ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS AND SEPARATION OF THEIR ZEROS

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Abstract. Let $\{p_n\}$ denote the orthonormal polynomials associated with a measure μ with compact support on the real line. In a recent paper, we showed there is a close relationship between the spacing of zeros of succesive orthogonal polynomials p_n, p_{n-1}, and uniform bounds on the orthogonal polynomials in subintervals of the support. In this paper, we show there is also a relationship between asymptotics for the spacing of zeros of p_{n-1}, p_n, and pointwise asymptotics for the orthogonal polynomials.

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1. Results

Let μ be a finite positive Borel measure with compact support, which we denote by supp$[\mu]$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

The zeros $\{x_{jn}\}$ of p_n are real and simple. We list them in decreasing order:

$$x_{1n} > x_{2n} > ... > x_{n-1,n} > x_{nn}.$$

The three term recurrence relation has the form

$$(x - b_n) p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x),$$

where for $n \geq 1$,

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} = \int x p_{n-1}(x) p_n(x) d\mu(x); \ b_n = \int x p_n^2(x) d\mu(x).$$

In a recent paper [7], we analyzed the relationship between the spacing of zeros of successive orthogonal polynomials p_{n-1}, p_n, namely

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$x_{jn} - x_{j,n-1}$ and uniform bounds on orthogonal polynomials in subintervals of the support. Spacing of zeros for the same orthogonal polynomial, namely $x_{j-1,n} - x_{jn}$, has been intensively studied for decades [6], [11], [15], [16]. Bounds on orthogonal polynomials is also a classic topic in orthogonal polynomials [1], [2], [4], [5], [9].

The results from [7] require more terminology: we let $dist\,(a, \mathbb{Z})$ denote the distance from a real number a to the integers. We say that μ is *regular* (in the sense of Stahl, Totik, and Ullmann) if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap\,(\text{supp}\,[\mu])},$$

where cap denotes logarithmic capacity. If the support consists of finitely many intervals, and $\mu' > 0$ a.e. in each subinterval, then μ is regular, though much less is required [13].

Recall that the equilibrium measure for the compact set supp$[\mu]$ is the probability measure that minimizes the energy integral

$$\int\int \log \frac{1}{|x-y|} d\nu\,(x)\, d\nu\,(y)$$

amongst all probability measures ν supported on supp$[\mu]$. If I is an interval contained in supp$[\mu]$, then the equilibrium measure is absolutely continuous in I, and moreover its density, which we denote throughout by ω, is positive and continuous in the interior I^o of I [10, p.216, Thm. IV.2.5]. Given sequences $\{x_n\}, \{y_n\}$ of non-0 real numbers, we write

$$x_n \sim y_n$$

if there exists $C > 1$ such that for $n \geq 1$,

$$C^{-1} \leq x_n / y_n < C.$$

Similar notation is used for functions and sequences of functions.

In [7, Theorem 1.1], we proved:

Theorem A

Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous. Let ω be the density of the equilibrium measure for the support of μ. Let $A > 0$. The following are equivalent:

(a) There exists $C > 0$ such that for $n \geq 1$ and $x_{jn} \in I$,

$$dist\,(n\omega\,(x_{jn})\,(x_{jn} - x_{j,n-1}), \mathbb{Z}) \geq C. \tag{1.1}$$

(b) There exists $C > 0$ such that for $n \geq 1$ and $x \in I$,

$$\|p_{n-1}\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \leq C. \tag{1.2}$$

Moreover, under either (a) or (b), we have

$$\sup_{n\geq 1}\sup_{x\in I}\left||x-b_n|^{1/2}\,p_n(x)\right|<\infty. \tag{1.3}$$

Under additional assumptions on the spacing of the zeros of p_{n-2} and p_n, the factor $|x-b_n|^{1/2}$ in (1.3) was removed.

