ON ZEROS, BOUNDS, AND ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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In Memory of A A Gonchar

Abstract. Let μ be a measure on the unit circle that is regular in the sense of Stahl Totik, and Ullmann. Let $\{\varphi_n\}$ be the orthonormal polynomials for μ and $\{z_{jn}\}$ their zeros. Let μ be absolutely continuous in an arc Δ of the unit circle, with μ' positive and continuous there. We show that uniform boundedness of the orthonormal polynomials in subarcs Γ of Δ is equivalent to certain asymptotic behavior of their zeros inside sectors that rest on Γ. Similarly the uniform limit $\lim_{n\to\infty} |\varphi_n(z)|^2 \mu'(z) = 1$ is equivalent to related asymptotics for the zeros in such sectors.

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1. Introduction

Let μ be a finite positive Borel measure on $[-\pi, \pi)$ (or equivalently on the unit circle) with infinitely many points in its support. Then we may define orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(z) \overline{\varphi_m(z)} d\mu(\theta) = \delta_{mn},$$

where $z = e^{i\theta}$. We denote the zeros of φ_n by $\{z_{jn}\}_{j=1}^{n}$. They lie inside the unit circle, and may not be distinct.

Soviet and Russian mathematicians have been leading lights in the theory of orthogonal polynomials ever since Chebyshev laid the foundations. Many in Gonchar's own school are world leaders, and their students continue that tradition. The celebrated work of Rakhmanov, Aptekarev, Denisov, and Tulyakov on the Steklov conjecture [3], [4], [18], [20] is just one of many examples. It is a privilege to pay tribute to Gonchar's memory.

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We shall assume that μ is *regular* in the sense of Stahl, Totik and Ullmann [25], so that

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1.$$

This is true if for example $\mu' > 0$ a.e. in $[-\pi, \pi)$, but there are pure jump and pure singularly continuous measures that are regular.

Many aspects of the zeros $\{z_{jn}\}$ have been studied down the years, for example, their distribution (often when projected onto the unit circle), and "clock spacing" of zeros of paraorthogonal polynomials. See Chapter 8 of Simon's monograph [23]. One result relevant to this paper, is due to Nevai and Totik [17], [23, Thm. 7.1.3, p. 383]. They relate the largest disk centered at the origin containing all zeros of the orthonormal polynomials to analytic continuation of the Szegő function inside the unit circle. In this case, μ' is infinitely differentiable on the unit circle. Another classic result of Mhaskar and Saff gives sufficient conditions in terms of the recurrence coefficients for the zero counting measures to converge weakly to the uniform distribution on the unit circle [15], [23, Thm. 8.1.2, p. 392]. Breuer and Selig [7] recently studied clock spacing of zeros of paraorthogonal polynomials, as did Simanek [21], [22]. See also the references there.

In a very interesting recent paper, Bessonov and Denisov [6, Theorem 3] showed that the distance of the zeros to the unit circle is intimately related to asymptotics of orthogonal polynomials. The following is a reformulation of one of their results:

Theorem
Let μ be a measure on the unit circle satisfying the Szegő condition

$$\int_{-\pi}^{\pi} \log \mu'\left(e^{it}\right) dt > -\infty.$$

For almost every ζ with $|\zeta| = 1$, the following are equivalent:
(I)

$$\lim_{n\to\infty} |\varphi_n(\zeta)|^2 \mu'(\zeta) = 1.$$

(II)

$$\lim_{n\to\infty} n \left(\inf_{1\le j\le n} |\zeta - z_{jn}| \right) = \infty.$$

We prove related equivalences for local bounds and asymptotics but for regular, rather than Szegő, measures:

Theorem 1.1
Let μ be a finite positive Borel measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Let Δ be an arc of the

unit circle in which μ is absolutely continuous, while μ' is positive and continuous there. The following are equivalent:
(I) In every proper subarc Γ of Δ,

$$\lim_{n\to\infty}\left(\inf\left\{n\left(1-|z_{jn}|\right):z_{jn}\neq 0,\frac{z_{jn}}{|z_{jn}|}\in\Gamma\right\}\right)=\infty.$$

(II) In every proper subarc Γ of Δ, *as* $n\to\infty$, *uniformly for* $z\in\Gamma$,

$$\lim_{n\to\infty}\left|\varphi_n\left(z\right)\right|^2\mu'\left(z\right)=1. \tag{1.1}$$

Remarks
(i) By a *proper* subarc, we mean that both endpoints of Γ are at a positive distance to the endpoints of Δ. All arcs in this paper are assumed to be closed arcs, so contain their endpoints.
(ii) We note that if μ is absolutely continuous on the whole unit circle, while μ' is positive and continuous there, then by applying the above result to two subarcs, we obtain the equivalence on the whole unit circle. As far as the author is aware, even that is new.
(iii) Asymptotics of orthogonal polynomials on the unit circle have been studied for at least a century, and there is an extensive literature. If $\log\mu'$ is integrable over the unit circle, then there is an L_2 asymptotic for φ_n [8, Chapter V], [9], [23, p. 132], [26, Chapter 10]. There are many sufficient conditions for pointwise asymptotics on subarcs of the unit circle, and their real line analogues and we cannot hope to review this here. The most general result for pointwise asymptotics on the unit circle, is almost certainly that of Badkov [5]. He showed that if $\log\mu'$ is integrable on the unit circle, and in some subarc, μ is absolutely continuous, while μ' satisfies there a Dini-Lipschitz condition, then we have a uniform asymptotic involving the Szegő function, and hence also (1.1).

One of the particularly significant results for non-Szegő weights is due to Rakhmanov [19, Thm 4, p. 151]: if μ is absolutely continuous on the unit circle, and μ' satisfies a Dini-Lipschitz condition on the unit circle, then (1.1) holds uniformly on each subarc of the circle where μ' is bounded below by a positive constant.

