\mu] \backslash \mathcal{H}.$$

Applying this to (2.11) gives

$$\sup_{|s|\leq r} \frac{K_n\left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi, \xi)} \leq C_4 e^{C_5 r}, \tag{2.13}$$

for $n \geq 1$, and $\xi \in \{\mu' > 0\} \backslash \mathcal{E}_n$, where

$$\mathcal{E}_n = (\{\mu' > 0\} \backslash E_n) \cup \mathcal{F}_n \cup \mathcal{H}$$

has measure $< \varepsilon$, recall (2.6) and (2.10). Moreover, C_4 and C_5 are independent of n and r. Finally, Cauchy-Schwarz gives

$$\frac{K_n^2(x, t)}{K_n^2(x, x)} \leq \frac{K_n(t, t)}{K_n(x, x)}$$

and (2.4) follows. ■

Proof of Theorem 1.2

Let $\varepsilon > 0$ and $r = \varepsilon^{-3}$. Now

$$\begin{aligned} & |G_n[f](x) - f(x)| \\ \leq & \int \frac{K_n^2(x, t)}{K_n(x, x)} |f(t) - f(x)| \, d\mu(t) \\ \leq & \left[\sup_{|t-x|\leq \frac{r}{\tilde{K}_n(x,x)}} \frac{K_n^2(x, t)}{K_n^2(x, x)}\right] K_n(x, x) \int_{|t-x|\leq \frac{r}{\tilde{K}_n(x,x)}} |f(t) - f(x)| \, d\mu(t) \\ & + 2\|f\|_{L_\infty(\mathbb{R})} \Psi_n(x, r). \end{aligned} \tag{2.14}$$

Here by Lemma 2.2, for $n \geq 1$, there is a set $\mathcal{E}_n$ of measure $< \varepsilon$ such that for $x \in \{\mu' > 0\} \backslash \mathcal{E}_n$,

$$\sup_{|t-x|\leq \frac{r}{\tilde{K}_n(x,x)}} \frac{K_n^2(x, t)}{K_n^2(x, x)} \leq C_1 e^{C_2 r}.$$

For $h>0$, let

$$M_h[f](x)=\frac{1}{2h}\int_{x-h}^{x+h}|f(t)-f(x)|\,d\mu(t).$$

For $n\geq 1$ and $x\in\{\mu'>0\}\backslash\mathcal{E}_n$, we obtain from (2.14),

$$\begin{aligned}&|G_n[f](x)-f(x)|\\ \leq\ & C_1e^{C_2r}\frac{2r}{\mu'(x)}M_{\frac{r}{\tilde{K}_n(x,x)}}[f](x)+2\|f\|_{L_\infty(\mathbb{R})}\Psi_n(x,r).\end{aligned}\tag{2.15}$$

Next, it is a simple consequence of Lebesgue's theory that

$$\lim_{h\to 0+}M_h[f](x)=0$$

at every Lebesgue point of f that is also a Lebesgue point of μ. Indeed, if μ_s denotes the singular part of μ,

$$\begin{aligned}M_h[f](x)\ \leq\ & \frac{\mu'(x)}{2h}\int_{x-h}^{x+h}|f(t)-f(x)|\,dt\\ &+\frac{2\|f\|_{L_\infty(\mathbb{R})}}{2h}\left\{\int_{x-h}^{x+h}d\mu_s(t)+\int_{x-h}^{x+h}|\mu'(t)-\mu'(x)|\,dt\right\}.\end{aligned}$$

Then also, as $K_n(x,x)\to\infty$ except at jumps of μ [5, p. 63, Th. II.2.1],

$$\lim_{n\to\infty}C_1e^{C_2r}\frac{r}{\mu'(x)}M_{\frac{r}{\tilde{K}_n(x,x)}}[f](x)=0$$

a.e. in $\{\mu'>0\}$. By Egorov's theorem, there is a set $\mathcal{G}$ of measure $\leq\varepsilon$, such that for large enough n, and all $\xi\in\{\mu'>0\}\backslash\mathcal{G}$,

$$C_1e^{C_2r}\frac{r}{\mu'(x)}M_{\frac{r}{\tilde{K}_n(x,x)}}[f](x)\leq\varepsilon.\tag{2.16}$$

