VARIATIONAL CONDITIONS ON EXTREMAL ORTHONORMAL POLYNOMIALS FOR RESTRICTED MEASURES

D. S. LUBINSKY

ABSTRACT. Suppose that ν is a given positive measure on $[-1,1]$. Let $\mathcal{M}(I,\Lambda)$ denote the set of all measures μ, whose restriction to $(-1,1)$ is ν, whose support is contained in a compact interval I, and whose total mass outside $(-1,1)$ is at most $\Lambda>0$. We analyze the measure(s) in $\mathcal{M}(I,\Lambda)$ whose orthonormal polynomials have largest absolute value among those in $\mathcal{M}(I,\Lambda)$ at given points.

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orthogonal polynomials, bounds

Dedicated to the 80th birthday of Ed Saff

1. Results

Let μ be a finite positive Borel measure on the real line with infinitely many points in its support, and all finite moments

$$\int t^j d\mu(t),\ j=0,1,2,\ldots .$$

Then we may define orthonormal polynomials

$$p_n(\mu,x)=\gamma_n(\mu)x^n+\ldots,\gamma_n(\mu)>0,$$

$n=0,1,2,\ldots$ satisfying the orthonormality conditions

$$\int p_n(\mu,x)\,p_m(\mu,x)\,d\mu(x)=\delta_{mn}.$$

The nth reproducing kernel is

$$K_n(\mu,x,t)=\sum_{j=0}^{n-1}p_j(\mu,x)\,p_j(\mu,t).$$

A central problem in the theory of orthonormal polynomials is to establish bounds on $p_n(\mu,x)$. Steklov conjectured in 1920 that if μ has support $[-1,1]$, and is absolutely continuous there, while μ' is bounded below by a positive constant, then $\{p_n(\mu,x)\}_{n\geq 0}$ is uniformly bounded in compact subsets of $(-1,1)$. Rakhmanov showed in 1979, 1981 [15], [16] that Steklov's conjecture is false, and in fact one can find a measure satisfying its hypotheses such that for infinitely many n,

$$|p_n(\mu,0)|\geq Cn^{1/2}/(\log n)^{3/2+\varepsilon}, \tag{1.1}$$

any $\varepsilon>0$. Rakhmanov's weight μ' was unbounded. Ambroladze [2] subsequently constructed continuous weights on the unit circle that satisfy a weak Dini condition, for which the corresponding sequence of orthonormal polynomials can be unbounded. There have been further major developments in recent years, due to

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Aptekarev, Denisov and Tulyakov[1], [3], [7], [8], [9], [10], [11]. The relationship between spacing of zeros of successive orthogonal polynomials and bounds has been explored in [13].

There are positive results that establish boundedness, at least away from the endpoints of the support, for fairly general weights. Probably the most general result for measures supported on $[-1,1]$ is due to Badkov [5]. He proved that if μ satisfies Szegő's condition and μ' satisfies a local Dini-Lipschitz condition, then the corresponding orthonormal polynomials are uniformly bounded in compact subsets of (a,b). This was a consequence of deeper pointwise asymptotics.

One might hope that when a measure is suitably restricted in some subinterval, one can say something about bounds in that subinterval. A recent result of the author [14] deals with this question. It involves $p_n\left(S^2\nu, y\right)$, the orthonormal polynomial for the measure $S^2\nu$, where S is a real polynomial.

Theorem 1.1
Let ν be a positive measure on $[-1,1]$, with infinitely many points in its support. Let $\mathcal{K}$ be a closed subset of $\mathbb{R}$ containing $(-1,1)$. Let $y\in\mathbb{R}$ and $n\geq 1$. Then

$$\begin{aligned} & \sup\left\{p_n^2(\mu,y) : \mu_{|(-1,1)} = \nu \text{ and } \mu\geq\nu \text{ and } \operatorname{supp}[\mu]\subseteq\mathcal{K}\right\} \\ = \; & \sup S_J^2(y)\, p_{n-J}^2\left(S_J^2\nu, y\right), \end{aligned}$$

where the supremum is taken over all $0\leq J\leq n$ and monic polynomials S_J of degree J with distinct zeros in $\mathcal{K}\backslash(-1,1)$.

In the case where $\mathcal{K}$ is an interval containing $[-1,1]$ in its interior, it was shown in [14] that the sup can grow like a power of n. We note that it was only implied in the proofs there that $\mu\geq\nu$, not explicitly stated. In the case where ν has no masspoints at $-1,1$, this follows from $\mu_{|(-1,1)}=\nu$. However, if μ has masspoints at ± 1, then it does not follow. It should have been stated explicitly there.

In this paper, we consider the case where the total mass of μ is bounded and in addition, supp$[\mu]$ is contained in an interval containing $[-1,1]$. Thus let $\Lambda>0$ and $I=[c,d]$ be an interval containing $[-1,1]$. Let

$$\mathcal{M}(I,\Lambda)=\left\{\mu : \mu_{|(-1,1)}=\nu;\operatorname{supp}[\mu]\subset I;\mu\left(I\backslash[-1,1]\right)\leq\Lambda\right\}.$$

For a given real x, let

$$\phi_n(I,\Lambda,x)=\sup\left\{|p_n(\mu,x)| : \mu\in\mathcal{M}(I,\Lambda)\right\}.$$

We shall derive necessary conditions for measures μ that attain the sup, via variational tools. Any such measure will be called an extremal measure.

Theorem 1.2
Let ν be a measure with support in $[-1,1]$ with infinitely many points in its support, and such that ± 1 are not masspoints. Let $d>1, c<-1$, $I=[c,d]$, $\Lambda>0$. Fix $x\in\mathbb{R}$.
(a) There exists a measure $\mu_I\in\mathcal{M}(I,\Lambda)$ such that

$$|p_n(\mu_I,x)|=\phi_n(I,\Lambda,x). \tag{1.2}$$

(b) Write this measure in the form $\mu_I=\nu+\rho_I$, and let

$$R_n(\mu_I,t)=\frac{p_n(\mu_I,t)}{p_n(\mu_I,x)}\left[K_{n+1}(\mu_I,x,t)+K_n(\mu_I,x,t)\right]. \tag{1.3}$$

Then

$$\int R_n\left(\mu_I, t\right) d\left(\omega-\rho_I\right)(t) \geq 0 \tag{1.4}$$

for every nonnegative measure ω that has support in $I \backslash(-1,1)$ and total mass $\leq \Lambda$.
(c) Let

$$\eta=\inf \left\{R_n\left(\mu_I, t\right): t \in I \backslash(-1,1)\right\}.$$

For each t_0 in the support of ρ_I,

$$R_n\left(\mu_I, t_0\right)=\eta \leq 0. \tag{1.5}$$

(d) The support of ρ_I is purely discrete and consists of at most $n+1$ mass points.
(e) If ρ_I has total mass less than Λ, then $\operatorname{supp}[\rho_I] \subset\{-1,1\}$ *and $R_n\left(\mu_I, \cdot\right)=0$ in* $\operatorname{supp}[\rho_I]$.

It is not clear if μ_I is unique. We next investigate how $\phi_n(I, \Lambda, x)$ varies as the right endpoint y of $I=[c, y]$ increases. Let us set

$$\psi_n(y)=\ln \frac{1}{\phi_n([c, y], \Lambda, x)^2}, y>1, \tag{1.6}$$

for fixed c, Λ and x.

Theorem 1.3
Let ν be a measure with support in $[-1,1]$ with infinitely many points in its support, and such that ± 1 are not masspoints. Let $c<-1$, $\Lambda>0$, and fix $x \in \mathbb{R}$. Then ψ_n is absolutely continuous in $(1, \infty)$, $\psi_n^{\prime}$ exists for a.e. $y>1$, and at every point where the derivative exists,

$$\psi_n^{\prime}(y)=\mu_{[c, y]}(\{y\}) R_n^{\prime}\left(\mu_{[c, y]}, y\right). \tag{1.7}$$

Here $\mu_{[c, y]}$ is any extremal measure for $[c, y]$.

We next investigate how $\phi_n(I, \Lambda, x)$ varies as the total mass Λ increases. Let

$$\chi_n(\Lambda)=\ln \frac{1}{\phi_n([c, d], \Lambda, x)^2}, \Lambda>0,$$

for fixed c, d and x, and study how this changes as Λ increases.

Theorem 1.4
Let ν be a measure with support in $[-1,1]$ with infinitely many points in its support, and such that ± 1 are not masspoints. Let $c<-1$, $d>1$, $I=[c, d]$, and fix $x \in \mathbb{R}$. Then $\chi_n^{\prime}(\Lambda)$ exists for a.e. *$\Lambda>0$ and at every point where the derivative exists,*

$$\chi_n^{\prime}(\Lambda)=\inf \left\{R_n\left(\mu_{I, \Lambda}, t\right): t \in I \backslash(-1,1)\right\}. \tag{1.8}$$

Here $\mu_{I, \Lambda}$ is any extremal measure for the given Λ and I. Moreover, the inf is continuous in Λ.

Note that it follows that the inf is independent of the particular extremal measure.

This paper is organized as follows: in Section 2, we present two variational results. In Section 3, we prove Theorem 1.2. We prove Theorem 1.3 in Section 4, and Theorem 1.4 in Section 5. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurrences.

Dedication

It is a privilege to dedicate this paper to the 80th birthday of Ed Saff. Ed has contributed greatly to many areas of mathematics, including orthogonal polynomials. His work on exponential weights and his development of potential theory for external fields [17] dramatically advanced the area. His service to the research community is unparalleled.