For bounds, we prove:

Theorem 1.2
Let μ be a finite positive Borel measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Let Δ be an arc of the unit circle in which μ is absolutely continuous, while μ' is positive and continuous there. The following are equivalent:

(I) In every proper subarc Γ *of* Δ*, there exists* $C_1 > 0$ *such that for* $n \geq 1$,

$$\inf \left\{ n\left(1 - |z_{jn}|\right) : z_{jn} \neq 0, \frac{z_{jn}}{|z_{jn}|} \in \Gamma \right\} \geq C_1.$$

(II) In every proper subarc Γ of Δ, *there exists* $C_2 > 0$ *such that for* $n \geq 1$,

$$\|\varphi_n\|_{L_\infty(\Gamma)} \leq C_2.$$

Remarks

(i) Again if μ is absolutely continuous on the whole unit circle, while μ' is positive and continuous there, then by applying the above result to two subarcs, we obtain the equivalence on the whole unit circle.

(ii) Bounds on orthogonal polynomials have also been investigated for a century, with one of the celebrated problems being Steklov's conjecture. It was E.A. Rakhmanov who resolved the conjecture, [18], [20], with definitive later contributions by Ambroladze [1], [2], Aptekarev, Denisov, and Tulyakov [3], [4]. There have been many who have contributed in a major way to the broader issue of bounds - for example, Badkov [5], Freud [8], Geronimus [9], Korous, Nevai [16]. Again, this is a very incomplete list.

(iii) The main ideas underlying the results of this paper are universality limits for reproducing kernels [10], [12], [27] and local limits for ratios of orthogonal polynomials [13].

(iv) For orthogonal polynomials on the real line, the analogous result to Theorem 1.2 involves the distance between zeros of orthogonal polynomials of successive degrees [11].

This paper is organized as follows: in Section 2, we present Theorems 2.1 and 2.2, which state more equivalences than those above. In Section 3, we present four preliminary lemmas. We prove Theorem 2.1 in Section 4, and Theorem 2.2 in Section 5.

We close this section with more notation. The sinc kernel is

$$\mathbb{S}(u) = \frac{\sin \pi u}{\pi u}.$$

We let

$$\varphi_n^*(z) = z^n \overline{\varphi_n\left(\frac{1}{\bar{z}}\right)}.$$

The nth reproducing kernel for μ is

$$K_n(z,u) = \sum_{j=0}^{n-1} \varphi_j(z) \overline{\varphi_j(u)}. \tag{1.2}$$

The Christoffel-Darboux formula asserts that for $z \neq u$ [23, p. 954]

$$K_n(z,u) = \frac{\overline{\varphi_n^*(u)}\varphi_n^*(z) - \overline{\varphi_n(u)}\varphi_n(z)}{1-\bar{u}z}. \tag{1.3}$$

We let

$$R_n(z) = \sum_{j=1}^{n} \frac{1-|z_{jn}|^2}{|z-z_{jn}|^2} \tag{1.4}$$

and

$$g_n(z) = \frac{z\varphi_n'(z)}{n\varphi_n(z)}. \tag{1.5}$$

If $z_{jn} = 0$, we set $\tau_{jn} = 0$, while if $z_{jn} \neq 0$, we set

$$\tau_{jn} = \frac{z_{jn}}{|z_{jn}|}. \tag{1.6}$$

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, z, ζ and polynomials P of degree $\leq n$. The same symbol need not denote the same constant in different occurrences. For sequences $\{x_n\}, \{y_n\}$ of non-zero real numbers, we write

$$x_n \sim y_n$$

if there exists $C > 1$ independent of n, but possibly depending on the sequences, such that

$$C^{-1} \leq x_n/y_n \leq C \text{ for all } n \geq 1.$$

2. Further Equivalences

Theorems 1.1 and 1.2 are special cases respectively of Theorem 2.1 and 2.2 below.

Theorem 2.1
Let μ be a finite positive Borel measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Let Δ be an arc of the unit circle in which μ is absolutely continuous, while μ' is positive and continuous there. The following are equivalent:
(a) uniformly for z in proper subarcs of Δ,

$$\lim_{n\to\infty} |\varphi_n(z)|^2 \mu'(z) = 1. \tag{2.1}$$

(b) uniformly for z in proper subarcs of Δ,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1-|z_{jn}|^2}{|z-z_{jn}|^2} = 1. \tag{2.2}$$

(c) uniformly for z in proper subarcs of Δ,

$$\lim_{n\to\infty} \operatorname{Re}\left(\frac{z\varphi_n'(z)}{n\varphi_n(z)}\right) = 1. \tag{2.3}$$

(d) uniformly for z in proper subarcs of Δ,

$$\lim_{n\to\infty} \frac{z\varphi_n'(z)}{n\varphi_n(z)} = 1. \tag{2.4}$$

(e) uniformly for z in proper subarcs of Δ,

$$\lim_{n\to\infty} \frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n(z)} = -1. \tag{2.5}$$

(f) uniformly for z in proper subarcs of Δ, and for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{\varphi_n\left(z\left(1+\frac{u}{n}\right)\right)}{\varphi_n(z)} = e^u. \tag{2.6}$$

(g) in proper subarcs Γ of Δ,

$$\lim_{n\to\infty} \left(\inf\left\{n\left(1-|z_{jn}|\right) : z_{jn}\neq 0, \frac{z_{jn}}{|z_{jn}|}\in\Gamma\right\}\right) = \infty. \tag{2.7}$$

(h) uniformly for z in proper subarcs of Δ,

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{|z-z_{jn}|^2} = 0. \tag{2.8}$$

Remarks
(i) Weaker versions of parts of Theorem 2.1 appear in Theorem 1.2 in [13], notably (b), (d), (e), (f), since we also made an unnecessary assumption (1.7) in [13] about $\operatorname{Im}\left(\varphi_n\left(ze^{\pm i\pi/n}\right)/\varphi_n(z)\right)$.
(ii) There was unfortunately an error in Lemma 4.2(a) in [13] that led to gaps in proofs later in that paper. These gaps were corrected in [14].