In this paper, we investigate the relationship between pointwise asymptotics of orthogonal polynomials, and the spacing $x_{jn}-x_{j,n-1}$. As a pointer to what might be possible, let us recall the form of the classical pointwise asymptotic for orthogonal polynomials inside $\mathrm{supp}[\mu]$. Let us suppose our support is $[-1,1]$, that μ satisfies Szegő's condition, and in some subinterval $I\subset(-1,1)$, μ is absolutely continuous, μ' is continuous, while the local modulus of continuity of μ' satisfies a suitable Dini condition. See [3] for a precise statement of the hypotheses. Badkov [3] generalized many earlier results, proving that as $n\to\infty$, uniformly in closed subintervals of I^0,

$$p_n(x)\,\mu'(x)^{1/2}\left(1-x^2\right)^{1/4}=\sqrt{\frac{2}{\pi}}\cos\left(n\theta+h(\theta)\right)+o(1), \tag{1.4}$$

where $x=\cos\theta$, and h is a continuous function. It is straightforward to prove:

Proposition 1.1
Assume the asymptotic (1.4) holds uniformly for x in I. Fix $k\geq 0,\ell\in\mathbb{Z}$. Let J be a closed subinterval of I. Then uniformly for x_{jn} in I,

$$n\left(x_{jn}-x_{j-\ell,n-k}\right)=\sqrt{1-x_{jn}^2}\left[k\arccos\left(x_{jn}\right)-\ell\pi\right]+o(1). \tag{1.5}$$

We shall prove this in Section 2. One can compare this to the much studied asymptotic for spacing of successive zeros of p_n when the support is $[-1,1]$,

$$n\left(x_{jn}-x_{j+1,n}\right)\frac{1}{\pi\sqrt{1-x_{jn}^2}}=1+o(1).$$

or for more general supports with equilibrium density ω, [6], [12], [15]

$$n\left(x_{jn}-x_{j+1,n}\right)\omega\left(x_{jn}\right)=1+o(1).$$

We prove the following partial converse:

Theorem 1.2
Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a

closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous. Let ω be the density of the equilibrium measure for the support of μ. Let $\mathcal{S}$ be an infinite sequence of positive integers. Assume that uniformly for $n \in \mathcal{S}$, $m = n, n+1$, and $x_{jn} \in I$,

$$m\omega\left(x_{jm}\right)\left(x_{jm} - x_{j,m-1}\right) = g\left(x_{jm}\right) + o\left(1\right), \tag{1.6}$$

where $g : I \to (0,1)$ is continuous. Let

$$f_n\left(x\right) = \omega\left(x_{jn}\right)\left(x - y_{jn}\right) + \frac{j}{n}, x \in \left[x_{j+1,n}, x_{jn}\right) \cap I. \tag{1.7}$$

Then uniformly for x in compact subsets of I, as $n \to \infty, n \in \mathcal{S}$,

$$\left|x - b_n\right|^{1/2} p_n\left(x\right) \mu'\left(x\right)^{1/2} = \sqrt{\frac{2}{\pi}} \left|\cot \pi g\left(x\right)\right|^{1/2} \left[\cos n\pi f_n\left(x\right) + o\left(1\right)\right]. \tag{1.8}$$

Corollary 1.3
If J is a compact subinterval of I, as $n \to \infty, n \in \mathcal{S}$,

$$\sup_{x \in J} \left| p_n\left(x\right) \mu'\left(x\right)^{1/2} \left|x - b_n\right|^{1/2} \right| = \sqrt{\frac{2}{\pi}} \sup_{x \in J} \left|\cot \pi g\left(x\right)\right|^{1/2} + o\left(1\right). \tag{1.9}$$

Remarks
(a) If $g\left(x\right) = \frac{1}{\pi} \arccos x$, as is the case in Proposition 1.1, while $b_n = 0$, Theorem 1.2 simplifies to

$$p_n\left(x\right) \mu'\left(x\right)^{1/2} \left(1 - x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} \left[\cos n\pi f_n\left(x\right) + o\left(1\right)\right], \tag{1.10}$$

uniformly in compact subsets of $I \setminus \{0\}$.
(b) Note that f_n is continuous: indeed, as shown by Lemma 3.1(c) below,

$$\lim_{x \to x_{jn}-} n f_n\left(x\right) = \frac{1}{2} + j = n f_n\left(x_{jn}\right).$$