2. Variational Formulae

In this section, we shall consider how $|p_n(\mu, x)|$ changes when we fix n and x, and perturb the measure μ. We shall consider two types of perturbations - the first involving adding a possibly signed measure, and the second involving shifting mass points. As far as we know, the formulae below are new, and we are believe are of independent interest.

Theorem 2.1
Let μ be a positive measure with support on the real line and infinitely many points in its support. Let ω be a possibly signed measure of finite total mass, with compact support. For $\varepsilon \geq 0$, let

$$\mu^{\varepsilon} = \mu + \varepsilon\omega \tag{2.1}$$

and assume that for small enough $\varepsilon \geq 0$, μ^{ε} is a nonnegative measure. Fix $n \geq 1$ and $x \in \mathbb{R}$ such that $p_n(\mu, x) \neq 0$. Then

$$\frac{\partial}{\partial \varepsilon}\left\{\ln \frac{1}{p_n(\mu^{\varepsilon}, x)^2}\right\}_{|\varepsilon=0} = \int R_n(\mu, t)\, d\omega(t), \tag{2.2}$$

where

$$R_n(\mu, t) = \frac{p_n(\mu, t)}{p_n(\mu, x)}\left\{K_n(\mu, x, t) + K_{n+1}(\mu, x, t)\right\}. \tag{2.3}$$

In (2.2), the derivative is a right derivative as μ_{ε} is only defined for $\varepsilon \geq 0$. In our second type of perturbation, which is more complicated, involving shifting point masses, we allow signed ε.

Theorem 2.2
Let μ be a positive measure with support on the real line and infinitely many points in its support. Let $L \geq 1, \left\{\beta_j\right\}_{j=1}^{L}$ be real numbers, and $\left\{\xi_j\right\}_{j=1}^{L}$ be distinct numbers. For each $1 \leq j \leq L$, let $\sigma\left(\xi_j\right) \in \mathbb{R}$. For real ε, let

$$\omega_{\varepsilon} = \sum_{j=1}^{L} \beta_j \delta_{\xi_j + \sigma(\xi_j)\varepsilon} \tag{2.4}$$

and

$$\mu^{\varepsilon} = \mu - \omega_0 + \omega_{\varepsilon}. \tag{2.5}$$

Assume that for small enough $|\varepsilon|$, μ^{ε} is a nonnegative measure. Fix $n \geq 1$, and $x \in \mathbb{R}$ such that $p_n(\mu, x) \neq 0$. Then with R_n as above,

$$\frac{\partial}{\partial \varepsilon}\left\{\ln \frac{1}{p_n(\mu^{\varepsilon}, x)^2}\right\}_{|\varepsilon=0} = \int R_n'(\mu, t)\, \sigma(t)\, d\omega_0(t). \tag{2.6}$$

We begin with the elementary assertion that orthogonal polynomials of fixed degree n, are smooth functions of the underlying moments:

Lemma 2.3
(a) Let μ^0 be a positive measure with infinitely points in its support and all finite moments. Assume that for small enough $|h|$, we are give a measure μ^h with the following property: for each fixed $j \geq 0$, there is a continuous function g_j such that for small enough $|h_1|$ and $|h_2|$, the jth moment satisfies

$$\int t^j d\mu^{h_1}(t) - \int t^j d\mu^{h_2}(t) = g_j(h_2)(h_1 - h_2) + o(h_1 - h_2).$$

Then for each fixed x, and n, $\frac{d}{dh}\left(p_n\left(\mu^h, x\right)\right)$ and $\frac{d}{dh}\gamma_n\left(\mu^h\right)$ exist and are continuous for small enough $|h|$.
(b) If we know only that the moments are continuous functions of h, then for fixed x, $\gamma_n\left(\mu^h\right)$ and $p_n\left(\mu^h, x\right)$ are continuous in h.
Proof
(a) This follows from the determinantal representation for orthogonal polynomials in terms of the moments: for a positive measure μ, write for $j \geq 0$,

$$c_j(\mu) = \int t^j d\mu$$

and for $m \geq 0$, let

$$D_m(\mu) = \det\left[c_{j+k}(\mu)\right]_{0 \leq j,k \leq m}. \tag{2.7}$$

It is known [19, Vol. 1, p. 15], [20, p. 27] that
(2.8)

$$p_n(\mu, x) = \frac{1}{\sqrt{D_{n-1}(\mu) D_n(\mu)}} \det \begin{bmatrix} c_0(\mu) & c_1(\mu) & c_2(\mu) & \cdots & c_n(\mu) \\ c_1(\mu) & c_2(\mu) & c_3(\mu) & \cdots & c_{n+1}(\mu) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1}(\mu) & c_n(\mu) & c_{n+1}(\mu) & \cdots & c_{2n-1}(\mu) \\ 1 & x & x^2 & \cdots & x^n \end{bmatrix}.$$

It follows from this representation and our hypothesis, that $p_n\left(\mu^h, x\right)$ is a differentiable function of h for small $|h|$; indeed we can expand each of the determinants as a product of finitely many terms involving the moments and powers of x. The one possible issue is that the determinants in the denominator approach 0, but this is impossible as $D_m\left(\mu^h\right) \to D_m\left(\mu^0\right) \neq 0$ as $h \to 0$. Since

$$\gamma_n(\mu) = \sqrt{\frac{D_{n-1}(\mu)}{D_n(\mu)}}, \tag{2.9}$$

it also follows that $\gamma_n\left(\mu^h\right)$ is differentiable with respect to h.
(b) is simpler. ■

Proof of Theorem 2.1
First note from Lemma 2.3 that $p_n(\mu^\varepsilon, x)$ is a continuously differentiable function of ε as the moments are linear functions of ε. Using the reproducing kernel property,

the definition (2.1) of μ^{ε}, and then orthogonality,

$$\begin{aligned} p_n(\mu,x) &= \int p_n(\mu,t) K_{n+1}(\mu^{\varepsilon},x,t)\, d\mu^{\varepsilon}(t) \\ &= p_n(\mu^{\varepsilon},x)\int p_n(\mu,t)\, p_n(\mu^{\varepsilon},t)\, d\mu(t) + 0 + \varepsilon \int p_n(\mu,t) K_{n+1}(\mu^{\varepsilon},x,t)\, d\omega(t). \end{aligned}$$

Differentiating with respect to ε gives

$$\begin{aligned} 0 &= \frac{\partial p_n(\mu^{\varepsilon},x)}{\partial\varepsilon}\int p_n(\mu,t)\, p_n(\mu^{\varepsilon},t)\, d\mu(t) + p_n(\mu^{\varepsilon},x)\int p_n(\mu,t)\,\frac{\partial p_n(\mu^{\varepsilon},t)}{\partial\varepsilon} d\mu(t) \\ &\quad + \int p_n(\mu,t) K_{n+1}(\mu^{\varepsilon},x,t)\, d\omega(t) + \varepsilon\int p_n(\mu,t)\,\frac{\partial K_{n+1}(\mu^{\varepsilon},x,t)}{\partial\varepsilon} d\omega(t). \end{aligned}$$

The previous lemma shows that all the derivatives exist and are continuous, so the interchanges are permissible. Letting $\varepsilon \to 0+$, we obtain

$$\begin{aligned} 0 &= \frac{\partial p_n(\mu^{\varepsilon},x)}{\partial\varepsilon}_{|\varepsilon=0}\int p_n^2(\mu,t)\, d\mu(t) + p_n(\mu,x)\int p_n(\mu,t)\,\frac{\partial p_n(\mu^{\varepsilon},t)}{\partial\varepsilon}_{|\varepsilon=0} d\mu(t) \\ &\quad + \int p_n(\mu,t) K_{n+1}(\mu,x,t)\, d\omega(t), \end{aligned}$$

so

(2.10)
$$\frac{\partial p_n(\mu^{\varepsilon},x)}{\partial\varepsilon}_{|\varepsilon=0} = -p_n(\mu,x)\int p_n(\mu,t)\,\frac{\partial p_n(\mu^{\varepsilon},t)}{\partial\varepsilon}_{|\varepsilon=0} d\mu(t) - \int p_n(\mu,t) K_{n+1}(\mu,x,t)\, d\omega(t).$$

Next, using orthogonality,

$$\begin{aligned} p_n(\mu^{\varepsilon},x) &= \int p_n(\mu^{\varepsilon},t) K_{n+1}(\mu,x,t)\, d\mu(t) \\ &= p_n(\mu,x)\int p_n(\mu^{\varepsilon},t)\, p_n(\mu,t)\, d\mu_{\varepsilon}(t) - \varepsilon\int p_n(\mu^{\varepsilon},t) K_{n+1}(\mu,x,t)\, d\omega(t). \end{aligned}$$

Differentiating with respect to ε gives

$$\begin{aligned} \frac{\partial p_n(\mu^{\varepsilon},x)}{\partial\varepsilon} &= p_n(\mu,x)\int \frac{\partial p_n(\mu^{\varepsilon},t)}{\partial\varepsilon} p_n(\mu,t)\, d\mu_{\varepsilon}(t) + p_n(\mu,x)\int p_n(\mu^{\varepsilon},t)\, p_n(\mu,t)\, d\omega(t) \\ &\quad - \int p_n(\mu^{\varepsilon},t) K_{n+1}(\mu,x,t)\, d\omega(t) - \varepsilon\int \frac{\partial p_n(\mu^{\varepsilon},t)}{\partial\varepsilon} K_{n+1}(\mu,x,t)\, d\omega(t). \end{aligned}$$