Theorem 2.2
Let μ be a finite positive Borel measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Let Δ be an arc of the unit circle in which μ is absolutely continuous, while μ' is positive and continuous there. The following are equivalent:
(a) for every proper subarc Γ of Δ,

$$\sup_{n\geq 1} \|\varphi_n\|_{L_\infty(\Gamma)} < \infty. \tag{2.9}$$

(b) for every proper subarc Γ *of* Δ,

$$\inf_{n\geq 1}\inf_{z\in\Gamma}\frac{1}{n}\sum_{j=1}^{n}\frac{1-\left|z_{jn}\right|^{2}}{\left|z-z_{jn}\right|^{2}}\geq C. \tag{2.10}$$

(c) for every proper subarc Γ *of* Δ*, there exists* n_0 *such that*

$$\inf_{n\geq n_0}\inf_{z\in\Gamma}\left|\operatorname{Re}\left(\frac{z\varphi_n'(z)}{n\varphi_n(z)}-\frac{1}{2}\right)\right|\geq C. \tag{2.11}$$

(d) for every proper subarc Γ *of* Δ*, there exists* n_0 *such that*

$$\inf_{n\geq n_0}\inf_{z\in\Gamma}\left|\operatorname{Re}\left(\frac{\varphi_n\left(ze^{\pm i\pi/n}\right)}{\varphi_n(z)}\right)\right|\geq C. \tag{2.12}$$

(e) for every proper subarc Γ *of* Δ,

$$\inf\left\{n\left(1-\left|z_{jn}\right|\right):n\geq 1,\ z_{jn}\neq 0,\frac{z_{jn}}{\left|z_{jn}\right|}\in\Gamma\right\}\geq C. \tag{2.13}$$

(f) for every proper subarc Γ *of* Δ,

$$\sup_{n\geq 1}\sup_{z\in\Gamma}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left|z-z_{jn}\right|^{2}}\leq C. \tag{2.14}$$

3. Preliminary Lemmas

Throughout, we assume the hypotheses of Theorem 1.1, namely that μ is regular in the sense of Stahl, Totik and Ullmann, while it is absolutely continuous in Δ, with μ' positive and continuous there. We first recall some asymptotics for Christoffel functions and universality and local limits.

Lemma 3.1
Let Γ *be a proper subarc of* Δ.
(a) Uniformly for $z\in\Gamma$,

$$\lim_{n\to\infty}\frac{1}{n}K_n(z,z)\,\mu'(z)=1. \tag{3.1}$$

(b) Uniformly for $z\in\Gamma$ *and* a,b *in compact subsets of* $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{K_n\left(z\left(1+\frac{i2\pi a}{n}\right),z\left(1+\frac{i2\pi\bar{b}}{n}\right)\right)}{K_n(z,z)}=e^{i\pi(a-b)}\mathbb{S}(a-b)\,. \tag{3.2}$$

(c) Let $\{\zeta_n\} \subset \Gamma$. *Assume that*

$$\sup_{n\geq 1} \frac{1}{n}\left|\sum_{j=1}^{n} \frac{1}{\zeta_n - z_{jn}}\right| < \infty \text{ and } \sup_{n\geq 1} \frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{\left|\zeta_n - z_{jn}\right|^2} < \infty. \tag{3.3}$$

From every infinite sequence of positive integers, we can choose an infinite subsequence $\mathcal{S}$ *such that uniformly for* u *in compact subsets of* $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)} = e^u + C\left(e^u - 1\right), \tag{3.4}$$

where

$$C = \lim_{n\to\infty, n\in\mathcal{S}} \left(\frac{\zeta_n}{n}\frac{\varphi_n'\left(\zeta_n\right)}{\varphi_n\left(\zeta_n\right)} - 1\right), \tag{3.5}$$

Proof

(a) See for example [24, p. 123, Thm. 2.16.1].
(b) See for example [10, Thm. 6.3, p. 559] or [24, p. 124, Thm. 2.16.1].
(c) This follows immediately from Theorem 1.3 in [13] as we have the universality limit (3.2). We note that there was a mistake in Lemma 4.2(a) in [13] that was corrected in [14]. However, the mistake did not in any way affect the proof of Theorem 1.3 there. ■

Some of the assertions in the following lemma appear in [13], but we include proofs for the reader's convenience. Recall that R_n and g_n were defined respectively by (1.4) and (1.5).

Lemma 3.2

Let Γ *be a proper subarc of* Δ.
(a) For $|z| = 1$,

$$\frac{1}{n}R_n\left(z\right) = \operatorname{Re}\left[2g_n\left(z\right) - 1\right]. \tag{3.6}$$

(b) Uniformly for $z \in \Gamma$, *and fixed real* α,

$$\lim_{n\to\infty} \operatorname{Im}\left[e^{i\pi\alpha}\varphi_n\left(z\right)\overline{\varphi_n\left(ze^{2\pi i\alpha/n}\right)}\right]\mu'\left(z\right) = -\sin\pi\alpha. \tag{3.7}$$

(c) Uniformly for $z \in \Gamma$, *and fixed real* α,

$$\begin{aligned}\lim\nolimits_{n\to\infty}\{&\operatorname{Re}\left[e^{\pi i\alpha}\varphi_n\left(z\right)\overline{\varphi_n\left(ze^{2\pi i\alpha/n}\right)}\right]\tfrac{1}{n}R_n\left(ze^{2\pi i\alpha/n}\right)\mu'\left(z\right) \\ &- \left(2\sin\pi\alpha\right)\left(\operatorname{Im} g_n\left(ze^{2\pi i\alpha/n}\right)\right)\} = \cos\pi\alpha.\end{aligned} \tag{3.8}$$

(d) Uniformly for $z \in \Gamma$,

$$\lim_{n\to\infty} \frac{1}{n}R_n\left(z\right)\left|\varphi_n\left(z\right)\right|^2\mu'\left(z\right) = 1. \tag{3.9}$$

(e) Uniformly for $z \in \Gamma$, and fixed real α,

$$\left(1-e^{-2\pi i\alpha}\right) g_n(z) = 1 - \frac{\varphi_n\left(ze^{-2\pi i\alpha/n}\right)}{\varphi_n(z)}\left(1+o(1)\right)+o(1). \tag{3.10}$$

Proof

Throughout this proof, we assume that

$$\zeta = ze^{2\pi i\alpha/n}$$

with α real or complex.

(a) Elementary manipulation shows that

$$\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2} = 2\,\mathrm{Re}\left(\frac{z}{z-z_{jn}}\right) - 1.$$

Dividing by n and adding for $j = 1, 2, ..., n$ gives (3.6).

(b) The Christoffel-Darboux formula (1.3) and universality limit (3.2) (as well as the uniformity of that limit) give uniformly for α in compact subsets of $\mathbb{C}$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\overline{\varphi_n^*(z)}\varphi_n^*(\zeta) - \overline{\varphi_n(z)}\varphi_n(\zeta)}{[1-\bar{z}\zeta]\,K_n(z,z)} \\
=\ &\lim_{n\to\infty} \frac{K_n(\zeta,z)}{K_n(z,z)} \\
=\ &\lim_{n\to\infty} \frac{K_n\left(z\left(1+\frac{2\pi i\alpha}{n}[1+o(1)]\right),z\right)}{K_n(z,z)} = e^{i\pi\alpha}\mathbb{S}(\alpha).
\end{aligned}$$

(3.11)

Here by (3.1),

$$\lim_{n\to\infty} [1-\bar{z}\zeta]\,K_n(z,z) = -2\pi i\alpha\mu'(z)^{-1}.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}\left[\overline{\varphi_n^*(z)}\varphi_n^*(\zeta) - \overline{\varphi_n(z)}\varphi_n(\zeta)\right]\mu'(z) &= -2\pi i\alpha e^{i\pi\alpha}\mathbb{S}(\alpha) \\
\text{(3.12)} \qquad &= -2ie^{\pi i\alpha}\sin\pi\alpha = 1-e^{2\pi i\alpha}.
\end{aligned}$$

Next, if α is real,

$$\overline{\varphi_n^*(z)}\varphi_n^*(\zeta) = e^{2\pi i\alpha}\varphi_n(z)\,\overline{\varphi_n(\zeta)}$$

so combining this and (3.12) gives

$$\lim_{n\to\infty} e^{\pi i\alpha}\left\{e^{i\pi\alpha}\varphi_n(z)\,\overline{\varphi_n(\zeta)} - e^{-\pi i\alpha}\overline{\varphi_n(z)}\varphi_n(\zeta)\right\}\mu'(z) = -2ie^{\pi i\alpha}\sin\pi\alpha.$$

Dividing by $2ie^{\pi i\alpha}$ gives (3.7).

(c) We go back to (3.12), which holds uniformly for α in compact subsets of $\mathbb{C}$. This uniformity allows us to differentiate with respect to α: after cancelling a factor of $2\pi i$, we obtain

$$\lim_{n\to\infty}\left[\overline{\varphi_n^*(z)}\varphi_n^{*\prime}(\zeta)-\overline{\varphi_n(z)}\varphi_n'(\zeta)\right]\frac{\zeta}{n}\mu'(z)=-e^{2\pi i\alpha}. \tag{3.13}$$

Now we again specialize to real α, and use that for $|\zeta|=1$,

$$\varphi_n^{*\prime}(\zeta)=n\zeta^{n-1}\overline{\varphi_n(\zeta)}-\zeta^{n-2}\overline{\varphi_n'(\zeta)}$$

so that using $\bar{z}\zeta=e^{2\pi i\alpha/n}$, and recalling the definition (1.5) of g_n

$$\overline{\varphi_n^*(z)}\varphi_n^{*\prime}(\zeta)\frac{\zeta}{n}=e^{2\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{2\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)g_n(\zeta)}.$$

Substituting in (3.13), and dividing by $e^{\pi i\alpha}$,

$$\begin{aligned}&\lim_{n\to\infty}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)g_n(\zeta)}-e^{-\pi i\alpha}\overline{\varphi_n(z)}\varphi_n(\zeta)g_n(\zeta)\right]\mu'(z)\\ =\;&-e^{\pi i\alpha}\end{aligned} \tag{3.14}$$

or

$$\lim_{n\to\infty}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-2\operatorname{Re}\left\{e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)g_n(\zeta)}\right\}\right]\mu'(z)=-e^{\pi i\alpha}.$$

Taking real parts,

$$\lim_{n\to\infty}\operatorname{Re}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}\left\{1-2\overline{g_n(\zeta)}\right\}\right]\mu'(z)=-\cos\pi\alpha.$$

Then using (3.7),

$$\begin{aligned}&\lim_{n\to\infty}\left\{\operatorname{Re}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}\mu'(z)\right]\operatorname{Re}\left[1-2g_n(\zeta)\right]+2\sin\pi\alpha\operatorname{Im}g_n(\zeta)\right\}\\ =\;&-\cos\pi\alpha.\end{aligned}$$

Finally apply (3.6).
(d) Here we set $\alpha=0$ in (3.8).
(e) From (a),

$$\overline{g_n(\zeta)}=2\operatorname{Re}g_n(\zeta)-g_n(\zeta)=\frac{1}{n}R_n(\zeta)+1-g_n(\zeta).$$

We substitute this in (3.14) and cancel a term:

$$\begin{aligned}&\lim_{n\to\infty}[-e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}\frac{1}{n}R_n(\zeta)+g_n(\zeta)\left\{e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{-\pi i\alpha}\overline{\varphi_n(z)}\varphi_n(\zeta)\right\}]\mu'(z)\\ =\;&-e^{\pi i\alpha}.\end{aligned}$$

Using (3.9) and (3.7), and continuity of μ', we obtain

$$\lim_{n\to\infty}\left[-e^{\pi i\alpha}\frac{\varphi_n(z)}{\varphi_n(\zeta)}(1+o(1))+g_n(\zeta)\,2i(-\sin\pi\alpha)\right]=-e^{\pi i\alpha}$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\varphi_n(z)}{\varphi_n(\zeta)}(1+o(1))+g_n(\zeta)\,2ie^{-\pi i\alpha}\sin\pi\alpha\right]=1.$$

Because of the uniformity, we can substitute $ze^{-2\pi i\alpha/n}$ for z so that ζ becomes z. ■

We now prove parts of Theorems 2.1, 2.2. Recall $\tau_{jn}=\frac{z_{jn}}{|z_{jn}|}$ as at (1.6).

Lemma 3.3

(a) The following are equivalent:

(i) in every proper subarc Γ of Δ,

$$\inf\{n(1-|z_{jn}|):\tau_{jn}\in\Gamma, n\geq 1\}\geq C>0.$$

(ii) in every proper subarc Γ of Δ,

$$\sup_{z\in\Gamma,n\geq 1}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{|z-z_{jn}|^2}<\infty.$$

(b) The following are equivalent:

(i) in every proper subarc Γ of Δ,

$$\lim_{n\to\infty}[\inf\{n(1-|z_{jn}|):\tau_{jn}\in\Gamma\}]=\infty.$$

(ii) in every proper subarc Γ of Δ,

$$\lim_{n\to\infty}\left[\sup_{z\in\Gamma}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{|z-z_{jn}|^2}\right]=0.$$

Proof

(a) (i)⇒(ii)

Let Γ,Γ_1 be proper subarcs of Δ such that Γ is a proper subarc of Γ_1. In particular, we assume that the distance from both endpoints of Γ to the endpoints of Γ_1 is positive. We have for $z\in\Gamma$,

$$\begin{aligned}\frac{1}{n^2}\sum_{\tau_{jn}\in\Gamma_1}\frac{1}{|z-z_{jn}|^2} &\leq \frac{1}{Cn}\sum_{\tau_{jn}\in\Gamma_1}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}\\ &\leq \frac{1}{Cn}R_n(z)\leq C_1|\varphi_n(z)|^{-2},\end{aligned}$$

by (3.9) and positivity and continuity of μ'. Next, we know from (3.7) with $\alpha=\frac{1}{2}$, that

$$|\varphi_n(z)|\left|\varphi_n\left(ze^{i\pi/n}\right)\right|\mu'(z)\geq 1+o(1)$$

so it follows that either

$$\frac{1}{n^2}\sum_{\tau_{jn}\in\Gamma_1}\frac{1}{|z-z_{jn}|^2}\leq C_1 \text{ or } \frac{1}{n^2}\sum_{\tau_{jn}\in\Gamma_1}\frac{1}{|ze^{i\pi/n}-z_{jn}|^2}\leq C_1$$

(or possibly both). But because of our hypothesis, for $\tau_{jn}\in\Gamma_1, z\in\Gamma$,

$$\begin{aligned}\left|\frac{z-z_{jn}}{ze^{i\pi/n}-z_{jn}}\right| &= \left|1+\frac{z\left(1-e^{i\pi/n}\right)}{ze^{i\pi/n}-z_{jn}}\right| \\ &\leq 1+\frac{2\sin\left(\pi/2n\right)}{1-|z_{jn}|}\leq C_2\end{aligned}$$

while a similar bound holds for the reciprocal. So for $z\in\Gamma$,

$$\frac{1}{n^2}\sum_{\tau_{jn}\in\Gamma_1}\frac{1}{|z-z_{jn}|^2}\leq C_3.$$

Then also for the remaining terms, as the distance from the boundary of Γ to that of Γ_1 is positive, and $z\in\Gamma$,

$$\frac{1}{n^2}\sum_{\tau_{jn}\notin\Gamma_1}\frac{1}{|z-z_{jn}|^2}\leq\frac{C_4 n}{n^2}=o\left(1\right).$$

Adding the two estimates gives the result.
(ii)$\Rightarrow$(i)
Choosing $z=\tau_{jn}\in\Gamma$ gives

$$\frac{1}{n^2\left(1-|z_{jn}|\right)^2}=\frac{1}{n^2\left|z-z_{jn}\right|^2}\leq\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{|z-z_{kn}|^2}\leq C,$$

by our hypothesis.
(b) (i)$\Rightarrow$(ii)
Let Γ,Γ_1 be as above. For $z\in\Gamma$,

$$\begin{aligned}\frac{1}{n^2}\sum_{\tau_{jn}\in\Gamma_1}\frac{1}{|z-z_{jn}|^2} &\leq \left(\frac{1}{n}\sum_{\tau_{jn}\in\Gamma_1}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}\right)\frac{1}{\inf_{\tau_{jn}\in\Gamma_1} n\left(1-|z_{jn}|^2\right)} \\ &= o\left(\frac{1}{n}R_n\left(z\right)\right).\end{aligned}$$

We can now proceed as in (a).
(b) (ii)$\Rightarrow$(i)
Again, proceed much as in (a). ■

Lemma 3.4
Assume that in every proper subarc Γ *of* Δ,

$$\sup_{n\leq 1}\|\varphi_n\|_{L_\infty(\Gamma)}<\infty.$$

Then in every proper subarc Γ *of* Δ, *there exists* $C>0$ *such that for* $n\geq 1$ *and* $z_{jn}\neq 0,\tau_{jn}\in\Gamma$,

$$n\left(1-|z_{jn}|\right)\geq C.$$

Proof
Let Γ be a proper subarc of Δ. Suppose the conclusion is false. Then we can can choose an infinite subsequence $\mathcal{S}$ of integers, and for $j=j\left(n\right)\in\mathcal{S}$, z_{jn} with $\tau_{jn}=z_{jn}/\left|z_{jn}\right|\in\Gamma$ such that

$$n\left(1-|z_{jn}|\right)\rightarrow 0.$$

Write

$$z_{jn}=\tau_{jn}\left(1+2\pi i\frac{\alpha_n}{n}\right);u=\tau_{jn}\left(1+2\pi i\frac{\bar{v}}{n}\right)$$

where $v=v\left(n\right)$, that is v depends on n and $\alpha_n\rightarrow 0$ as $n\rightarrow\infty$. Then from the universality limit (3.2), (which by our assumptions holds in a larger arc than Γ), uniformly for v in compact sets,

$$\begin{aligned}\frac{K_n\left(z_{jn},u\right)}{K_n\left(\tau_{jn},\tau_{jn}\right)} &= e^{i\pi(\alpha_n-v)}\mathbb{S}\left(\alpha_n-v\right)+o\left(1\right)\\ &= e^{-i\pi v}\mathbb{S}\left(v\right)+o\left(1\right).\end{aligned}\tag{3.15}$$

Next from the Christoffel-Darboux formula (1.3),

$$\overline{\varphi_n^*\left(u\right)}\varphi_n^*\left(z_{jn}\right)=\left\{K_n\left(\tau_{jn},\tau_{jn}\right)\left(1-\bar{u}z_{jn}\right)\right\}\frac{K_n\left(z_{jn},u\right)}{K_n\left(\tau_{jn},\tau_{jn}\right)}\tag{3.16}$$

and setting $u=\tau_{jn}$ so that $v=0$ in both formulas, and using (3.1), as well as the fact that $n\left(1-|z_{jn}|\right)\rightarrow 0$, gives

$$\begin{aligned}\overline{\varphi_n^*\left(\tau_{jn}\right)}\varphi_n^*\left(z_{jn}\right) &= K_n\left(\tau_{jn},\tau_{jn}\right)\left(1-|z_{jn}|\right)\frac{K_n\left(z_{jn},\tau_{jn}\right)}{K_n\left(\tau_{jn},\tau_{jn}\right)}\\ &= o\left(1\right)\left(1+o\left(1\right)\right)=o\left(1\right).\end{aligned}\tag{3.17}$$

Now apply (3.15), (3.16) with $u=\tau_{jn}e^{i\pi/n}$, so that in u above, $v=\frac{1}{2}+o\left(1\right)$, and

$$\frac{K_n\left(z_{jn},u\right)}{K_n\left(\tau_{jn},\tau_{jn}\right)}=e^{-i\pi/2}\mathbb{S}\left(\frac{1}{2}\right)+o\left(1\right),$$

while

$$\begin{aligned}
&\overline{\varphi_n^*(\tau_{jn}e^{i\pi/n})}\varphi_n^*(z_{jn}) \\
= \ & \left\{K_n(\tau_{jn},\tau_{jn})\left(1-e^{-i\pi/n}\left[1+2\pi i\frac{\alpha_n}{n}\right]\right)\right\}\frac{K_n(z_{jn},u)}{K_n(\tau_{jn},\tau_{jn})} \\
= \ & \left\{K_n(\tau_{jn},\tau_{jn})\left(1-e^{-i\pi/n}+o\left(\frac{1}{n}\right)\right)\right\}\left\{e^{-i\pi/2}\mathbb{S}\left(\frac{1}{2}\right)+o(1)\right\}
\end{aligned}$$

so that using (3.1),

$$\left|\varphi_n^*\left(\tau_{jn}e^{i\pi/n}\right)\varphi_n^*(z_{jn})\right|\sim 1.$$

Dividing (3.17) by this, gives

$$\left|\frac{\varphi_n(\tau_{jn})}{\varphi_n(\tau_{jn}e^{i\pi/n})}\right|=\left|\frac{\varphi_n^*(\tau_{jn})}{\varphi_n^*(\tau_{jn}e^{i\pi/n})}\right|=o(1).$$

But from (3.7) with $\alpha=\frac{1}{2}$,

$$\left|\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\varphi_n(\tau_{jn})\right|\geq 1+o(1),$$

so

$$\begin{aligned}
\left|\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\right|^2 &= \left|\frac{\varphi_n(\tau_{jn})}{\varphi_n(\tau_{jn}e^{i\pi/n})}\right|^{-1}\left|\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\varphi_n(\tau_{jn})\right| \\
&\to \infty \text{ as } n\to\infty,
\end{aligned}$$

contradicting the assumed boundedness of $\{\varphi_n\}$. ■

4. Proof of Theorem 2.1

Proof of Theorem 2.1

(a)⇔(b)

This is immediate from (3.9).

(b)⇔(c)

This is immediate from the identity (3.6).

(c)⇔(d)

It is immediate that (2.4)⇒(2.3). Now assume (2.3) holds. We must show that $\operatorname{Im} g_n(z)\to 0$ as $n\to\infty$, uniformly in Γ. From the established equivalence (b)⇔(c), we have (2.2), so from (3.8) with $\alpha=\frac{1}{2}$, and (3.9),

$$\lim_{n\to\infty}\left\{-\operatorname{Im}\left[\varphi_n(z)\overline{\varphi_n(ze^{i\pi/n})}\mu'(z)\right]-2\left(\operatorname{Im} g_n\left(ze^{\pi i/n}\right)\right)\right\}=0. \tag{4.1}$$

Next, we have already proved that (c) is equivalent to (a), so using (2.1) for $\zeta=z,ze^{i\pi/n}$ and continuity of μ',

$$\lim_{n\to\infty}\left|\varphi_n(z)\overline{\varphi_n(ze^{i\pi/n})}\right|\mu'(z)=1,$$

while from (3.7) with $\alpha = \frac{1}{2}$,

$$\lim_{n\to\infty} \operatorname{Re}\left[\varphi_n(z)\overline{\varphi_n\left(ze^{i\pi/n}\right)}\right]\mu'(z) = -1.$$

Hence

$$\lim_{n\to\infty} \operatorname{Im}\left[\varphi_n(z)\overline{\varphi_n\left(ze^{i\pi/n}\right)}\right]\mu'(z) = 0.$$

Then (4.1) gives

$$\lim_{n\to\infty} \operatorname{Im} g_n\left(ze^{\pi i/n}\right) = 0. \tag{4.2}$$

All of the above limits hold uniformly in Γ, and even in a larger subarc of Δ. Because of the uniformity in z, we may replace $ze^{i\pi/n}$ by z. So indeed (2.3)$\Rightarrow$(2.4).

(d) $\Leftrightarrow$(e)

From (3.10) with $\alpha = -\frac{1}{2}$,

$$2g_n(z) = 1 - \frac{\varphi_n\left(ze^{\pi i/n}\right)}{\varphi_n(z)}(1+o(1)) + o(1)$$

and so (2.4)$\Leftrightarrow$(2.5).

(a)$\Leftrightarrow$(f)

Let Γ_1 be a proper subarc of Δ properly containing Γ. Assume first (2.1) holds. We apply Lemma 3.1(c) with all $\zeta_n = z$, so must verify (3.3) there. The first condition in (3.3), with all $\zeta_n = z$, follows immediately from (2.4) - and in turn, we have proved that follows from (2.1). For the second, observe first from Lemma 3.4 and our hypothesis, that

$$\inf\{n(1-|z_{jn}|) : \tau_{jn} \in \Gamma\} \geq C.$$

Then Lemma 3.3(a) shows that the second condition in (3.3) holds with all $\zeta_n = z$. From Lemma 3.1(c), we obtain that every subsequence of positive integers contains a further subsequence $\mathcal{S}$ such that uniformly for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(z\left(1+\frac{u}{n}\right)\right)}{\varphi_n(z)} = e^u + C\left(e^u - 1\right),$$

where

$$C = \lim_{n\to\infty, n\in\mathcal{S}} \left(\frac{z}{n}\frac{\varphi_n'(z)}{\varphi_n(z)} - 1\right).$$

But from (2.4) (which we know follows from(a)), $C = 0$, so the limit is independent of the subsequence, and we have (2.6).

Now conversely assume we have the local limit (2.6). Then setting $u = i\pi/n$ and using the uniformity,

$$\lim_{n\to\infty} \frac{\varphi_n(ze^{i\pi/n})}{\varphi_n(z)} = \lim_{n\to\infty} \frac{\varphi_n(z\left(1+\frac{i\pi}{n}[1+o(1)]\right))}{\varphi_n(z)} = e^{i\pi} = -1,$$

so we have (2.5) and hence the result from the established equivalence (a)$\Leftrightarrow$(e).

(f)$\Rightarrow$(g)

This is a consequence of the fact that e^u has no zeros. Indeed, if there were a subsequence of zeros z_{jn}, $n \in \mathcal{S}$, $j = j(n)$, with $\tau_{jn} \in \Gamma$, and $1 - |z_{jn}| = O\left(\frac{1}{n}\right)$, then writing $z_{jn} = \tau_{jn}(1 + i\alpha_n/n)$, we have $\alpha_n = O(1)$, and by the local limit,

$$0 = \frac{\varphi_n(\tau_{jn}(1+i\alpha_n/n))}{\varphi_n(\tau_{jn})} = e^{\pi i\alpha_n} + o(1),$$

leading to a contradiction.

(g)$\Leftrightarrow$(h)

This is Lemma 3.3(b).

(h)$\Rightarrow$(f)

Now

$$\begin{aligned}\frac{1}{n}g_n'(z) &= \frac{1}{n^2}\frac{d}{dz}\left(\sum_{j=1}^{n}\left[1+\frac{z_{jn}}{z-z_{jn}}\right]\right)\\ &= -\frac{1}{n^2}\sum_{j=1}^{n}\frac{z_{jn}}{(z-z_{jn})^2}.\end{aligned}$$

Let $A > 0$. Our hypothesis gives uniformly for $z \in \Gamma$,

$$\frac{1}{n}g_n'(z) = o(1)$$

and hence for $\zeta, z \in \Gamma$ with $|\zeta - z| \le A/n$,

$$|g_n(z) - g_n(\zeta)| = o(1). \tag{4.3}$$

Next (3.10) with $\alpha = -\frac{1}{2}$ gives

$$2g_n(z) = 1 - \frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n(z)}(1+o(1)) + o(1).$$

Also, replacing z by $ze^{i\pi/n}$ and using (3.10) with $\alpha = \frac{1}{2}$ gives

$$2g_n\left(ze^{i\pi/n}\right) = 1 - \frac{\varphi_n(z)}{\varphi_n\left(ze^{i\pi/n}\right)}(1+o(1)) + o(1).$$