Next, since supp$[\mu]$ is bounded, $\sup_n\frac{\gamma_{n-1}}{\gamma_n}<\infty$. Then, from Theorem 2.1(c), if

$$\mathcal{F}_n=\{x\in\operatorname{supp}[\mu]:\Psi_n(x,r)>\varepsilon\},$$

and recalling our choice $r=\varepsilon^{-3}$,

$$meas\{\mathcal{F}_n\}\leq C_1\varepsilon,$$

with C_1 independent of n,ε. Thus for $x\in\{\mu'>0\}\backslash(\mathcal{E}_n\cup\mathcal{F}_n\cup\mathcal{G})$, (2.15) and (2.16) give

$$\begin{aligned}&|G_n[f](x)-f(x)|\\ \leq\ &\left(1+2\|f\|_{L_\infty(\mathbb{R})}\right)\varepsilon.\end{aligned}$$

Since meas$(\mathcal{E}_n\cup\mathcal{F}_n\cup\mathcal{G})\leq(2+C_1)\varepsilon$, this proves the convergence in measure. As

$$|G_n[f]-f|\leq 2\|f\|_{L_\infty(\mathbb{R})},$$

the convergence in L_p norms also follows. ■

3. Christoffel Functions

In this section, we prove Theorem 1.1. We begin with the elementary inequality [12, Theorem 5, p. 77]

$$\frac{\lambda_n\left(g\ d\mu,x\right)P^2\left(x\right)}{\lambda_{n-m}\left(d\mu,x\right)}\leq G_{n-m}\left[gP^2\right]\left(x\right). \tag{3.1}$$

Here g is a non-negative function that is integrable with respect to μ, and P is a polynomial of degree $\leq m$, while $m\leq n$.

Proof of Theorem 1.1
Let $\varepsilon>0$. Now by Lusin's Theorem, g equals a continuous function on a compact set $\mathcal{A}$ such that meas$(\{\mu'>0\}\setminus\mathcal{A})<\varepsilon$. Moreover, $g^{\pm1}$ is bounded. We can then choose a polynomial P such that

$$\left(1+\varepsilon\right)^{-1}\leq\left(gP^2\right)\left(x\right)\leq1+\varepsilon \tag{3.2}$$

on compact $\mathcal{A}$. Suppose that P has degree m. Write, in $\mathcal{A}$,

$$\begin{aligned}
&\frac{\lambda_n\left(g\ d\mu,x\right)}{\lambda_n\left(d\mu,x\right)g\left(x\right)}\\
=&\ \left[\frac{\lambda_n\left(g\ d\mu,x\right)P^2\left(x\right)}{\lambda_{n-m}\left(d\mu,x\right)}\right]\left[\frac{1}{\left(gP^2\right)\left(x\right)}\right]\left[\frac{\lambda_{n-m}\left(d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}\right]\\
\leq&\ G_{n-m}\left[gP^2\right]\left(x\right)\left(1+\varepsilon\right)\left[\frac{\lambda_{n-m}\left(d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}\right],
\end{aligned} \tag{3.3}$$

by (3.1) and (3.2). Here from Theorem 1.2, as $n\to\infty$,

$$meas\left\{x\in\{\mu'>0\}:G_{n-m}\left[gP^2\right]\left(x\right)\geq\left(gP^2\right)\left(x\right)+\varepsilon\right\}\to0.$$

In view of (3.2), this also shows that as $n\to\infty$,

$$meas\left\{x\in\mathcal{A}\cap\{\mu'>0\}:G_{n-m}\left[gP^2\right]\left(x\right)\geq1+2\varepsilon\right\}\to0. \tag{3.4}$$