Letting $\varepsilon \to 0+$ gives

$$\begin{aligned} &\frac{\partial p_n(\mu^{\varepsilon},x)}{\partial\varepsilon}_{|\varepsilon=0} = p_n(\mu,x)\int \frac{\partial p_n(\mu^{\varepsilon},t)}{\partial\varepsilon}_{|\varepsilon=0} p_n(\mu,t)\, d\mu(t) \\ &+ p_n(\mu,x)\int p_n^2(\mu,t)\, d\omega(t) - \int p_n(\mu,t) K_{n+1}(\mu,x,t)\, d\omega(t). \end{aligned}$$

Adding this and (2.10) gives

$$2\frac{\partial p_n(\mu^{\varepsilon},x)}{\partial\varepsilon}_{|\varepsilon=0} = p_n(\mu,x)\int p_n^2(\mu,t)\, d\omega(t) - 2\int p_n(\mu,t) K_{n+1}(\mu,x,t)\, d\omega(t).$$

Then

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left\{\ln\frac{1}{p_n\left(\mu^\varepsilon,x\right)^2}\right\}_{|\varepsilon=0} &= -\frac{2}{p_n\left(\mu,x\right)}\frac{\partial p_n\left(\mu^\varepsilon,x\right)}{\partial\varepsilon}_{|\varepsilon=0}\\ &= \int\left[-p_n^2\left(\mu,t\right)+\frac{2K_{n+1}\left(\mu,x,t\right)}{p_n\left(\mu,x\right)}p_n\left(\mu,t\right)\right]d\omega\left(t\right)\\ &= \int\frac{p_n\left(\mu,t\right)}{p_n\left(\mu,x\right)}\left[K_{n+1}\left(\mu,x,t\right)+K_n\left(x,t\right)\right]d\omega\left(t\right).\end{aligned}$$

■

Proof of Theorem 2.2
First note from Lemma 2.3 that $p_n\left(\mu^\varepsilon,x\right)$ is a continuously differentiable function of ε as the moments are differentiable functions of ε. Using the reproducing kernel property, the definition (2.5) of μ^ε, and then orthogonality,

$$\begin{aligned}p_n\left(\mu,x\right) &= \int p_n\left(\mu,t\right)K_{n+1}\left(\mu^\varepsilon,x,t\right)d\mu^\varepsilon\left(t\right)\\ &= p_n\left(\mu^\varepsilon,x\right)\int p_n\left(\mu,t\right)p_n\left(\mu^\varepsilon,t\right)d\mu\left(t\right)\\ &\quad -\int p_n\left(\mu,t\right)K_{n+1}\left(\mu^\varepsilon,x,t\right)d\omega_0\left(t\right)+\int p_n\left(\mu,t\right)K_{n+1}\left(\mu^\varepsilon,x,t\right)d\omega_\varepsilon\left(t\right).\end{aligned}$$

Differentiating with respect to ε and recalling the definition (2.4) of ω_ε gives

$$\begin{aligned}0 &= \frac{\partial p_n\left(\mu^\varepsilon,x\right)}{\partial\varepsilon}\int p_n\left(\mu,t\right)p_n\left(\mu^\varepsilon,t\right)d\mu\left(t\right)+p_n\left(\mu^\varepsilon,x\right)\int p_n\left(\mu,t\right)\frac{\partial p_n\left(\mu^\varepsilon,t\right)}{\partial\varepsilon}d\mu\left(t\right)\\ &\quad -\int p_n\left(\mu,t\right)\frac{\partial K_{n+1}\left(\mu^\varepsilon,x,t\right)}{\partial\varepsilon}d\omega_0\left(t\right)+\int p_n\left(\mu,t\right)\frac{\partial K_{n+1}\left(\mu^\varepsilon,x,t\right)}{\partial\varepsilon}d\omega_\varepsilon\left(t\right)\\ &\quad +\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n\left(\mu,t\right)K_{n+1}\left(\mu^\varepsilon,x,t\right)\right\}_{|t=\xi_j+\sigma\left(\xi_j\right)\varepsilon}\sigma\left(\xi_j\right)\end{aligned}$$

so letting $\varepsilon\to 0+$, and using continuity of the derivatives,

$$\begin{aligned}\frac{\partial p_n\left(\mu^\varepsilon,x\right)}{\partial\varepsilon}_{|\varepsilon=0} &= -p_n\left(\mu,x\right)\int p_n\left(\mu,t\right)\frac{\partial p_n\left(\mu^\varepsilon,t\right)}{\partial\varepsilon}_{|\varepsilon=0}d\mu\left(t\right)\\ &\quad -\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n\left(\mu,t\right)K_{n+1}\left(\mu,x,t\right)\right\}_{|t=\xi_j}\sigma\left(\xi_j\right).\end{aligned}\tag{2.11}$$

Next,

$$\begin{aligned}p_n\left(\mu^\varepsilon,x\right) &= \int p_n\left(\mu^\varepsilon,t\right)K_{n+1}\left(\mu,x,t\right)d\mu\left(t\right)\\ &= p_n\left(\mu,x\right)\int p_n\left(\mu^\varepsilon,t\right)p_n\left(\mu,t\right)d\mu_\varepsilon\left(t\right)\\ &\quad +\int p_n\left(\mu^\varepsilon,t\right)K_{n+1}\left(\mu,x,t\right)d\omega_0\left(t\right)-\int p_n\left(\mu^\varepsilon,t\right)K_{n+1}\left(\mu,x,t\right)d\omega_\varepsilon\left(t\right).\end{aligned}$$

Differentiating with respect to ε and recalling the definition (2.4) of ω_ε gives

$$\begin{aligned}
&\frac{\partial p_n(\mu^\varepsilon,x)}{\partial\varepsilon}\\
=\;& p_n(\mu,x)\int\frac{\partial p_n(\mu^\varepsilon,t)}{\partial\varepsilon}p_n(\mu,t)\,d\mu_\varepsilon(t)\\
&+p_n(\mu,x)\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n(\mu^\varepsilon,t)\,p_n(\mu,t)\right\}_{|t=\xi_j+\sigma(\xi_j)\varepsilon}\sigma\left(\xi_j\right)+\int\frac{\partial p_n(\mu^\varepsilon,t)}{\partial\varepsilon}K_{n+1}(\mu,x,t)\,d\omega_0(t)\\
&-\int\frac{\partial p_n(\mu^\varepsilon,t)}{\partial\varepsilon}K_{n+1}(\mu,x,t)\,d\omega_\varepsilon(t)-\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n(\mu^\varepsilon,t)\,K_{n+1}(\mu,x,t)\right\}_{|t=\xi_j+\sigma(\xi_j)\varepsilon}\sigma\left(\xi_j\right)
\end{aligned}$$

so letting $\varepsilon\to 0+$, and using the continuity of the derivatives in ε,

$$\begin{aligned}
\frac{\partial p_n(\mu^\varepsilon,x)}{\partial\varepsilon}_{|\varepsilon=0} &= p_n(\mu,x)\int\frac{\partial p_n(\mu^\varepsilon,t)}{\partial\varepsilon}_{|\varepsilon=0}p_n(\mu,t)\,d\mu(t)+p_n(\mu,x)\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n^2(\mu,t)\right\}_{|t=\xi_j}\sigma\left(\xi_j\right)\\
&\quad-\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n(\mu,t)\,K_{n+1}(\mu,x,t)\right\}_{|t=\xi_j}\sigma\left(\xi_j\right).
\end{aligned}$$

Adding this and (2.11) gives

$$\begin{aligned}
&2\frac{\partial p_n(\mu^\varepsilon,x)}{\partial\varepsilon}_{|\varepsilon=0}\\
=\;& p_n(\mu,x)\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n^2(\mu,t)\right\}_{|t=\xi_j}\sigma\left(\xi_j\right)-2\sum_{j=1}^{L}\beta_j\frac{\partial}{\partial t}\left\{p_n(\mu,t)\,K_{n+1}(\mu,x,t)\right\}_{|t=\xi_j}\sigma\left(\xi_j\right)\\
=\;& \int\frac{\partial}{\partial t}\left\{p_n(\mu,x)\,p_n^2(\mu,t)-2p_n(\mu,t)\,K_{n+1}(\mu,x,t)\right\}\sigma(t)\,d\omega_0(t)
\end{aligned}$$

so

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}\left\{\ln\frac{1}{p_n(\mu^\varepsilon,x)^2}\right\} &= \int\frac{\partial}{\partial t}\left[-p_n^2(\mu,t)+\frac{2K_{n+1}(\mu,x,t)}{p_n(\mu,x)}p_n(\mu,t)\right]\sigma(t)\,d\omega_0(t)\\
&= \int R_n'(\mu,t)\,\sigma(t)\,d\omega_0(t).
\end{aligned}$$

■

3. Proof of Theorem 1.2

Proof of Theorem 1.2(a)
We note that since $\mu\geq\nu$,

$$p_n\left(\mu^2,x\right)\leq K_{n+1}(\mu,x,x)\leq K_{n+1}(\nu,x,x).$$

Thus

$$\phi_n(I,\Lambda,x)=\sup\left\{|p_n(\mu,x)|:\mu\in\mathcal{M}(I,\Lambda)\right\}\leq K_{n+1}(\nu,x,x).$$

Next, choose a sequence $\left\{\mu^{(m)}\right\}_{m\geq1}$ of measures in $\mathcal{M}(I,\Lambda)$ such that

$$\lim_{m\to\infty}\left|p_n\left(\mu^{(m)},x\right)\right|=\phi_n(I,\Lambda,x).$$