(c) Lemma 3.4 below shows that we can recast the asymptotic as

$$\begin{aligned} & \left|x - b_n\right|^{1/2} p_n\left(x\right) \mu'\left(x\right)^{1/2} \\ (1.11) \quad = & \sqrt{\frac{2}{\pi}} \left| \frac{\left(\frac{x - b_{n-1}}{2a_n}\right)\left(\frac{x - b_n}{2a_n}\right)}{1 - \left(\frac{x - b_{n-1}}{2a_n}\right)\left(\frac{x - b_{n-1}}{2a_n}\right)} \right|^{1/4} \left[\cos n\pi f_n\left(x\right) + o\left(1\right)\right] \end{aligned}$$

and hence that

$$\begin{aligned}
&\sqrt[4]{1-\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_n}{2a_n}\right)}p_n(x)\,\mu'(x)^{1/2} \\
=\ &\sqrt{\frac{1}{a_n\pi}}\left|\frac{x-b_{n-1}}{x-b_n}\right|^{1/4}\left[\cos n\pi f_n(x)+o(1)\right].
\end{aligned}$$

except close to zeros of $\cos\pi g(x)$.

We prove Proposition 1.1 in the next section and Theorem 1.2 and Corollary 1.3 in Section 3. We close this section with some notation. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, θ. The same symbol does not necessarily denote the same constant in different occurences. We denote the zeros of p_n' by y_{jn}, ordered so that

$$y_{jn}\in\left(x_{j+1,n},x_{jn}\right),\ 1\le j\le n-1. \tag{1.12}$$

The nth reproducing kernel for μ is

$$K_n(x,y)=\sum_{k=0}^{n-1}p_k(x)\,p_k(y).$$

2. Proof of Proposition 1.1

We turn to

Proof of Proposition 1.1
Write

$$x_{jn}=\cos(\theta_{jn}).$$

Note that $\{\theta_{jn}\}$ lie in a closed subinterval of $(0,\pi)$ as I is a closed subinterval of $(-1,1)$. Then

$$\cos(n\theta_{jn}+h(\theta_{jn}))=o(1).$$

This gives for some integer $j_1=j_1(j,n)$ that may depend on both j and n,

$$n\theta_{jn}+h(\theta_{jn})=-\frac{\pi}{2}+j_1\pi+o(1)$$

so

$$\theta_{jn}=\frac{1}{n}\left(-\frac{\pi}{2}+j_1\pi-h(\theta_{jn})\right)+o\left(\frac{1}{n}\right).$$

Also then from interlacing of the zeros and the ordering,

$$\theta_{j-\ell,n-k}=\frac{1}{n-k}\left(-\frac{\pi}{2}+(j_1-\ell)\pi-h(\theta_{j-\ell,n-k})\right)+o\left(\frac{1}{n}\right).$$

As ℓ, k are fixed, it follows that

$$\frac{1}{2}\left(\theta_{j-\ell,n-k}+\theta_{jn}\right)=\theta_{jn}+O\left(\frac{1}{n}\right)$$

and hence also, as $\sin\theta_{jn}$ is bounded away from 0,

$$\sin\left(\frac{1}{2}\left(\theta_{j-\ell,n-k}+\theta_{jn}\right)\right)=\left(\sin\theta_{jn}\right)\left(1+O\left(\frac{1}{n}\right)\right). \tag{2.1}$$

Also

$$\begin{aligned}
&\frac{1}{2}\left(\theta_{j-\ell,n-k}-\theta_{jn}\right)\\
=\;&\left(-\frac{\pi}{2}+j_1\pi\right)\frac{1}{2}\left(\frac{1}{n-k}-\frac{1}{n}\right)-\frac{\ell\pi}{2\left(n-k\right)}+\frac{1}{2}\left(\frac{h\left(\theta_{jn}\right)}{n}-\frac{h\left(\theta_{j-\ell,n-k}\right)}{n-k}\right)+o\left(\frac{1}{n}\right)\\
=\;&\left(-\frac{\pi}{2}+j_1\pi\right)\frac{k}{2}\frac{1}{n\left(n-k\right)}-\frac{\ell\pi}{2n}+o\left(\frac{1}{n}\right)\\
=\;&\frac{k}{2\left(n-k\right)}\left[\theta_{jn}+\frac{h\left(\theta_{jn}\right)}{n}\right]-\frac{\ell\pi}{2n}+o\left(\frac{1}{n}\right)\\
=\;&\frac{k\theta_{jn}}{2n}-\frac{\ell\pi}{2n}+o\left(\frac{1}{n}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&x_{jn}-x_{j-\ell,n-k}\\
=\;&\cos\left(\theta_{jn}\right)-\cos\left(\theta_{j-\ell,n-k}\right)\\
=\;&-2\sin\left(\frac{1}{2}\left(\theta_{j-\ell,n-k}+\theta_{jn}\right)\right)\sin\left(\frac{1}{2}\left(\theta_{jn}-\theta_{j-\ell,n-k}\right)\right)\\
=\;&2\left(\sin\theta_{jn}\right)\left(1+O\left(\frac{1}{n}\right)\right)\sin\left(\frac{k\theta_{jn}}{2n}-\frac{\ell\pi}{2n}+o\left(\frac{1}{n}\right)\right)\\
=\;&\left(\sin\theta_{jn}\right)\left(1+O\left(\frac{1}{n}\right)\right)\left[\frac{k\theta_{jn}}{n}-\frac{\ell\pi}{n}+o\left(\frac{1}{n}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
&n\left(x_{jn}-x_{j-\ell,n-k}\right)\\
=\;&\left(\sin\theta_{jn}\right)\left[k\theta_{jn}-\ell\pi+o\left(1\right)\right]+o\left(1\right)\\
=\;&\sqrt{1-x_{jn}^2}\left[k\arccos\left(x_{jn}\right)-\ell\pi\right]+o\left(1\right).
\end{aligned}$$