Next, we claim that as $n\to\infty$,

$$meas\left\{x\in\{\mu'>0\}:\frac{\lambda_{n-m}\left(d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}\geq1+\varepsilon\right\}\to0. \tag{3.5}$$

To see this, note that

$$\begin{aligned}
&\left(\frac{\lambda_{n-m}\left(d\mu,x\right)}{\lambda_n\left(d\mu,x\right)}-1\right)\mu'\\
=&\ \lambda_{n-m}\left(d\mu;x\right)\left(\sum_{k=n-m}^{n-1}p_k^2\left(x\right)\right)\mu'.
\end{aligned}$$

Here

$$\int\left(\sum_{k=n-m}^{n-1}p_k^2\right)\mu'\leq m,$$

so

$$meas\left\{x\in\mathbb{R}:\left(\sum_{k=n-m}^{n-1}p_k^2\right)\mu'\geq\frac{m}{\varepsilon}\right\}\leq\varepsilon.$$

Moreover, we know that as $n \to \infty$, $\lambda_{n-m}(d\mu,,x) = K_{n-m}^{-1}(d\mu,x,x) \to 0$, except at point masses of μ, so (for example) Egoroff's Theorem shows that for large enough n,

$$meas\left\{x \in \{\mu' > 0\} : \lambda_{n-m}(d\mu;x) \geq \frac{\varepsilon^2}{m}\right\} \leq \varepsilon.$$

It follows that for large enough n,

$$meas\left\{x \in \{\mu' > 0\} : \left(\frac{\lambda_{n-m}(d\mu,x)}{\lambda_n(d\mu,x)} - 1\right)\mu'(x) \geq \varepsilon\right\} \leq 2\varepsilon.$$

Thus $\left(\frac{\lambda_{n-m}(d\mu,\cdot)}{\lambda_n(d\mu,\cdot)} - 1\right)\mu'(\cdot)$ converges in measure to 0 in $\{\mu' > 0\}$ as $n \to \infty$. Inasmuch as $\mu' > 0$ in $\{\mu' > 0\}$, this also easily implies (3.5). Together (3.3), (3.4), and (3.5) imply that for large enough n,

$$meas\left\{x \in \mathcal{A} \cap \{\mu' > 0\} : \frac{\lambda_n(g\ d\mu,x)}{\lambda_n(d\mu,x)\,g(x)} \geq (1+2\varepsilon)(1+\varepsilon)(1+\varepsilon)\right\} \leq 4\varepsilon.$$

Inasmuch as $\varepsilon > 0$ is arbitrary, and meas$(\{\mu' > 0\} \setminus \mathcal{A}) < \varepsilon$, this gives that as $n \to \infty$,

$$meas\left\{x \in \{\mu' > 0\} : \frac{\lambda_n(g\ d\mu,x)}{\lambda_n(d\mu,x)\,g(x)} \geq 1+\varepsilon\right\} \to 0.$$

In a similar way, we can establish the relation

$$meas\left\{x \in \{\mu' > 0\} : \frac{\lambda_n(d\mu,x)}{\lambda_n(g\ d\mu,x)\,g(x)^{-1}} \geq 1+\varepsilon\right\} \to 0.$$

For this, one uses Nevai's operators for the measure $g\ d\mu$ rather than $d\mu$, and the fact that g^{-1} is bounded in supp$[\mu]$. These last two relations establish that $\frac{\lambda_n(g\ d\mu,x)}{\lambda_n(d\mu,x)g(x)} \to 1$ in measure in $\{\mu' > 0\}$. Since $g^{\pm 1}$ is bounded in supp$[\mu]$, and since

$$\inf_{\mathrm{supp}[\mu]} g \leq \frac{\lambda_n(g\ d\mu,x)}{\lambda_n(d\mu,x)} \leq \sup_{\mathrm{supp}[\mu]} g,$$

we obtain the result on convergence in L_p as well. ■

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