Since $\left\{\mu^{(m)}\right\}$ are measures supported on I and all have total mass $\leq\Lambda+\nu([-1,1])$, we can choose a subsequence that converges weakly to some measure μ_I supported on I and of total mass $\leq \Lambda + \nu([-1,1])$. Since each $\mu^{(m)} = \nu$ in $(-1,1)$, the same is true of μ_I. So $\mu_I \in \mathcal{M}(I,\Lambda)$. For notational simplicity, we assume that the full sequence $\{\mu^{(m)}\}$ converges weakly to μ_I. Then for each fixed $j \geq 0$,

$$\lim_{m\to\infty} \int t^j d\mu^{(m)}(t) = \int t^j d\mu_I(t).$$

From Lemma 2.3(b),

$$|p_n(\mu_I, x)| = \lim_{m\to\infty} \left| p_n\left(\mu^{(m)}, x\right)\right| = \phi_n(I, \Lambda, x).$$

■

Proof of Theorem 1.2(b)

Given $\varepsilon \in [0,1]$ and a nonnegative measure ω supported on $I\backslash(-1,1)$ with total mass $\leq \Lambda$, and the extremal measure $\mu = \mu_I = \nu + \rho_I$ above, we shall consider

$$\begin{aligned} \mu_I^\varepsilon &= \mu_I + \varepsilon(\omega - \rho_I) \\ &= \nu + (1-\varepsilon)\rho_I + \varepsilon\omega. \end{aligned} \tag{3.1}$$

This is a positive measure, and belongs to $\mathcal{M}(I,\Lambda)$. Observe that the function

$$f(\varepsilon) = \ln \frac{1}{p_n(\mu_I^\varepsilon, x)^2}, \varepsilon \in [0,1],$$

has its minimum in $[0,1]$ at $\varepsilon = 0$. It is also a differentiable function of ε, at least for sufficiently small ε, since the moments

$$\int t^j d\mu^\varepsilon(t) = \int t^j d\nu(t) + \varepsilon \int t^j d(\omega - \rho_I)(t)$$

are linear functions of ε, so we may apply Lemma 2.3. Then $f'(0+) \geq 0$. By Theorem 2.1, with ω replaced there by $\omega - \rho_I$,

$$0 \leq f'(0+) = \frac{\partial}{\partial\varepsilon}\left(\ln \frac{1}{p_n(\mu_I^\varepsilon, x)^2}\right)_{|\varepsilon=0} = \int R_n(\mu_I, t)\, d(\omega - \rho_I)(t).$$

■

Proof of Theorem 1.2(c)

Suppose t_0 is in the support of ρ_I and $\delta > 0$. Then we set $\omega = \rho_I$ outside $[t_0-\delta, t_0+\delta]$ and $\omega = 0$ in $[t_0-\delta, t_0+\delta]$. This is still valid even if $[t_0-\delta, t_0+\delta]$ is not contained in $I\backslash(-1,1)$. From (1.4),

$$-\int_{t_0-\delta}^{t_0+\delta} R_n(\mu_I, t)\, d\rho_I(t) \geq 0 \Rightarrow \inf_{[t_0-\delta, t_0+\delta]} R_n(\mu_I, t) \leq 0,$$

as $\rho_I([t_0-\delta, t_0+\delta]) > 0$. Letting $\delta \to 0+$ gives

$$R_n(\mu_I, t_0) \leq 0.$$

Next, let $\delta > 0$, $t_1 \in I\backslash((-1,1) \cup [t_0-\delta, t_0+\delta])$, and $\Delta = \rho_I([t_0-\delta, t_0+\delta])$, and

$$\omega = \rho_I - \rho_{I|[t_0-\delta,t_0+\delta]} + \Delta\delta_{t_1}.$$

Then ω is a nonnegative measure supported in $I\backslash(-1,1)$, and still $\omega(I) \leq \Lambda$. Also then from (1.4),

$$\begin{aligned} & \int_{t_0-\delta}^{t_0+\delta} R_n(\mu_I,t)\,d(-\rho_I)(t) + \Delta R_n(\mu_I,t_1) \geq 0 \\ \Rightarrow \quad & -\Delta \inf_{[t_0-\delta,t_0+\delta]} R_n(\mu_I,t) + \Delta R_n(\mu_I,t_1) \geq 0 \\ \Rightarrow \quad & R_n(\mu_I,t_1) \geq \inf_{[t_0-\delta,t_0+\delta]} R_n(\mu_I,t). \end{aligned}$$

Letting $\delta \to 0+$, we obtain for any t_0 in the support of ρ_I, and any $t_1 \in I\backslash(-1,1)$,

$$R_n(\mu_I,t_1) \geq R_n(\mu_I,t_0),$$

which implies (1.5). ■

In the following lemma, the zeros of $p_n(\mu_I,\cdot)$ are denoted by

$$(c<)\, x_{nn} < x_{n-1,n} < x_{n-2,n} < ... < x_{1n}\, (<d).$$

These are of course not in general the zeros of $p_n(\nu,x)$. For the given x, let

$$U_n(t) = (t-x)\left[K_{n+1}(\mu_I,x,t) + K_n(\mu_I,x,t)\right].$$

Lemma 3.1
(a) U_n has simple zeros $\{y_{jn}\}_{0\leq j\leq n}$, where $y_{jn} \in (x_{j+1,n}, x_{jn})$ if $1 \leq j \leq n-1$, while $y_{0n} > x_{1n}$ and $y_{nn} < x_{nn}$.
(b) R_n defined by (1.3) has $2n$ simple zeros and at most 2 lie outside I^o. There are n zeros at those of $p_n(\mu_I,\cdot)$ and another n at the zeros of U_n, other than x. At most 1 zero can lie in $[d,\infty)$ and at most one in $(-\infty,c]$.
(c) R_n' has $2n-1$ simple zeros. For $1 \leq j \leq n-1$, it has one zero in $(x_{j+1,n}, y_{jn})$ and another in (y_{jn}, x_{jn}) as long as $x \notin [x_{j+1,n}, x_{jn}]$. If $x \in [x_{j+1,n}, x_{jn}]$, then this interval contains one zero of R_n'. In addition there is one zero in (x_{1n}, y_{0n}), provided $x \neq y_{0n}$ and one zero in (y_{nn}, x_{nn}), provided $x \neq y_{nn}$.
(d) If $x \neq y_{0n}$, then R_n is positive and increasing in (y_{0n},∞), so there is a local minimum in (x_{1n}, y_{0n}). If $x = y_{0n}$, then R_n is positive and increasing in (x_{1n},∞).

Proof
(a) First note that from (1.2), $p_n(\mu_I,x) \neq 0$. We use a result of Shohat [18, p. 472, Theorem VII], which is more clearly stated as Theorem 4 in [6, p. 161, Theorem 4]. To apply this, note that

$$\begin{aligned} & U_n(t) / \left(\frac{\gamma_n}{\gamma_{n+1}} p_n(\mu_I,x)\right) \\ = \; & p_{n+1}(\mu_I,t) + \left(\frac{-\frac{\gamma_n}{\gamma_{n+1}} p_{n+1}(\mu_I,x) + \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(\mu_I,x)}{\frac{\gamma_n}{\gamma_{n+1}} p_n(\mu_I,x)}\right) p_n(\mu_I,t) - \left(\frac{\frac{\gamma_{n-1}}{\gamma_n}}{\frac{\gamma_n}{\gamma_{n+1}}}\right) p_{n-1}(\mu_I,t) \\ = \; & p_{n+1}(\mu_I,t) + C_n p_n(\mu_I,t) + D_n p_{n-1}(\mu_I,t), \text{ say.} \end{aligned}$$

As $D_n < 0$, $C_n \in \mathbb{R}$, then the result of Shohat shows that $p_{n+1}(\mu_I,t) + C_n p_n(\mu_I,t) + D_n p_{n-1}(\mu_I,t)$ has all real simple zeros and at most two lie outside (c,d).

Next, Theorem 5 of [6, p. 161], shows that the interval $(x_{nn}(\mu_I), x_{1n}(\mu_I)) \subset (c,d)$ contains $n-1$ zeros of U_n and there is one zero to the left of $x_{nn}(\mu_I)$ and one to the right of $x_{1n}(\mu_I)$.

(b) From (a), $K_{n+1}(\mu_I, x, t) + K_n(\mu_I, x, t)$ has simple zeros. If we can show that $p_n(\mu_I, t)$ and $K_{n+1}(\mu_I, x, t) + K_n(\mu_I, x, t)$ don't have common zeros, then we have proved that R_n has simple zeros, since $p_n(\mu_I, \cdot)$ has n simple zeros in I^o.

So suppose that $p_n(\mu_I, t)$ and $K_{n+1}(\mu_I, x, t) + K_n(\mu_I, x, t)$ have a common zero y. Firstly this cannot be x, since $K_{n+1}(\mu_I, x, x) + K_n(\mu_I, x, x) > 0$. So $y \neq x$. Then

$$\begin{aligned} 0 &= K_{n+1}(\mu_I, x, y) + K_n(\mu_I, x, y) \\ &= 2K_n(\mu_I, x, y) = 2\frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(\mu_I, x)\, p_{n-1}(\mu_I, y)}{x - y}. \end{aligned}$$

Since $p_n(\mu_I, x) \neq 0$, so $p_{n-1}(\mu_I, y) = p_n(\mu_I, y) = 0$, which is impossible. Thus R_n has $2n$ simple zeros.