Thus from (4.3),

$$\left|\frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n\left(z\right)}\left(1+o\left(1\right)\right)-\frac{\varphi_n\left(z\right)}{\varphi_n\left(ze^{i\pi/n}\right)}\left(1+o\left(1\right)\right)\right|=o\left(1\right)$$

so that

$$\left(\frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n\left(z\right)}\right)^2=1+o\left(1\right).$$

In view of (3.7) with $\alpha=\frac{1}{2}$, necessarily

$$\lim_{n\to\infty}\frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n\left(z\right)}=-1.$$

Then we have the conclusion (2.5) and the established equivalence (e)$\Leftrightarrow$(f) gives the result. Note that all the limits above hold uniformly in Γ, so we have uniformity in (2.5). ■

5. Proof of Theorem 2.2

Proof of Theorem 2.2

We note that several of the equivalences hold automatically for finitely many n. So we should deal with large enough n.

(a)$\Leftrightarrow$(b)

This is immediate from (3.9) and the continuity of μ'.

(b)$\Leftrightarrow$(c)

This is immediate from (3.6).

(c)$\Leftrightarrow$(d)

This follows from (3.10) with $\alpha=-\frac{1}{2}$, which implies

$$2g_n\left(z\right)-1=-\frac{\varphi_n\left(ze^{\pm i\pi/n}\right)}{\varphi_n\left(z\right)}\left(1+o\left(1\right)\right)+o\left(1\right).$$

(a)$\Rightarrow$(e)

This was proved in Lemma 3.4.

(e)$\Leftrightarrow$(f)

This was proved in Lemma 3.3(a).

(f)$\Rightarrow$(a)

Assume the result is false. Then we can choose $\Gamma\subset\Delta$, a sequence $\mathcal{S}$ of positive integers, and for $n\in\mathcal{S}$, $\zeta_n\in\Gamma$ such that

$$\left|\varphi_n\left(\zeta_n\right)\right|\to\infty.$$

Then from (3.9),

$$\frac{1}{n}R_n\left(\zeta_n\right)\to 0.$$

Let $C > 0$. Let $\{u_n\}$ be a sequence on the unit circle such that $|\zeta_n - u_n| \leq \frac{C}{n}$, $n \geq 1$. We claim that

$$\frac{1}{n} R_n(u_n) \to 0. \tag{5.1}$$

Indeed, let Γ_1 contain Γ as a proper subarc. If $z_{jn} \in \Gamma_1$, then using (2.13) (which we may because of our equivalence (e)$\Leftrightarrow$(f))

$$\left| \frac{u_n - z_{jn}}{\zeta_n - z_{jn}} \right| \leq 1 + \frac{|u_n - \zeta_n|}{1 - |z_{jn}|} \leq C,$$

with a similar lower bound. The terms with $z_{jn} \notin \Gamma_1$ are easier, so $R_n(\zeta_n) \sim R_n(u_n)$ and we have (5.1). Next, using our equivalence (e)$\Leftrightarrow$(f),

$$\begin{aligned} \frac{1}{n^2} \sum_{\tau_{jn} \in \Gamma_1} \frac{1}{|u_n - z_{jn}|^2} &\leq \frac{C}{n} \sum_{\tau_{jn} \in \Gamma_1} \frac{1 - |z_{jn}|^2}{|u_n - z_{jn}|^2} \\ &\leq \frac{C}{n} R_n(u_n) = o(1) \end{aligned}$$

while the tail sum admits the estimate

$$\frac{1}{n^2} \sum_{\tau_{jn} \notin \Gamma_1} \frac{1}{|u_n - z_{jn}|^2} \leq C \frac{n}{n^2} = o(1)$$

so

$$\frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{|u_n - z_{jn}|^2} = o(1).$$

We now proceed much as in the proof of (h)$\Rightarrow$(f) in Theorem 2.1. We have

$$\frac{1}{n} g_n'(u_n) = -\frac{1}{n^2} \sum_{j=1}^{n} \frac{z_{jn}}{(u_n - z_{jn})^2} = o(1).$$

It follows that if $|u_n - \zeta_n| \leq C/n$, and $u_n, \zeta_n \in \Gamma$,

$$|g_n(u_n) - g_n(\zeta_n)| = o(1).$$

Then from (3.10) with $\alpha = \frac{1}{2}$, and appropriate choices of u_n,

$$\begin{aligned} &\left| \frac{\varphi_n\left(\zeta_n e^{i\pi/n}\right)}{\varphi_n(\zeta_n)} (1 + o(1)) - \frac{\varphi_n(\zeta_n)}{\varphi_n\left(\zeta_n e^{i\pi/n}\right)} (1 + o(1)) \right| = o(1) \\ &\Rightarrow \left(\frac{\varphi_n\left(\zeta_n e^{i\pi/n}\right)}{\varphi_n(\zeta_n)} \right)^2 = 1 + o(1) \\ &\Rightarrow \frac{\varphi_n\left(\zeta_n e^{i\pi/n}\right)}{\varphi_n(\zeta_n)} = -1 + o(1), \end{aligned}$$

in view of (3.7) with $\alpha = \frac{1}{2}$. Next, using (3.10),

$$g_n(\zeta_n) = 1 + o(1)$$

which gives using (3.6) that

$$\frac{1}{n} R_n(\zeta_n) = 1 + o(1)$$

and hence using (3.9), that

$$|\varphi_n(\zeta_n)|^2 \mu'(\zeta_n) = 1 + o(1).$$

This contradicts our assumption that $\{\varphi_n(\zeta_n)\}$ is unbounded.■

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