■

3. The Converse

We begin with some established asymptotics and bounds for orthogonal polynomials:

Lemma 3.1
Assume that μ is a regular measure with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of the interior I^o of I. Let ω denote the equilibrium density for the support of μ.
(a) Then

$$\lim_{n\to\infty} \frac{p_n\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)}{p_n\left(y_{jn}\right)} = \cos\pi z \tag{3.1}$$

uniformly for $y_{jn} \in J$ and z in compact subsets of $\mathbb{C}$. Here ω is the equilibrium density for the support of μ.
(b) Uniformly for $x \in J$,

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(x,x\right)\mu'\left(x\right) = \omega\left(x\right). \tag{3.2}$$

(c) Uniformly for $y_{jn} \in J$,
(3.3)
$$n\omega\left(x_{jn}\right)\left(x_{jn}-y_{jn}\right) = \frac{1}{2}+o\left(1\right); n\omega\left(x_{jn}\right)\left(y_{jn}-x_{j+1,n}\right) = \frac{1}{2}+o\left(1\right).$$

(d) Uniformly for $y_{jn} \in J$,
(3.4)
$$n\omega\left(x_{jn}\right)\left(x_{jn}-x_{j+1,n}\right) = 1+o\left(1\right); n\omega\left(x_{jn}\right)\left(y_{jn}-y_{j+1,n}\right) = 1+o\left(1\right).$$

Proof
(a) See [8, Theorem 1.1].
(b) See [14].
(c), (d) See [7, Lemma 3.1]. ■

Lemma 3.2
Assume that μ is a regular measure with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of the interior I^o of I. Let ω denote the equilibrium density for the support of μ. Assume in addition the hypothesis (1.6).

(a) Uniformly for $y_{jn} \in J$,

$$n\omega\left(x_{jn}\right)\left(x_{jn}-y_{j,n-1}\right)=g\left(x_{jn}\right)+\frac{1}{2}+o\left(1\right); \tag{3.5}$$

$$n\omega\left(x_{jn}\right)\left(y_{jn}-y_{j,n-1}\right)=g\left(x_{jn}\right)+o\left(1\right). \tag{3.6}$$

(b) Fix $A>0$. *Uniformly for* $n \geq 1$,

$$\sup_{x\in J}\left\|p_n\right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\left\|p_{n-1}\right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\leq C. \tag{3.7}$$

(c)

$$\left\|p_n\right\|_{L_\infty(J)}=o\left(n^{-1/2}\right). \tag{3.8}$$

Proof
(a) Using continuity of ω, and our hypothesis (1.6),

$$\begin{aligned}n\omega\left(x_{jn}\right)\left(x_{jn}-y_{j,n-1}\right) &= n\omega\left(x_{jn}\right)\left(x_{jn}-x_{j,n-1}\right)+n\omega\left(x_{j,n-1}\right)\left(x_{j,n-1}-y_{j,n-1}\right)+o\left(1\right)\\ &= g\left(x_{jn}\right)+\frac{1}{2}+o\left(1\right).\end{aligned}$$

Next, from this last asymptotic and (3.3),

$$n\omega\left(x_{jn}\right)\left(y_{jn}-y_{j,n-1}\right)=n\omega\left(x_{jn}\right)\left(y_{jn}-x_{jn}+x_{jn}-y_{j,n-1}\right)=g\left(x_{jn}\right)+o\left(1\right).$$

(b) See [7, Theorem 1.1]. Note that our hypothesis (1.6) implies the spacing hypotheses required for Theorem 1.1 there.
(c) This follows from the asymptotic (3.2) for the Christoffel function: as $n \to \infty$, uniformly for $x \in J$,

$$\begin{aligned}\frac{1}{n}p_n^2\left(x\right)\mu'\left(x\right) &= \left(1+\frac{1}{n}\right)\frac{1}{n+1}K_{n+1}\left(x,x\right)\mu'\left(x\right)-\frac{1}{n}K_n\left(x,x\right)\mu'\left(x\right)\\ &\to \omega\left(x\right)-\omega\left(x\right)=0.\end{aligned}$$

■

Here is the main ingredient for our Theorem 1.2:

Lemma 3.3
Uniformly for $y_{jn} \in I$,
(3.9)

$$\left|y_{jn}-b_n\right|^{1/2}p_n\left(y_{jn}\right)\mu'\left(y_{jn}\right)^{1/2}\left(-1\right)^j=\sqrt{\frac{2}{\pi}}\left|\cot \pi g\left(y_{jn}\right)\right|^{1/2}+o\left(1\right).$$

Proof
We multiply the recurrence relation by $p_n\left(y_{jn}\right)$:

$$\left(y_{jn}-b_n\right)p_n^2\left(y_{jn}\right)=a_{n+1}\left(p_{n+1}p_n\right)\left(y_{jn}\right)+a_n\left(p_np_{n-1}\right)\left(y_{jn}\right). \tag{3.10}$$

We use the local limit (3.1) and the Christoffel-Darboux formula to simplify the right-hand side. First, from the confluent form of the Christoffel-Darboux formula,

$$K_n(y_{jn}, y_{jn}) = -a_n p_{n-1}'(y_{jn}) p_n(y_{jn}). \tag{3.11}$$

Since the local limit (3.1) holds uniformly in compact subsets of the plane, we can differentiate it:

$$\lim_{n\to\infty} \frac{p_{n-1}'\left(y_{j,n-1} + \frac{z}{(n-1)\omega(y_{j,n-1})}\right)}{p_{n-1}(y_{j,n-1})(n-1)\omega(y_{j,n-1})} = -\pi \sin \pi z.$$

Using this and (3.6),

$$\begin{aligned} p_{n-1}'(y_{jn}) &= p_{n-1}'(y_{j,n-1} + (y_{jn} - y_{j,n-1})) \\ &= p_{n-1}'\left(y_{j,n-1} + \frac{g(x_{jn}) + o(1)}{n\omega(y_{j,n-1})}\right) \\ &= -p_{n-1}(y_{j,n-1})(n-1)\omega(y_{j,n-1})\pi(\sin \pi g(x_{jn}) + o(1)) \\ &= -\pi p_{n-1}(y_{j,n-1}) n\omega(y_{j,n-1})(\sin \pi g(x_{jn}))(1 + o(1)), \end{aligned} \tag{3.12}$$

recall that $\sin \pi g(x_{jn})$ is bounded away from 0. We substitute this in (3.11), multiply by $\frac{1}{n}\mu'(y_{jn})$, and use the asymptotic (3.2):

$$\omega(y_{jn}) + o(1) = a_n \pi p_{n-1}(y_{j,n-1}) p_n(y_{jn}) \mu'(y_{jn}) \omega(y_{j,n-1})(\sin \pi g(x_{jn}))(1 + o(1))$$

and hence

$$a_n \pi p_{n-1}(y_{j,n-1}) p_n(y_{jn}) \mu'(y_{jn})(\sin \pi g(x_{jn})) = 1 + o(1)). \tag{3.13}$$

To replace $p_{n-1}(y_{j,n-1})$ by $p_{n-1}(y_{jn})$, we again use (3.1):

$$p_{n-1}(y_{jn}) = p_{n-1}(y_{j,n-1})[\cos \pi g(x_{jn}) + o(1)]. \tag{3.14}$$

Thus (3.13) can be recast as

$$\begin{aligned} & a_n \pi p_n(y_{jn}) p_{n-1}(y_{jn}) \mu'(y_{jn}) \sin \pi g(x_{jn}) \\ = {} & \cos \pi g(x_{jn}) + o(1) + o(p_n(y_{jn}) p_{n-1}(y_{j,n-1})) \\ = {} & \cos \pi g(x_{jn}) + o(1), \end{aligned}$$

by Lemma 3.2(b). Since $\sin \pi g(x_{jn})$ is bounded away from 0,

$$a_n \pi p_n(y_{jn}) p_{n-1}(y_{jn}) \mu'(y_{jn}) = \cot \pi g(x_{jn}) + o(1). \tag{3.15}$$

Next, replacing n by $n+1$ in (3.15) and using continuity of μ', g gives

$$a_{n+1} \pi p_{n+1}(y_{j,n+1}) p_n(y_{j,n+1}) \mu'(y_{jn}) = \cot \pi g(x_{jn}) + o(1). \tag{3.16}$$

Using the local limit (3.1) on p_n and then p_{n+1} gives

$$\begin{aligned} p_{n+1}(y_{j,n+1}) p_n(y_{j,n+1}) &= p_{n+1}(y_{j,n+1})(\cos \pi g(x_{jn}) + o(1)) p_n(y_{jn}) \\ &= (p_{n+1} p_n(y_{jn})) + o(1), \end{aligned}$$

by Lemma 3.2(b) again. Thus (3.16) yields

$$a_{n+1}\pi\left(p_{n+1}p_n\left(y_{jn}\right)\right)\mu'\left(y_{jn}\right)=\cot\pi g\left(x_{jn}\right)+o\left(1\right). \tag{3.17}$$

We substitute this and (3.15) into the recurrence (3.10):

$$\left(y_{jn}-b_n\right)p_n^2\left(y_{jn}\right)=\frac{2}{\pi}\cot\pi g\left(x_{jn}\right)+o\left(1\right). \tag{3.18}$$

Finally as $y_{jn}\in\left(x_{j+1,n},x_{jn}\right)$, so

$$\left(-1\right)^j p_n\left(y_{jn}\right)>0, 1\leq j\leq n-1,$$

and we obtain the result on taking square roots. ■

Proof of Theorem 1.2

Now from (3.1), (3.9), for $x\in\left[x_{j+1,n},x_{jn}\right)\cap I$,

$$\begin{aligned}&\left|y_{jn}-b_n\right|^{1/2}p_n\left(x\right)\mu'\left(x\right)^{1/2}\\ =&\left[\left(-1\right)^j\sqrt{\frac{2}{\pi}}\left|\cot\pi g\left(y_{jn}\right)\right|^{1/2}+o\left(1\right)\right]\left[\cos\left(\pi n\omega\left(x_{jn}\right)\left(x-y_{jn}\right)\right)+o\left(1\right)\right]\\ =&\sqrt{\frac{2}{\pi}}\left|\cot\pi g\left(x\right)\right|^{1/2}\cos\left(\pi n\left[\omega\left(x\right)\left(x-y_{jn}\right)+\frac{j}{n}\right]\right)+o\left(1\right),\end{aligned}$$

by continuity of g,ω. Next, uniformly for $x\in J$,

$$\begin{aligned}\left|x-b_n\right|^{1/2}&=\left|y_{jn}-b_n\right|^{1/2}+O\left(\left|x-y_{jn}\right|^{1/2}\right)\\ &=\left|y_{jn}-b_n\right|^{1/2}+O\left(n^{-1/2}\right),\end{aligned}$$

so

$$\begin{aligned}&\left|x-b_n\right|^{1/2}p_n\left(x\right)\mu'\left(x\right)^{1/2}\\ =&\sqrt{\frac{2}{\pi}}\left|\cot\pi g\left(x\right)\right|^{1/2}\cos\left(\pi n\left[\omega\left(x\right)\left(x-y_{jn}\right)+\frac{j}{n}\right]\right)+o\left(1\right)+O\left(n^{-1/2}\left|p_n\left(x\right)\right|\right)\\ =&\sqrt{\frac{2}{\pi}}\left|\cot\pi g\left(x\right)\right|^{1/2}\cos\left(\pi n f_n\left(x\right)\right)+o\left(1\right),\end{aligned}$$