It remains to show that R_n can have at most one zero in $[c, \infty)$. We know that any such zero must be a zero of $K_{n+1}(\mu_I, x, t) + K_n(\mu_I, x, t)$ as $p_n(\mu_I, \cdot) > 0$ in $[c, \infty)$. Theorem 5 of [6, p. 161], shows that the interval $(x_{nn}(\mu_I), x_{1n}(\mu_I)) \subset (c, d)$ contains $n - 1$ zeros of $U_n(t)$ and there is one zero to the left of $x_{nn}(\mu_I)$ and one to the right of $x_{1n}(\mu_I)$. Since $x_{1n}(\mu) < c$, it follows that there is at most one zero of $K_{n+1}(\mu_I, x, t) + K_n(\mu_I, x, t)$ in $[c, \infty)$.
(c) As R_n has $2n$ simple zeros, so R_n' has $2n - 1$ simple zeros, that interlace the zeros of R_n. The remaining statements follow from (b).
(d) We have as $t \to \infty$,

$$R_n(\mu_I, t) = p_n^2(\mu_I, t)(1 + o(1)),$$

so $R_n(\mu_I, t) > 0$ for $t > y_{0n}$. Then if $x \neq y_{0n}$, the zero of R_n' in (x_{1n}, y_{0n}) must be a local minimum of R_n. If $x = y_{0n}$, then the largest zero of $R_n(\mu_I, \cdot)$ is x_{1n} and R_n is increasing and positive in (x_{1n}, ∞). ■

Proof of Theorem 1.2(d)
We begin with three observations:
(I) $R_n(\mu_I, \cdot)$ has exactly n local minima in $(-\infty, \infty)$.
Proof
This follows from the fact that R_n has $2n$ simple zeros and $R_n(\mu_I, t)$ is positive for large enough $|t|$.

(II) If at least one of $-1, 1$ is a support point of ρ_I, then $R_n(\mu_I, t)$ has a local minimum in $[-1, 1]$ with the derivative vanishing at that point.
Proof
We know that $R_n(\pm 1) \geq \eta = \inf\{R_n(\mu_I, t) : t \in I \backslash (-1, 1)\}$. If for example 1 is a support point, then we know $R_n(\mu_I, 1) = \eta$. If $R_n'(\mu_I, 1) = 0$, then 1 is a local minimum. Now suppose the derivative is not zero. Now necessarily $R_n(\mu_I, \cdot) > \eta$ in $(1, 1 + \varepsilon)$ for some $\varepsilon > 0$ and hence also $R_n(\mu_I, \cdot) < \eta$ in $(1 - \varepsilon, 1)$ for some $\varepsilon > 0$. As $R_n(\mu_I, -1) \geq \eta$, there is then a local minimum in $(-1, 1)$.

(III) If d is a support point of ρ_I, then R_n has a local minimum in $[d, \infty)$. Similarly for c.
Proof
Now $R_n(\mu_I, d) = \eta,$. If $R_n'(\mu_I, d) = 0$, then d must be a local minimum. Suppose the derivative is not 0. Then $R_n(\mu_I, \cdot) > \eta$ in $(d - \varepsilon, d)$ for some $\varepsilon > 0$. Then also

$R_n(\mu_I, \cdot) < \eta$ in $(d, d+\varepsilon)$ for some $\varepsilon > 0$. But $R_n(\mu_I, t) \to \infty$ as $t \to \infty$, so there must be a local minimum in (d, ∞).

We can now proceed to the proof proper. Let ℓ be the number of support points amongst $\{c, d\}$. Let $k = 1$ if at least one of $-1, 1$ is a support point and 0 otherwise. Then from our observations above, there are $\leq \ell + k$ local minima of $R_n(\mu_I, \cdot)$ in $[-1, 1] \cup (-\infty, c] \cup [d, \infty)$. Then there are at most $(n - \ell - k)$ local minima of $R_n(\mu_I)$ in $(c, d) \setminus (-1, 1)$ and hence at most $n - \ell - k$ support points there. This gives altogether at most

$$(n - \ell - k) + \ell + k + 1 = n + 1$$

support points. The extra 1 arises if both $-1, 1$ are support points. ■

Proof of Theorem 1.2(e)
Suppose ρ_I has total mass $T < \Lambda$. Then pick any $t_0 \in I \setminus (-1, 1)$ and let

$$\omega = \rho_I + (\Lambda - T)\,\delta_{t_0}.$$

Then from (4.4),

$$0 \leq \int R_n(\mu_I, t)\, d(\omega - \rho_I)(t) = (\Lambda - T)\, R_n(\mu_I, t_0).$$

So $R_n \geq 0$ in $I \setminus (-1, 1)$. In particular, for $t_0 \in$supp$[\rho_I]$, we have

$$R_n(\mu_I, t_0) \geq 0.$$

But we already know that $R_n(\mu_I, t_0) \leq 0$. So $R_n = 0$ in supp$[\rho_I]$. But then from (c),

$$\inf \{R_n(\mu_I, s) : s \in I\} = 0$$

so that $R_n \geq 0$ in $I \setminus (-1, 1)$. As R_n changes sign at its zeros, it follows that R_n cannot have zeros in $I^o \setminus [-1, 1]$ and $R_n > 0$ there. Hence the support of ρ_I does not intersect $I^o \setminus [-1, 1]$. So

$$\operatorname{supp}[\rho_I] \subset \{-1, 1, c, d\}.$$

Next suppose that d lies in the support. Then $R_n(\mu_I, d) = 0$ and by Lemma 3.1(b), there are no more zeros in (d, ∞), so that R_n is of one sign there. But since $R_n \geq 0$ to the left of d and R_n changes sign at d, we have $R_n < 0$ in (d, ∞). However, as $t \to \infty$, it follows that

$$R_n(\mu_I, t) = p_n^2(\mu_I, t)(1 + o(1))$$

so we have a contradiction. Thus d is not in the support and similarly, c is not in the support. ■

4. Proof of Theorem 1.3

Recall that

$$\psi_n(y) = \ln \frac{1}{\phi_n([c, y], \Lambda, x)^2}, y > 1.$$

Let

$$\phi_n(y) = \phi_n([c, y], \Lambda, x).$$

We begin by showing that ϕ_n is continuous:

Lemma 4.1

(a) Let $y>1, I=[c,y]$, and $\{h_m\}_{m\geq 1}$ be a sequence of numbers with limit 0. For $m\geq 1$, let $I_m=[c,y+h_m]$ and *μ_{I_m} be an extremal measure for I_m, and assume that μ_{I_m} converges weakly to $\hat{\mu}_I$ as $m\to\infty$. Then $\hat{\mu}_I$ is an extremal measure for $I=[c,y]$ and*

$$\phi_n(y)=\lim_{m\to\infty}\phi_n(y+h_m). \tag{4.1}$$

(b) ϕ_n is continuous in $[1,\infty)$.

Proof

(a) It suffices to consider sequences in which all h_m are of one sign.

Case I: All $h_m\geq 0$.

Recall that ϕ_n is an increasing function. As $\hat{\mu}_I\in\mathcal{M}(I,\Lambda)$,

$$\begin{aligned}|p_n(\hat{\mu}_I,x)| &\leq \phi_n(y)\leq\liminf_{m\to\infty}\phi_n(y+h_m)\\ &\leq \limsup_{m\to\infty}\phi_n(y+h_m)=\lim_{m\to\infty}\left|p_n\left(\mu_{I_m},x\right)\right|=|p_n(\hat{\mu}_I,x)|\end{aligned}$$

as in Lemma 2.3(b), since μ_{I_m} converges weakly to $\hat{\mu}_I$. Then we have (4.1) and that $\hat{\mu}_I$ is extremal for I.

Case II: All $h_m<0$

Let μ_I be an extremal measure for $[c,y]$, so that it has the form $\nu+\rho_I$, where for some $L\leq n+1$,

$$\rho_I=\sum_{j=1}^{L}\mu_I\left(\{\xi_j\}\right)\delta_{\xi_j},$$

where all $\xi_j\in I\backslash(-1,1)$. We assume that ξ_1 is smallest and ξ_L is largest. Then for large enough m, and $1\leq j\leq L-1$, $\xi_j\in[c,y+h_m]$. Define

$$\rho_{I,m}=\rho_I+\rho_I(\{\xi_L\})\left[\delta_{\xi_L+h_m}-\delta_{\xi_L}\right]$$

so that (recall $h_m<0$)

$$\mu^{(m)}=\nu+\rho_{I,m}\in\mathcal{M}(I_m,\Lambda).$$

Also as $m\to\infty$, $\mu^{(m)}\to\mu_I$ weakly, so

$$\lim_{m\to\infty}\left|p_n\left(\mu^{(m)},x\right)\right|=|p_n(\mu_I,x)|=\phi_n(y)$$

so

$$\begin{aligned}\liminf_{n\to\infty}\phi_n(d+h_m) &\geq \liminf_{n\to\infty}\left|p_n\left(\mu^{(m)},x\right)\right|\\ &= \phi_n(y)\geq\limsup_{n\to\infty}\phi_n(y+h_m).\end{aligned}$$

Then we have (4.1). Also, it then follows that for the given extremal measures $\left\{\mu_{I_m}\right\}$ converging weakly to $\hat{\mu}_I$, necessarily,

$$|p_n(\hat{\mu}_I,x)|=\lim_{m\to\infty}\left|p_n\left(\mu_{I_m},x\right)\right|=\lim_{m\to\infty}\phi_n(y+h_m)=\phi_n(y)$$

so $\hat{\mu}_I$ is extremal for I.

(b) The continuity of ϕ_n in $(1,\infty)$ follows directly from (a), while Case I there shows that ϕ_n is right continuous at 1. ■

Our next step is to show that ψ_n' is given a.e. by (1.7):

Lemma 4.2
ψ_n' exists a.e. in $(1,\infty)$. At every point where the derivative exists, it is given by

$$\psi_n'(y) = \mu_{[c,y]}(\{y\})\, R_n'\left(\mu_{[c,y]}, y\right),$$

where $\mu_{[c,y]}$ is any extremal measure. In particular, the right-hand side is independent of the particular $\mu_{[c,y]}$.

Proof
Let μ_I be an extremal measure for the given y, so that it has a representation

$$\mu_{[c,y]} = \nu + \sum_{j=1}^{L} \mu_{[c,y]}\left(\{\xi_j\}\right)\delta_{\xi_j} \tag{4.2}$$

where all $\{\xi_j\}$ lie in $I\backslash(-1,1)$, while $L \le n+1$, and the sum of the $\mu_I\left(\{\xi_j\}\right)$ is at most Λ. Let

$$\omega_h = \mu_{[c,y]}(\{y\})\,\delta_{y+h}, \tag{4.3}$$

so that ω_h consists of the point mass of $\mu_{[c,y]}$ at y (if any), shifted to $y+h$. Then for $h>0$,

$$\mu^{(h)} = \mu_{[c,y]} + \omega_h - \omega_0 \in \mathcal{M}\left([c, y+h], \Lambda\right). \tag{4.4}$$

Then for small $h>0$, as $\mu_{[c,y]}$ is extremal for $[c,y]$,

$$\begin{aligned}
&\psi_n(y+h) - \psi_n(y)\\
\le\ & \left[\ln \frac{1}{p_n\left(\mu^{(h)}, x\right)^2} - \ln \frac{1}{p_n\left(\mu_I, x\right)^2}\right]\\
=\ & \left(\int R_n'\left(\mu_{[c,y]}, t\right) sign(t)\, d\omega_0(t)\right) h + o(h)\\
=\ & \left(\mu_{[c,y]}(\{y\})\, R_n'\left(\mu_{[c,y]}, y\right)\right) h + o(h),
\end{aligned} \tag{4.5}$$

by Theorem 2.2 and our choice of measure. So

$$\begin{aligned}
\limsup\nolimits_{h\to 0+}\ & [\psi_n(y+h) - \psi_n(y)]/h\\
\le\ & \mu_{[c,y]}(\{y\})\, R_n'\left(\mu_{[c,y]}, y\right).
\end{aligned} \tag{4.6}$$

Next, let ω_h be defined by (4.3) and for small negative h, define $\mu^{(h)}$ by (4.4). As before, we have (4.5), but since $h<0$, so

$$\begin{aligned}
&\liminf_{h\to 0-} [\psi_n(y+h) - \psi_n(y)]/h\\
\ge\ & \int R_n'\left(\mu_{[c,y]}, t\right) sign(t)\, d\omega_0(t)\\
=\ & \mu_{[c,y]}(\{y\})\, R_n'\left(\mu_{[c,y]}, y\right).
\end{aligned}$$

In summary, we have shown that for $y>1$, and any extremal measure $\mu_{[c,y]}$,

$$\begin{aligned}
&\liminf_{h\to 0-} [\psi_n(y+h) - \psi_n(y)]/h\\
\ge\ & \mu_{[c,y]}(\{y\})\, R_n'\left(\mu_{[c,y]}, y\right)\\
\ge\ & \limsup_{h\to 0+} [\psi_n(y+h) - \psi_n(y)]/h.
\end{aligned} \tag{4.7}$$

Since ψ_n is decreasing, it is differentiable a.e. and at such points y, this last relation shows that

$$\psi_n'(y) = \mu_{[c,y]}(\{y\}) R_n'\left(\mu_{[c,y]}, y\right),$$

for any extremal measure $\mu_{[c,y]}$. Of course the last right-hand side is independent of the particular extremal measure. ■

Next we prove uniform boundedness of R_n. In the sequel, we let

$$K_n^{(1,1)}(\nu, t, t) = \sum_{j=0}^{n-1} p_j'(\nu, t)^2.$$

Lemma 4.3
Fix $n \geq 1$, $x \in \mathbb{R}$, and $\mu \in \mathcal{M}(I, \Lambda)$. Assume that

$$|p_n(\mu, x)| \geq r > 0.$$

Let

$$R_n(\mu, t) = \frac{p_n(\mu, t)}{p_n(\mu, x)} \left[K_{n+1}(\mu, x, t) + K_{n+1}(\mu, x, t)\right].$$

Then
(a)

$$|R_n(\mu, t)| \leq \frac{2}{r} K_{n+1}(\nu, x, x)^{1/2} K_{n+1}(\nu, t, t). \tag{4.8}$$

(b)

$$|R_n'(\mu, t)| \leq \frac{4}{r}\left[K_{n+1}(\nu, x, x) K_{n+1}(\nu, t, t) K_{n+1}^{(1,1)}(\nu, t, t)\right]^{1/2}. \tag{4.9}$$

In particular these bounds do not depend on μ or Λ.
Proof
(a) Firstly, as $\mu \geq \nu$,

$$p_n^2(\mu, t) \leq K_{n+1}(\mu, t, t) \leq K_{n+1}(\nu, t, t),$$

while by Cauchy-Schwarz,

$$(K_{n+1}(\mu, x, t))^2 \leq K_{n+1}(\nu, t, t) K_{n+1}(\nu, x, x).$$

Then

$$|R_n(\mu, t)| \leq \frac{K_{n+1}(\nu, t, t)^{1/2}}{r} 2\left(K_{n+1}(\nu, t, t) K_{n+1}(\nu, x, x)\right)^{1/2}.$$

(b) We use the well known extremal property of reproducing kernels and easy consequence of Cauchy-Schwarz: for any polynomial P of degree $\leq n-1$, and any real t

$$P'(t)^2 \leq K_n^{(1,1)}(\nu, t, t) \int P^2 \, d\nu.$$

Then

$$\begin{aligned}\left(\frac{\partial}{\partial t}K_n(\mu,x,t)\right)^2 &\leq K_n^{(1,1)}(\nu,t,t)\int K_n(\mu,x,t)^2\,d\nu(t)\\ &\leq K_n^{(1,1)}(\nu,t,t)\int K_n(\mu,x,t)^2\,d\mu(t)\\ &= K_n^{(1,1)}(\nu,t,t)\,K_n(\mu,x,x)\\ &\leq K_n^{(1,1)}(\nu,t,t)\,K_n(\nu,x,x)\end{aligned}$$

as $\mu \geq \nu$ and the reproducing kernel K_n is decreasing in the measure. Then

$$\begin{aligned}|R_n'(\mu,t)| &\leq \left|\frac{p_n'(\mu,t)}{p_n(\mu,x)}\right|\left|K_{n+1}(\mu,x,t)+K_{n+1}(\mu,x,t)\right|\\ &\quad+\left|\frac{p_n(\mu,t)}{p_n(\mu,x)}\right|\left|\frac{\partial}{\partial t}\left(K_{n+1}(\mu,x,t)+K_{n+1}(\mu,x,t)\right)\right|\\ &\leq \frac{1}{r}K_{n+1}^{(1,1)}(\nu,t,t)^{1/2}\,2\left(K_{n+1}(\nu,t,t)\,K_{n+1}(\nu,x,x)\right)^{1/2}\\ &\quad+\frac{1}{r}K_{n+1}(\nu,t,t)^{1/2}\,2\left(K_{n+1}^{(1,1)}(\nu,t,t)\,K_{n+1}(\nu,x,x)\right)^{1/2}.\end{aligned}$$

Then (4.9) follows. ■

To prove that ψ_n is absolutely continuous, is more complicated. We first need

Lemma 4.4
Let $y > 1$, $I = [c, y]$, and for $|h| \leq 1$,

$$\omega_h = \sum_{j=1}^{N} \beta_j \delta_{\xi_j + h\sigma(\xi_j)} \tag{4.10}$$

where $N \geq 1$, all $\{\xi_j\}_{j=1}^N$ are distinct and lie in $[c, y]\setminus(-1,1)$, all $\sigma(\sigma_j) = \pm 1$, and all β_j are real with

$$\sum_{j=1}^{N} |\beta_j| \leq \Lambda. \tag{4.11}$$

Fix $n \geq 1$, $x \in (-1,1)$, and $\mu \in \mathcal{M}(I,\Lambda)$. For $|h| \leq 1$ let

$$\mu^{(h)} = \mu - \omega_0 + \omega_h. \tag{4.12}$$

(a) Then there exist numbers F, $C > 0$, $h_0 > 0$ such that for $|h| \leq h_0$, subject to $\mu^{(h)}$ being a nonnegative measure,

$$\left|p_n\left(\mu^{(h)}, x\right) - p_n(\mu, x) - Fh\right| \leq Ch^2. \tag{4.13}$$

Here the number F is continuous in the parameters $\{\beta_j\}, \{\xi_j\}$. The constant C depends only on y, Λ, ν, n and not on the particular $\mu, \{\beta_j\}, \{\xi_j\}, N, \sigma$.
(b) Fix $r > 0$, and assume that

$$|p_n(\mu, x)| \geq r. \tag{4.14}$$

Then there exists h_1 such that for $|h| \leq h_1$ for which $\mu^{(h)}$ is a nonnegative measure,

$$\left| \ln \frac{1}{p_n\left(\mu^{(h)}, x\right)^2} - \ln \frac{1}{p_n\left(\mu, x\right)^2} - \left(\int R_n'\left(\mu, t\right) \sigma\left(t\right) d\omega_0\left(t\right) \right) h \right| \leq C_1 h^2. \tag{4.15}$$

The constants h_1 and C_1 depend on y, ν, n, Λ and the lower bound r but not on the particular $\mu, \left\{\beta_j\right\}, \left\{\xi_j\right\}, N, \sigma$.