by Lemma 3.2(c). ■

Proof of Corollary 1.3

This is immediate from Theorem 1.2. ■

Lemma 3.4

Uniformly for x in compact subsets of I omitting zeros of $\cos\pi g$,

(a)

$$(x-b_n)(x-b_{n-1}) = 4a_n^2\cos^2(\pi g(x)) + o(1). \tag{3.19}$$

(b)

$$|\cot \pi g(x)| = \frac{\left|\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_n}{2a_n}\right)\right|^{1/2}}{\sqrt{1-\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_{n-1}}{2a_n}\right)}} + o(1) + o(1). \tag{3.20}$$

Proof

From the recurrence relation,

$$(y_{jn}-b_{n-1})(p_{n-1}p_n)(y_{jn}) = a_n p_n^2(y_{jn}) + a_{n-1}(p_{n-2}p_n)(y_{jn}). \tag{3.21}$$

We now replace the terms on both sides. First we multiply by $a_n\pi\mu'(y_{jn})$ and use (3.15):

$$\begin{aligned} & (y_{jn}-b_{n-1})[\cot \pi g(x_{jn}) + o(1)] \\ = \; & a_n^2\pi p_n^2(y_{jn})\mu'(y_{jn}) + a_{n-1}a_n\pi(p_{n-2}p_n)(y_{jn})\mu'(y_{jn}). \end{aligned} \tag{3.22}$$

Next, from (3.13),

$$a_n\pi p_n(y_{jn})p_{n-1}(y_{j,n-1})\mu'(y_{jn})\pi\sin\pi g(x_{jn}) = 1 + o(1).$$

Replacing n by $n-1$:

$$a_{n-1}\pi p_{n-1}(y_{j,n-1})p_{n-2}(y_{j,n-2})\mu'(y_{j,n-1})\sin\pi g(x_{j,n-1}) = 1 + o(1).$$

Dividing the two relations, and using continuity of μ', g, and the fact that $\sin\pi g$ is bounded away from 0, gives

$$\frac{a_n p_n(y_{jn})}{a_{n-1}p_{n-2}(y_{j,n-2})} = 1 + o(1). \tag{3.23}$$

Next, our local limit (3.1) and the spacing (3.4) give

$$p_{n-2}(y_{jn}) = p_{n-2}(y_{j,n-2})(\cos(2\pi g(x_{jn})) + o(1)).$$

So

$$a_n p_n(y_{jn})(\cos(2\pi g(x_{jn})) + o(1)) = a_{n-1}p_{n-2}(y_{jn})(1 + o(1)).$$

Then (3.22) becomes

$$\begin{aligned} & (y_{jn}-b_{n-1})[\cot \pi g(x_{jn}) + o(1)] \\ = \; & a_n^2\pi p_n^2(y_{jn})\mu'(y_{jn})\{1 + \cos(2\pi g(x_{jn})) + o(1)\}. \end{aligned}$$

Multiplying by $(y_{jn}-b_n)$ and using Lemma 3.3 gives

$$\begin{aligned} & (y_{jn}-b_n)(y_{jn}-b_{n-1})\left[\cot \pi g(x_{jn})+o(1)\right] \\ = \; & 4a_n^2\left\{\cot \pi g(y_{jn})+o(1)\right\}\left\{\cos^2(\pi g(x_{jn}))+o(1)\right\} \end{aligned}$$

so that if $\cos \pi g(x_{jn})$ is bounded away from 0,

$$(y_{jn}-b_n)(y_{jn}-b_{n-1}) = 4a_n^2\cos^2(\pi g(x_{jn}))+o(1).$$

Then the result follows from the continuity of g, the density of the $\{y_{jn}\}$ and the boundedness of the $\{b_n\}$.
(b) Away from zeros of $\cos \pi g(x)$,

$$\begin{aligned} |\cot \pi g(x)| &= \frac{|\cos \pi g(x)|}{\sqrt{1-\cos^2 \pi g(x)}} \\ &= \frac{\left|\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_n}{2a_n}\right)\right|^{1/2}}{\sqrt{1-\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_{n-1}}{2a_n}\right)}}+o(1). \end{aligned}$$

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