Proof

(a) Observe that for each fixed k,

$$\begin{aligned} & \int t^k d\mu^{(h)}\left(t\right) - \int t^k d\mu\left(t\right) \\ = & \left(\sum_{j=1}^N \beta_j k \xi_j^{k-1} \sigma\left(\xi_j\right) \right) h + O\left(h^2\right), \end{aligned}$$

where the sum in the last line is uniformly continuous in $\left\{\beta_j\right\}$ and $\left\{\xi_j\right\}$ when the $\beta_j \geq 0$ and satisfy (4.11), while the $\left\{\xi_j\right\}$ lie in $[c, d] \setminus (-1, 1)$. Moreover the constant in the order term depends only on Λ, y, L, k. Then if D_m is the determinant defined by (2.7), we see that also

$$D_m\left(\mu^{(h)}\right) - D_m\left(\mu\right) = E_m h + O\left(h^2\right),$$

where E_m is uniformly continuous in $\left\{\beta_j\right\}$ and $\left\{\xi_j\right\}$ when the $\beta_j \geq 0$ and satisfy (4.11), while the $\left\{\xi_j\right\}$ lie in $[c, y] \setminus (-1, 1)$. Moreover the constant in the order term depends only on Λ, c, y, m. Then a similar assertion is true for the numerator in the determinant representation (2.8) of $p_n\left(\mu, x\right)$. Finally, we know that as each $\mu^{(h)} \geq \nu$, each $D_m\left(\mu^{(h)}\right)$ is bounded below by a constant depending only on $D_m\left(\nu\right)$, at least for small enough $|h|$. Then the result follows from the representation (2.8).

(b) From (a) and the lower bound for $p_n\left(\mu, x\right)$,

$$\left| \ln \frac{1}{p_n\left(\mu^{(h)}, x\right)^2} - \ln \frac{1}{p_n\left(\mu, x\right)^2} - Gh \right| \leq C_1 h^2,$$

where C_1 and h_1 satisfy the uniformity conclusions in (b). Also then from Theorem 2.2,

$$G = \frac{\partial}{\partial h} \left\{ \ln \frac{1}{p_n\left(\mu^{(h)}, x\right)^2} \right\}_{|h=0} = \int R_n'\left(\mu, t\right) \sigma\left(t\right) d\omega_0\left(t\right).$$

■

We next show that the left and right-hand derivatives exist at each $y > 1$.

Lemma 4.5

(a) For $y \in (1, \infty)$ there exists

$$D_+ \psi_n\left(y\right) = \lim_{h \to 0+} \frac{\psi_n\left(y+h\right) - \psi_n\left(y\right)}{h} = \inf_{\mu_{[c,y]}} \mu_{[c,y]}\left(\{y\}\right) R_n'\left(\mu_{[c,y]}, y\right).$$

(b) For $y \in (1, \infty)$ there exists

$$D_- \psi_n\left(y\right) = \lim_{h \to 0-} \frac{\psi_n\left(y+h\right) - \psi_n\left(y\right)}{h} = \sup_{\mu[c,y]} \mu_{[c,y]}\left(\{y\}\right) R_n'\left(\mu_{[c,y]}, y\right).$$

Proof
(a) We proved in Lemma 4.2 that

$$\begin{aligned} & \limsup_{h\to 0+}\left[\psi_n\left(y+h\right)-\psi_n\left(y\right)\right]/h \\ \leq & \inf_{\mu_{[c,y]}} \mu_{[c,y]}\left(\{y\}\right)R_n'\left(\mu_{[c,y]},y\right). \end{aligned} \tag{4.16}$$

Indeed in (4.7), $\mu_{[c,y]}$ was any extremal measure. Next choose a sequence $\{h_m\}$ of positive numbers with limit 0 such that

$$\begin{aligned} & \lim_{m\to\infty}\left[\psi_n\left(y+h_m\right)-\psi_n\left(y\right)\right]/h_m \\ = & \liminf_{h\to 0+}\left[\psi_n\left(y+h\right)-\psi_n\left(y\right)\right]/h. \end{aligned} \tag{4.17}$$

By Theorem 1.2, each extremal measure μ_{I_m} for $I_m=[c,y+h_m]$ has a representation of the form

$$\mu_{I_m}=\nu+\sum_{j=1}^{L}\alpha_{jm}\delta_{\xi_{jm}} \tag{4.18}$$

We may assume that $L\leq n+1$ is the same for all m by passing to a subsequence. By passing to another subsequence, we can assume that μ_{I_m} converges weakly as $m\to\infty$ to some measure $\hat{\mu}_I$. Then $\hat{\mu}_I$ is an extremal measure for $I=[c,y]$ by Lemma 4.1. Next, fix a small $\delta>0$ such that $y-\delta$ is not a mass point of $\hat{\mu}_I$, nor of $\mu_{I_m}, m\geq 1$. For $m\geq 1$, $|h|\leq c$, let

$$\omega_{m,h}=\sum_{j:\xi_{jm}\in[y-\delta,y+h_m]}\alpha_{jm}\delta_{\xi_{jm}-h} \tag{4.19}$$

and

$$\mu^{(m)}=\mu_{[c,y+h_m]}-\omega_{m,0}+\omega_{m,h_m}. \tag{4.20}$$

Thus we are shifting all possible mass points of μ_{c+h_m} in the interval $[y-\delta,y+h_m]$ to the left by h_m. Then $\mu^{(m)}\in\mathcal{M}\left(I,\Lambda\right)$. By Lemma 4.4(b), for large enough m (recall $\left|p_n\left(\mu_{I_m},x\right)\right|$ is bounded below for large enough m as it is maximal over the class $\mathcal{M}\left(\nu,S,I_m\right)$ and approaches $\left|p_n\left(\hat{\mu}_I,x\right)\right|=\phi_n\left(y\right)$)
(4.21)

$$\left|\ln\frac{1}{p_n\left(\mu^{(m)},x\right)^2}-\ln\frac{1}{p_n\left(\mu_{I_m},x\right)^2}-\int R_n'\left(\mu_{I_m},t\right)\left(-sign\left(t\right)\right)d\omega_{m,0}\left(t\right)h_m\right|\leq C_1h_m^2$$

where C_1 is independent of m. Here we are using the unformity in the measure in Lemma 4.4, which allows us to apply that lemma to μ_{I_m} uniformly in m. Then

$$\begin{aligned} & \left(\ln\frac{1}{p_n\left(\mu^{(m)},x\right)^2}-\ln\frac{1}{p_n\left(\mu_{I_m},x\right)^2}\right)/h_m \\ = & -\int R_n'\left(\mu_{I_m},t\right)sign\left(t\right)d\omega_{m,0}\left(t\right)+O\left(h_m\right) \\ = & -\int_{y-\delta}^{y+h_m}R_n'\left(\mu_{I_m},t\right)d\mu_{I_m}\left(t\right)+O\left(h_m\right). \end{aligned}$$

Of course the possible mass points of μ_{I_m} at the endpoint $y + h_m$ are included in the integrals. As μ_{I_m} converges weakly to $\hat{\mu}_I$ and $y - \delta$ is not a mass point of $\hat{\mu}_I$, and by the continuity established in Lemma 4.4,

$$\begin{aligned}
& \lim_{m\to\infty} \left(\ln \frac{1}{p_n\left(\mu^{(m)}, x\right)^2} - \ln \frac{1}{p_n\left(\mu_{I_m}, x\right)^2} \right) / h_m \\
(4.22) \qquad &= -\int_{y-\delta}^{y} R_n'\left(\hat{\mu}_I, t\right) d\hat{\mu}_I\left(t\right).
\end{aligned}$$

Then from (4.17), using $\phi_n\left(y\right)^2 \geq p_n\left(\mu^{(m)}, x\right)^2$,

$$\begin{aligned}
& \liminf_{h\to 0+} \left[\psi_n\left(y+h\right) - \psi_n\left(y\right)\right]/h \\
&= \lim_{m\to\infty} \left[\ln \frac{1}{\phi_n\left(y+h_m\right)^2} - \ln \frac{1}{\phi_n\left(y\right)^2} \right] / h_m \\
&\geq \lim_{m\to\infty} \left[\ln \frac{1}{p_n\left(\mu_{I_m}, x\right)^2} - \ln \frac{1}{p_n\left(\mu^{(m)}, x\right)^2} \right] / h_m \\
&= \int_{y-\delta}^{y} R_n'\left(\hat{\mu}_I, t\right) d\hat{\mu}_I\left(t\right),
\end{aligned}$$

by (4.22). Here $\delta > 0$ can be taken arbitrarily small, subject only to $y - \delta$ not being a mass point of $\hat{\mu}_I$ and μ_{I_m} for all m, so we obtain

$$\begin{aligned}
& \liminf_{h\to 0+} \left[\psi_n\left(y+h\right) - \psi_n\left(y\right)\right]/h \\
&\geq R_n'\left(\hat{\mu}_I, y\right) \hat{\mu}_I\left(\{y\}\right) \\
&\geq \inf_{\mu_{[c,y]}} R_n'\left(\mu_{[c,y]}, y\right) \mu_{\mu_{[c,y]}}\left(\{y\}\right).
\end{aligned}$$

Together with (4.16), this gives the result.

(b) Again let $\mu_{[c,y]}$ be an extremal measure for the given y. As before, we have (4.7), so that

$$\begin{aligned}
& \liminf_{h\to 0-} \left[\psi_n\left(y+h\right) - \psi_n\left(y\right)\right]/h \\
(4.23) \qquad &\geq \sup_{\mu_{[c,y]}} \mu_{[c,y]}\left(\{y\}\right) R_n'\left(\mu_{[c,y]}, y\right).
\end{aligned}$$

Next choose a sequence $\{h_m\}$ of negative numbers with limit 0 such that

$$\begin{aligned}
& \lim_{m\to\infty} \left[\psi_n\left(y+h_m\right) - \psi_n\left(y\right)\right]/h_m \\
&= \limsup_{h\to 0-} \left[\psi_n\left(y+h\right) - \psi_n\left(y\right)\right]/h.
\end{aligned}$$

By Theorem 1.2, each extremal measure μ_{I_m} for $I_m = [c, y + h_m]$ has a representation of the form (4.18). As above, by passing to a subsequence, we can assume that μ_{I_m} converges weakly as $m \to \infty$ to some measure $\hat{\mu}_I$. Then $\hat{\mu}_I$ is an extremal measure for $I = [c, y]$ by Lemma 4.1. Next, fix a small $\delta > 0$ such that $y - \delta$ is not a mass point of $\hat{\mu}_I$ nor any μ_{I_m}. For $m \geq 1$, define $\omega_{m,h}$ and $\mu^{(m)}$ by (4.19) and (4.20). Thus we are shifting all possible mass points of μ_{I_m} in the interval $[y-\delta, y+h_m]$ to the right by $|h_m|$. Then $\mu^{(m)} \in \mathcal{M}(I,\Lambda)$. As above, and recalling that $h_m < 0$,

$$\begin{aligned}
& \limsup_{h\to 0-} \left[\psi_n(y+h) - \psi_n(y)\right]/h \\
= & \lim_{m\to\infty} \left[\ln \frac{1}{\phi_n(y+h_m)^2} - \ln \frac{1}{\phi_n(y)^2}\right]/h_m \\
\leq & \lim_{m\to\infty} \left[\ln \frac{1}{p_n\left(\mu_{I_m}, x\right)^2} - \ln \frac{1}{p_n\left(\mu^{(m)}, x\right)^2}\right]/h_m \\
= & \int_{y-\delta}^{y} R_n'\left(\hat{\mu}_I, t\right) d\hat{\mu}_I(t).
\end{aligned}$$

As above we can let $\delta \to 0+$ through a suitable subsequence and hence

$$\begin{aligned}
& \limsup_{h\to 0-} \left[\psi_n(y+h) - \psi_n(y)\right]/h \\
\leq & R_n'\left(\hat{\mu}_I, y\right)\hat{\mu}_I(\{y\}) \\
\leq & \sup_{\mu_{[c,y]}} R_n'\left(\mu_{[c,y]}, y\right)\mu_{[c,y]}(\{y\}).
\end{aligned}$$

Together with (4.23), this gives the result. ■

Proof of Theorem 1.3

We have already proved in Lemma 4.2 that $\psi_n'(y)$ is given by (1.7) at every point where the derivative eixsts and in particular a.e. Now we prove the absolute continuity of ψ in $[S,T]$ where $1 < S < T < \infty$. This follows from the uniform boundedness of the left and right derivatives of ψ in compact intervals. Recall that from Lemma 4.3,

$$|R_n'(\mu, y)| \leq \frac{4}{\left|p_n\left(\mu_{[c,y]}, x\right)\right|} \left[K_{n+1}(\nu, x, x)\, K_{n+1}(\nu, y, y)\, K_{n+1}^{(1,1)}(\nu, y, y)\right]^{1/2},$$

so is uniformly bounded for $y \in [S,T]$. Also

$$\mu_{[c,y]}(\{y\}) \leq \Lambda.$$

Then the previous lemma shows that for each $y \in [S,T]$, there exists $\varepsilon_y > 0$ such that

$$0 \geq \frac{\psi_n(y+h) - \psi_n(y)}{h} \geq -C,\ h \in [0, \varepsilon_y]$$

and

$$0 \leq \frac{\psi_n(y-h) - \psi_n(y)}{-h} \leq C,\ h \in [0, \varepsilon_y]$$

Here C is independent of y. Thus

$$|\psi_n(y+h) - \psi_n(y)| \leq C\,|h| \text{ for all } |h| \leq \varepsilon_y.$$

Let $S \leq s < t \leq T$. Since the intervals $\left\{\left(y - \frac{1}{2}\varepsilon_y, y + \frac{1}{2}\varepsilon_y\right) : y \in [S,T]\right\}$ cover $[s,t]$, we can find finitely many that cover $[s,t]$. Then for some $\{y_j\}$,

$$s = y_0 < y_1 < ... < y_m = t$$

with $y_{j+1} - y_j \leq \varepsilon_{y_j}$ for all j. Then

$$\begin{aligned} 0 &\leq \psi_n(s) - \psi_n(t) = \sum_{j=0}^{m-1} (\psi_n(y_j) - \psi_n(y_{j+1})) \\ &\leq C \sum_{j=0}^{m-1} (y_{j+1} - y_j) = C(t-s). \end{aligned}$$

So ψ_n satisfies a Lipschitz condition of order 1 in $[S, T]$ with Lipschitz constant C. By standard results, ψ_n is absolutely continuous in $[S, T]$. ■

5. Proof of Theorem 1.4

Recall that we fix $I = [c, d]$ and x, and let

$$\chi_n(\Lambda) = \ln \frac{1}{\phi_n(I, \Lambda, x)^2}, \Lambda > 0.$$

Proof of Theorem 1.4

Let $\mu_{I,\Lambda} = \nu + \rho_{I,\Lambda}$ denote an extremal measure for the given $\Lambda > 0$ and for the given interval I. Let $\tau = \rho_{I,\Lambda}(I)$. For $\varepsilon > 0$, let

$$\mu^\varepsilon = \mu_{I,\Lambda} + \frac{\varepsilon}{\tau}\rho_{I,\Lambda} = \nu + \left(1 + \frac{\varepsilon}{\tau}\right)\rho_{I,\Lambda}$$

Then $\mu^\varepsilon \in M(\nu, \Lambda + \varepsilon, I)$, and so by Theorem 2.1, with $\omega = \frac{1}{\tau}\rho_I$, as $\varepsilon \to 0+$,

$$\begin{aligned} &\frac{\chi_n(\Lambda + \varepsilon) - \chi_n(\Lambda)}{\varepsilon} \\ \leq\ & \frac{1}{\varepsilon}\left[\ln \frac{1}{|p_n(\mu^\varepsilon, x)|^2} - \ln \frac{1}{|p_n(\mu_I, x)|^2}\right] \\ =\ & \frac{1}{\tau}\left(\int R_n(\mu_{I,\Lambda}, t)\, d\rho_{I,\Lambda}(t)\right) + o(1) \\ =\ & \inf\left\{R_n(\mu_{I,\Lambda}, t) : t \in I\right\}, \end{aligned}$$

by Theorem 1.2(c). In the other direction, for small $\varepsilon > 0$, let

$$\mu^\varepsilon = \mu_{I,\Lambda} + \frac{\varepsilon}{\tau}(-\rho_{I,\Lambda}) = \nu + \left(1 - \frac{\varepsilon}{\tau}\right)\rho_{I,\Lambda}.$$

By Theorem 2.1, with $\omega = -\frac{1}{\tau}\rho_{I,\Lambda}$, as $\varepsilon \to 0+$,

$$\begin{aligned} &\frac{\chi_n(\Lambda - \varepsilon) - \chi_n(\Lambda)}{-\varepsilon} \\ \geq\ & \frac{1}{-\varepsilon}\left[\ln \frac{1}{|p_n(\mu^\varepsilon, x)|^2} - \ln \frac{1}{|p_n(\mu_I, x)|^2}\right] \\ =\ & -\frac{1}{\tau}\left(\int R_n(\mu_{I,\Lambda}, t)\, d(-\rho_{I,\Lambda})(t)\right) + o(1) \\ =\ & \inf\left\{R_n(\mu_{I,\Lambda}, t) : t \in I\right\}, \end{aligned}$$

by Theorem 1.2 (c) again. Thus at every $\Lambda > 0$,

(5.1)
$$\limsup_{h\to 0+} \frac{\chi_n(\Lambda+h)-\chi_n(\Lambda)}{h} \le \inf\left\{R_n\left(\mu_{I,\Lambda},t\right) : t\in I\right\} \le \liminf_{h\to 0-} \frac{\chi_n(\Lambda+h)-\chi_n(\Lambda)}{h}.$$

Then if $\chi_n'(\Lambda)$ exists, we obtain,

$$\chi_n'(\Lambda) = \inf\left\{R_n\left(\mu_{I,\Lambda},t\right) : t\in I\right\}.$$

As χ_n is decreasing in Λ, this holds a.e. The continuity of the inf in Λ follows easily from Lemma 4.1. ■

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu