POINTWISE ASYMPTOTICS FOR ORTHONORMAL POLYNOMIALS AT THE ENDPOINTS OF THE INTERVAL VIA UNIVERSALITY

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Abstract. We show that universality limits and bounds for orthonormal polynomials imply pointwise asymptotics for orthonormal polynomials at the endpoints of the interval of orthonormality. As a consequence, we show that if μ is a regular measure supported on $[-1,1]$, and in a neighborhood of 1, μ is absolutely continuous, while for some $\alpha > -1$, $\mu'(t) = h(t)(1-t)^{\alpha}$, where $h(t) \to 1$ as $t \to 1-$, then the corresponding orthonormal polynomials $\{p_n\}$ satisfy the asymptotic

$$\lim_{n\to\infty} \frac{p_n\left(1-\frac{z^2}{2n^2}\right)}{p_n(1)} = \frac{J_\alpha^*(z)}{J_\alpha^*(0)}$$

uniformly in compact subsets of the plane. Here $J_\alpha^*(z) = J_\alpha(z)/z^\alpha$ is the normalized Bessel function of order α. These are by far the most general conditions for such endpoint asymptotics.

1. Results

Let μ be a finite positive Borel measure with compact support, contianing infinitely many points. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

We denote the zeros of p_n by

$$x_{nn} < x_{n-1,n} < ... < x_{2n} < x_{1n}.$$

The $\{p_n\}$ satisfy the three term recurrence relation

$$xp_{n-1}(x) = a_n p_n(x) + b_{n-} p_{n-}(x) + a_{n-1} p_{n-2}(x),$$

where $a_n = \frac{\gamma_{n-1}}{\gamma_n}$ and $b_n \in \mathbb{R}$.

Asymptotics for p_n as $n \to \infty$ are a much studied subject, and have numerous applications. The asymptotic in the interior of the support of μ, is quite different from that at the edges, or in the exterior. In this paper, we focus on asymptotics at the edges.

The best known such asymptotic is the Mehler-Heine formula for classical Jacobi polynomials $\left\{P_n^{(\alpha,\beta)}\right\}$, which are orthogonal with respect to the Jacobi weight

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta, \; x \in (-1,1), \tag{1.1}$$

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and are normalized by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

It has the form [10, p. 192]

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(1-\frac{z^2}{2n^2}\right) = 2^{\alpha} J_{\alpha}^{*}(z),$$

uniformly for z in compact subsets of the plane. Here, J_α is the usual Bessel function of the first kind and order α,

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\alpha}}{n!\Gamma(\alpha+n+1)}, \tag{1.2}$$

and J_α^* is the normalized Bessel function

$$J_\alpha^*(z) = J_\alpha(z)/z^\alpha = 2^{-\alpha}\sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(\alpha+n+1)}. \tag{1.3}$$

For the orthonormal Jacobi polynomials $\left\{p_n^{(\alpha,\beta)}\right\}$, this may be put into the form [1, p. 36]

$$\lim_{n\to\infty} n^{-\left(\alpha+\frac{1}{2}\right)} p_n^{(\alpha,\beta)}\left(1-\frac{z^2}{2n^2}\right) = J_\alpha^*(z).$$

Beyond these and results obtained from the Riemann-Hilbert method, there is not as much known as inside the support (at the endpoints, approximation by Bernstein-Szegő weights does not work, because of the square root factor $\sqrt{1-t^2}$ in such weights).

There is one beautiful general result, due to S. Aptekarev, whose hypotheses involve the recurrence relation. Recall that the *Nevai-Blumenthal class* $\mathcal{M}$ is the set of measures for which

$$\lim_{n\to\infty} a_n = \frac{1}{2} \text{ and } \lim_{n\to\infty} b_n = 0.$$

In particular, Rakhmanov's theorem asserts that this is true when μ is supported on $[-1,1]$ and $\mu' > 0$ a.e. on $[-1,1]$.

Theorem A [1]
Let μ be a measure of class $\mathcal{M}$. Assume that for some $\alpha > 0$, we have as $n\to\infty$,

$$\frac{p_{n+1}(1)}{p_n(1)} = 1 + \frac{\alpha+\frac{1}{2}}{n} + o\left(\frac{1}{n}\right).$$

Then uniformly in compact subsets of the plane

$$\lim_{n\to\infty} n^{-\left(\alpha+\frac{1}{2}\right)} p_n\left(1-\frac{z^2}{2n^2}\right) = J_\alpha^*(z).$$

One of the two main results of this paper is the following, which requires the concept of a regular measure. We say that μ is *regular* (in the sense of Ullmann, Stahl and Totik) [9], if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap\,(\text{supp}\,[\mu])},$$

where *cap* denotes logarithmic capacity, and supp$[\mu]$ denotes the support of μ. In particular, if the support of μ consists of fintely many intervals, and $\mu' > 0$ a.e. in the support, then μ is regular.

Theorem 1.1
Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Assume that for some $\rho > 0$, μ is absolutely continuous in $J = [1-\rho, 1]$, and in J, its absolutely continuous component has the form $w = hw^{(\alpha,0)}$, where $\alpha > -1$ and

$$\lim_{t \to 1-} h(t) = 1. \tag{1.4}$$

Then uniformly for z in compact subsets of $\mathbb{C}$, we have

$$\lim_{n\to\infty} \frac{p_n\left(1 - \frac{z^2}{2n^2}\right)}{p_n(1)} = \frac{J_\alpha^*(z)}{J_\alpha^*(0)}. \tag{1.5}$$

At first this result is surprising, perhaps even suspicious, since one normally expects pointwise asymptotics of orthonormal polynomials to be associated with weights in the Szegő class, with additional conditions. The class of regular weights is far larger than the Szegő class, or even the Nevai-Blumenthal class $\mathcal{M}$. However, on reflection asymptotics at the endpoints are closer to exterior asymptotics, and moreover, we are dividing by $p_n(1)$, which allows for more generality.

Corollary 1.2
Under the hypotheses of Theorem 1.1,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{1 - x_{jn}} = \frac{1}{2\alpha + 2}.$$

Theorem 1.1 is deduced from a result of the author on universality limits in random matrices. The latter involve the reproducing kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)\, p_k(y)$$

and its normalized cousin

$$\widetilde{K}_n(x,y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x,y).$$

On the set of linear Lebesgue measure where $\mu'(x)$ does not exist, we set $\mu'(x) = 0$. We also define the Christoffel function

$$\lambda_n(x) = 1/K_n(x,x).$$

There are different universality limits inside the support of μ (the "bulk" of the spectrum) and at the edges of the support. Kuijlaars and Vanlessen [3] used the Deift-Zhou Riemann-Hilbert method to establish universality limits for Jacobi type weights both inside the support and at the endpoints. Let μ be absolutely continuous, and μ have the form

$$d\mu(x) = h(x)\, w^{(a,\beta)}(x)\, dx = h(x)(1-x)^\alpha (1+x)^\beta\, dx,$$

where h is positive and analytic in $[-1,1]$. At the endpoint 1, they showed that uniformly for a, b in bounded subsets of $(0,\infty)$, as $n \to \infty$, the limit involves the Bessel kernel of order α:

$$\frac{1}{2n^2} \tilde{K}_n\left(1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a,b) + O\left(\frac{a^{\alpha/2} b^{\alpha/2}}{n}\right).$$

Here if $u \neq v$,

$$\mathbb{J}_\alpha(u,v) = \frac{J_\alpha(\sqrt{u})\sqrt{v}J'_\alpha(\sqrt{v}) - J_\alpha(\sqrt{v})\sqrt{u}J'_\alpha(\sqrt{u})}{2(u-v)}, \tag{1.6}$$

while

$$\mathbb{J}_\alpha(u,u) = \frac{1}{4}\left\{J_\alpha^2(\sqrt{u}) - J_{\alpha+1}\left(\sqrt{u}\right)J_{\alpha-1}\left(\sqrt{u}\right)\right\}. \tag{1.7}$$

We shall also need the normalized Bessel kernel

$$\mathbb{J}^*_\alpha(z,v) = \mathbb{J}_\alpha(z,v)/\left\{z^{\alpha/2}v^{\alpha/2}\right\}. \tag{1.8}$$

In [4], we used a comparison method to prove endpoint universality:

Theorem B [4]
Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Assume that for some $\rho > 0$, μ is absolutely continuous in $J = [1-\rho, 1]$, and in J, its absolutely continuous component has the form $w = hw^{(\alpha,\beta)}$, where $\alpha, \beta > -1$. Assume that

$$\lim_{t\to 1-} h(t) = 1. \tag{1.9}$$

Then uniformly for a, b in compact subsets of $(0,\infty)$, we have

$$\lim_{n\to\infty} \frac{1}{2n^2}\tilde{K}_n\left(1-\frac{a}{2n^2}, 1-\frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a,b). \tag{1.10}$$

If $\alpha \geq 0$, we may allow compact subsets of $[0,\infty)$.

In a subsequent paper, we treated more general measures, using a normality method, and proved equivalence of universality on the diagonal and in general:

Theorem C [5]
Let μ have compact support, and that for some $\varepsilon_0 > 0$, the interval $(1, 1+\varepsilon_0)$ lies outside the support. Assume that for some $\rho > 0$, μ is absolutely continuous in $J = [1-\rho, 1]$, and in J, its absolutely continuous component has the form $w = hw^{(\alpha,0)}$, where $\alpha > -1$ and (1.6) holds. The following are equivalent:
(I) For each real a

$$\lim_{n\to\infty} \frac{K_n\left(1-a^2\eta_n, 1-a^2\eta_n\right)}{K_n(1,1)} = \frac{\mathbb{J}^*_\alpha\left(a^2,a^2\right)}{\mathbb{J}^*_\alpha(0,0)}. \tag{1.11}$$

(II) Uniformly for a, b in compact subsets of the complex plane,

$$\lim_{n\to\infty} \frac{K_n\left(1-a^2\eta_n, 1-b^2\eta_n\right)}{K_n(1,1)} = \frac{\mathbb{J}^*_\alpha\left(a^2,b^2\right)}{\mathbb{J}^*_\alpha(0,0)}, \tag{1.12}$$

where

$$\eta_n = \left(\frac{\mathbb{J}^*_\alpha(0,0)}{K_n(1,1)}\right)^{1/(\alpha+1)}. \tag{1.13}$$

Note that for Jacobi weights $w^{(\alpha,\beta)}$,

$$\left(\frac{\mathbb{J}^*_\alpha(0,0)}{K_n(1,1)}\right)^{1/(\alpha+1)} = \frac{1}{2n^2}(1+o(1)).$$

One way to establish these universality limits is to apply asymptotics for orthonormal polynomials at endpoints of the interval of orthogonality. Indeed, the Riemann-Hilbert methods yield that and much more. The possibility of a partial converse, namely of establishing asymptotics for orthonormal polynomial from universality limits, seems much more remote, especially at the endpoints of the interval of orthogonality. In this paper, we show that it is achievable.

Theorem 1.1 is a consequence of a more general result for sequences of measures. Its formulation requires more notation. For $n \geq 1$, let μ_n be a measure with support on the real line. $K_n(\mu_n, x, y)$ will denote the nth reproducing kernel for μ_n, while $p_n(\mu_n, z)$ denotes the orthonormal polynomial of degree n for μ_n. We denote the zeros of $p_n(\mu_n, z)$ by

$$-\infty < x_{nn,n} < x_{n-1,n,n} < \ldots < x_{1n,n} < \infty.$$

Theorem 1.3

Let $a \in (-\infty, 1)$. For $n \geq 1$, let μ_n be a positive measure with support in $[a, 1]$ and infinitely many points in its support. Assume that uniformly for z, w in compact subsets of $\mathbb{C}$, we have

$$\lim_{n\to\infty} \frac{K_n\left(1-\frac{z^2}{2n^2}, 1-\frac{w^2}{2n^2}\right)}{K_n(1,1)} = \frac{\mathbb{J}^*_\alpha(z^2, w^2)}{\mathbb{J}^*_\alpha(0,0)}. \tag{1.14}$$

Then the following are equivalent:
(I)

$$\sup_{n\geq 1} \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{1-x_{jn,n}} < \infty. \tag{1.15}$$

(II)

$$\sup_{n\geq 1} \frac{1}{n^2} \frac{p_n'(\mu_n, 1)}{p_n(\mu_n, 1)} < \infty. \tag{1.16}$$

(III) For each $R > 0$,

$$\sup_{n\geq 1} \sup_{|z|\leq R} \frac{\left|p_n\left(\mu_n, 1+\frac{z}{n^2}\right)\right|}{p_n(\mu_n, 1)} < \infty. \tag{1.17}$$

(IV) Uniformly for z in compact subsets of $\mathbb{C}$, we have

$$\lim_{n\to\infty} \frac{p_n\left(\mu_n, 1-\frac{z^2}{2n^2}\right)}{p_n(\mu_n, 1)} = \frac{J^*_\alpha(z)}{J^*_\alpha(0)}. \tag{1.18}$$

An obvious question is whether we can replace $p_n(1)$ in (1.11) by some multiple of $n^{\alpha+\frac{1}{2}}$. We prove the following as a small step. $[x]$ denotes the greatest integer $\leq x$.

Theorem 1.4

Assume that μ is a measure satisfying the hypotheses of Theorem 1.1. Assume also that μ lies in the Nevai-Blumenthal class. Let

$$d_n = \left|\frac{p_n(1)}{n^{\alpha+\frac{1}{2}}} - \frac{1}{2^{\alpha/2}\Gamma(\alpha+1)}\right|. \tag{1.19}$$

Then

$$\lim_{r\to 1-}\left(\limsup_{n\to\infty}\left(\inf_{[nr]\le j\le n} d_j\right)\right)=0. \tag{1.20}$$

In particular,

$$\liminf_{n\to\infty} d_n=0. \tag{1.21}$$

We note that when there exists n_0 such that $a_n\le\frac{1}{2}$ and $b_n\le 0$ for $n\ge n_0$; or $a_n\ge\frac{1}{2}$ and $b_n\ge 0$ for $n\ge n_0$, then one can show that there exists n_2 such that $\{p_n(1)\}_{n\ge n_2}$ is either increasing or decreasing, and consequently

$$\lim_{n\to\infty} d_n=0.$$

This paper is organised as follows. In the next section, we prove Theorem 1.3. In Section 3, we deduce Theorem 1.1 and Corollary 1.2. In Section 4, we prove Theorem 1.4. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of $n, x, \ldots$. The same symbol does not necessarily denote the same constant in different occurences.

2. Proof of Theorem 1.3

We begin with some more notation. For a given α, we denote the positive zeros of J_α (and hence of J_α^*) by

$$0<j_{\alpha,1}<j_{\alpha,2}<j_{\alpha,3}<\ldots$$

The zeros are all simple, so also

$$J_\alpha^{*\prime}(j_{a,k})\neq 0,\ k\ge 1.$$

We denote the leading coefficient of $p_n(\mu_n,z)$ by $\gamma_n(\mu_n)$. Throughout this section, we assume the hypotheses of Theorem 1.3, and in particular, the universality limit (1.12). The main ideas are contained in the following lemma:

Lemma 2.1
Assume that $\mathcal{S}$ is an infinite subsequence of integers such that uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{\mathcal{S}}\frac{p_n\left(1-\frac{z^2}{2n^2}\right)}{p_n(1)}=f(z). \tag{2.1}$$

(a) Assume $u,z,w\in\mathbb{C}$. Then

$$\mathbb{J}_\alpha^*\left(z^2,w^2\right)\left(z^2-w^2\right)f(u)=\mathbb{J}_\alpha^*\left(u^2,z^2\right)\left(z^2-u^2\right)f(w)+\mathbb{J}_\alpha^*\left(w^2,u^2\right)\left(u^2-w^2\right)f(z). \tag{2.2}$$

(b) Either $f(j_{\alpha,k})=0$ for all k, or $f(j_{\alpha,k})\neq 0$ for all k and for all k,ℓ

$$\frac{f(j_{\alpha,k})}{f(j_{\alpha,\ell})}=\frac{j_{\alpha,k}J_\alpha^{*\prime}(j_{\alpha,k})}{j_{\alpha,\ell}J_\alpha^{*\prime}(j_{\alpha,\ell})}. \tag{2.3}$$

(c) Let

$$G(w,u)=\frac{f(w)}{f(u)}-\frac{J_\alpha^*(w)}{J_\alpha^*(u)}$$

provided $f(u)\, J_\alpha^*(u) \neq 0$. *Then for* $u, z, w \in \mathbb{C}$ *with* $f(u)\, J_\alpha^*(u)\, f(z)\, J_\alpha^*(z) \neq 0$,

$$0 = \mathbb{J}_\alpha^*\left(u^2, z^2\right)\left(z^2 - u^2\right) G(w, u) + \mathbb{J}_\alpha^*\left(w^2, u^2\right)\left(u^2 - w^2\right) G(z, u). \tag{2.4}$$

(d) If $f(j_{\alpha,k}) \neq 0$ *for all* k, *then for all* k, ℓ

$$\frac{f(j_{\alpha,k})}{f(j_{\alpha,\ell})} = \frac{J_\alpha^{*\prime}(j_{\alpha,k})}{J_\alpha^{*\prime}(j_{\alpha,\ell})}. \tag{2.5}$$

(e)

$$f(z) = \frac{J_\alpha^*(z)}{J_\alpha^*(0)}. \tag{2.6}$$

Proof

(a) Now

$$\frac{p_{n-1}}{p_n}(z) - \frac{p_{n-1}}{p_n}(w) = \left[\frac{p_{n-1}}{p_n}(z) - \frac{p_{n-1}}{p_n}(u)\right] + \left[\frac{p_{n-1}}{p_n}(u) - \frac{p_{n-1}}{p_n}(w)\right].$$

We multiply by $\frac{\gamma_{n-1}(\mu_n)}{\gamma_n(\mu_n)}$ and deduce from the Christoffel-Darboux formula that

$$\frac{K_n(z, w)}{p_n(z)\, p_n(w)}(w - z) = \frac{K_n(u, z)}{p_n(z)\, p_n(u)}(u - z) + \frac{K_n(w, u)}{p_n(u)\, p_n(w)}(w - u).$$

Now we replace u, z, w respectively by $1 - \frac{u^2}{2n^2}, 1 - \frac{z^2}{2n^2}, 1 - \frac{w^2}{2n^2}$. Then divide each numerator by $K_n(1,1)$ and each denominator by $(p_n(1))^2$ and then take limits as $n \to \infty$ through $\mathcal{S}$. Assuming $f(z)\, f(u)\, f(w) \neq 0$, we obtain from (1.12) and (2.1),

$$\frac{\mathbb{J}_\alpha^*\left(z^2, w^2\right)\left(z^2 - w^2\right)}{f(z)\, f(w)} = \frac{\mathbb{J}_\alpha^*\left(u^2, z^2\right)\left(z^2 - u^2\right)}{f(z)\, f(u)} + \frac{\mathbb{J}_\alpha^*\left(w^2, u^2\right)\left(u^2 - w^2\right)}{f(u)\, f(w)}.$$

Multiplying by $f(u)\, f(z)\, f(w)$ gives (2.2) when these do not vanish. Analytic continuation gives the result even when they do.

(b) In (2.2), set $z = j_{\alpha,k}$ and $w = j_{\alpha,\ell}$ where k, ℓ are different. The left-hand side vanishes, and we obtain

$$0 = \mathbb{J}_\alpha^*\left(u^2, j_{\alpha,k}^2\right)\left(j_{a,k}^2 - u^2\right) f(j_{\alpha,\ell}) + \mathbb{J}_\alpha^*\left(j_{\alpha,\ell}^2, u^2\right)\left(u^2 - j_{\alpha,\ell}^2\right) f(j_{\alpha,k}).$$

The definition of $\mathbb{J}_\alpha^*$ gives for $u \neq j_{\alpha,k}, j_{\alpha,\ell}$,

$$0 = -\left\{J_\alpha^*(u)\, j_{\alpha,k} J_\alpha^{*\prime}(j_{\alpha,k})\right\} f(j_{\alpha,\ell}) + \left\{J_\alpha^*(u)\, j_{\alpha,\ell} J_\alpha^{*\prime}(j_{\alpha,\ell})\right\} f(j_{\alpha,k})$$

so choosing u such that $J_\alpha^*(u) \neq 0$, and assuming that $f(j_{\alpha,\ell}) \neq 0$, we obtain

$$\frac{j_{\alpha,k} J_\alpha^{*\prime}(j_{\alpha,k})}{j_{\alpha,\ell} J_\alpha^{*\prime}(j_{\alpha,\ell})} = \frac{f(j_{\alpha,k})}{f(j_{\alpha,\ell})}.$$

If some $f(j_{\alpha,\ell}) = 0$ then this also gives $f(j_{\alpha,k}) = 0$ for all k.

(c) Dividing by $f(u)$ in (a),

$$\begin{aligned} \mathbb{J}_\alpha^*\left(z^2, w^2\right)\left(z^2 - w^2\right) &= \mathbb{J}_\alpha^*\left(u^2, z^2\right)\left(z^2 - u^2\right) G(w, u) + \mathbb{J}_\alpha^*\left(w^2, u^2\right)\left(u^2 - w^2\right) G(z, u) \\ &\quad + \mathbb{J}_\alpha^*\left(u^2, z^2\right)\left(z^2 - u^2\right)\frac{J_\alpha^*(w)}{J_\alpha^*(u)} + \mathbb{J}_\alpha^*\left(w^2, u^2\right)\left(u^2 - w^2\right)\frac{J_\alpha^*(z)}{J_\alpha^*(u)}. \end{aligned}$$

(2.7)

Here

$$
\begin{aligned}
& \mathbb{J}_{\alpha}^{*}\left(u^{2}, z^{2}\right)\left(z^{2}-u^{2}\right) \frac{J_{\alpha}^{*}(w)}{J_{\alpha}^{*}(u)}+\mathbb{J}_{\alpha}^{*}\left(w^{2}, u^{2}\right)\left(u^{2}-w^{2}\right) \frac{J_{\alpha}^{*}(z)}{J_{\alpha}^{*}(u)} \\
= & \frac{1}{2 J_{\alpha}^{*}(u)}\left\{\left[J_{\alpha}^{*}(z) u J_{\alpha}^{*\prime}(u)-J_{\alpha}^{*}(u) z J_{\alpha}^{*\prime}(z)\right] J_{\alpha}^{*}(w)+\left[J_{\alpha}^{*}(u) w J_{\alpha}^{*\prime}(w)-J_{\alpha}^{*}(w) u J_{\alpha}^{*\prime}(u)\right] J_{\alpha}^{*}(z)\right\} \\
= & \frac{1}{2}\left\{-z J_{\alpha}^{*\prime}(z) J_{\alpha}^{*}(w)+w J_{\alpha}^{*\prime}(w) J_{\alpha}^{*}(z)\right\}=\mathbb{J}_{\alpha}^{*}\left(z^{2}, w^{2}\right)\left(z^{2}-w^{2}\right) .
\end{aligned}
$$

Thus after cancellation in (2.7), we obtain (2.4).
(d) We let $u=j_{\alpha,k}$ and $z=j_{\alpha,\ell}$ in the identity in (c) and use l'Hospital's rule to define $G\left(j_{\alpha,k}, j_{\alpha,\ell}\right)$, recall J_{α}^{*} has only simple zeros. Assuming that no $j_{\alpha,k}$ is a zero of f, we obtain for all w,

$$
\mathbb{J}_{\alpha}^{*}\left(w^{2}, j_{\alpha,k}^{2}\right)\left(j_{\alpha,k}^{2}-w^{2}\right) G\left(j_{\alpha,k}, j_{\alpha,\ell}\right)=0 .
$$

Assume that we choose $w \neq j_{\alpha,k}$ such that $\mathbb{J}_{\alpha}^{*}\left(w^{2}, j_{\alpha,k}^{2}\right) \neq 0$. We then obtain $G\left(j_{\alpha,k}, j_{\alpha,\ell}\right)=0$, so

$$
\frac{f\left(j_{\alpha,k}\right)}{f\left(j_{\alpha,\ell}\right)}=\frac{J_{\alpha}^{*\prime}\left(j_{\alpha,k}\right)}{J_{\alpha}^{*\prime}\left(j_{\alpha,\ell}\right)} .
$$

(e) Using (b), this gives,

$$
\frac{j_{\alpha,k}}{j_{\alpha,\ell}}=1
$$

for all k, ℓ, a contradiction. It follows that f must vanish at all $j_{\alpha,k}$. Next, set $w=j_{\alpha,k}$ and $u=0$ in (2.2). Since $f(0)=1$, this gives

$$
\mathbb{J}_{\alpha}^{*}\left(z^{2}, j_{\alpha,k}^{2}\right)\left(z^{2}-j_{\alpha,k}^{2}\right)=\mathbb{J}_{\alpha}^{*}\left(j_{\alpha,k}^{2}, 0\right)\left(-j_{\alpha,k}^{2}\right) f(z)
$$

so

$$
\left\{J_{\alpha}^{*}(z) j_{\alpha,k} J_{\alpha}^{*\prime}\left(j_{\alpha,k}\right)\right\}=\left\{J_{\alpha}^{*}(0) j_{\alpha,k} J_{\alpha}^{*\prime}\left(j_{\alpha,k}\right)\right\} f(z)
$$

so

$$
f(z)=\frac{J_{\alpha}^{*}(z)}{J_{\alpha}^{*}(0)} .
$$

■

We note that taking scaling limits in the usual form of the Christoffel-Darboux formula does not yield (2.6) - one obtains an extra factor of n, which is cancelled out in taking the difference.

Proof of Theorem 1.3
We start with
(III)⇒(IV)
The normality assumed in (III) ensures that from every subsequence of integers, we can choose another subsequence S for which (2.1) holds. From Lemma 2.1, we have the limit (2.6). Since the limit is independent of the subsequence, we obtain the limit for the full sequence of positive integers.
(IV)⇒(III)
The limit (1.16) implies the uniform boundedness in (1.15).

(I)⇒(III)
For $|z| \leq R$,

$$\begin{aligned}
\log \frac{\left|p_n\left(1+\frac{z}{n^2}\right)\right|}{p_n(1)} &= \sum_{j=1}^{n} \log\left|1+\frac{z}{n^2\left(1-x_{jn,n}\right)}\right| \\
&\leq \sum_{j=1}^{n} \log\left(1+\frac{|z|}{n^2\left(1-x_{jn,n}\right)}\right) \\
&\leq \frac{R}{n^2}\sum_{j=1}^{n} \frac{1}{1-x_{jn,n}}.
\end{aligned}$$

Then (1.13) implies the uniform boundedness in (1.15). Of course, we are also using that all zeros lie in $(a,1)$.
(III)⇒(II)
The uniform boundedness in compact subsets of $\{f_n\}$, where

$$f_n(z) = \frac{p_n\left(1+\frac{z}{n^2}\right)}{p_n(1)}$$

also implies the uniform boundedness in compact subsets of $\{f_n'\}$. In particular, then

$$\sup_n |f_n'(0)| < \infty,$$

that is

$$\sup_n \frac{1}{n^2}\left|\frac{p_n'(1)}{p_n(1)}\right| < \infty.$$

(II)⇒(I)
We use the identity

$$\frac{p_n'(1)}{p_n(1)} = \sum_{j=1}^{n} \frac{1}{1-x_{jn,n}}$$

so (1.13) follows from (1.14). ■

3. Proof of Theorem 1.1 and Corollary 1.2

In the next two lemmas, we assume the hypotheses of Theorem 1.1. We begin by recalling Christoffel function limits and estimates:

Lemma 3.1
(a) Let $R>0$. Uniformly for $a \in [0,R]$,

$$\lim_{n\to\infty} \lambda_n\left(1-\frac{a}{2n^2}\right) n^{2\alpha+2} = 2^{\alpha+1}\mathbb{J}_\alpha^*(a,a). \tag{3.1}$$

(b) There exists $\eta'>0$ and $C>0$ such that for $n \geq 1$ and $x \in [1-\eta',1]$,

$$\lambda_n(x) \geq \frac{C}{n}\left(1-x+\frac{1}{n^2}\right)^{\alpha+\frac{1}{2}}.$$

Proof
(a) From Theorem A,

$$\lim_{n\to\infty}\left(\frac{1}{2n^2}\right)^{1+\alpha} K_n\left(1-\frac{a}{2n^2}, 1-\frac{a}{2n^2}\right) = \mathbb{J}_\alpha^*(a,a),$$

which is equivalent to the stated result.
(b) Choose η_1 such that

$$\mu'(x) \geq \frac{1}{2}(1-x)^{\alpha}, \; x \in [1-\eta_1, 1].$$

Define the measure ν on $[1-\eta_1, 1]$ by

$$\nu'(x) = (1-x)^{\alpha}, \; x \in [1-\eta_1, 1].$$

This is a Jacobi weight after translation of the interval and multiplication by a constant. Using estimates of the Christoffel functions of Jacobi weights [7, p. 94, 108], and translating the interval, we obtain for any $0 < \eta' < \eta_1$,

$$\lambda_n(x) \geq \lambda_n(\nu, x) \geq \frac{C}{n}\left(1 - x + \frac{1}{n^2}\right)^{\alpha+\frac{1}{2}}, \; x \in [1-\eta', 1].$$

■

Lemma 3.2
There exists $\varepsilon > 0$ such that for $n \geq 1$ and polynomials P of degree $\leq n-1$,

$$\int_{1-\varepsilon n^{-2}}^{1} P^2(x)\, d\mu(x) \leq \frac{1}{2}\int_a^1 P^2(x)\, d\mu(x) \tag{3.2}$$

Proof
Using the variational property of Christoffel functions, namely

$$P^2(x) \leq \lambda_n^{-1}(x) \int_a^1 P^2(x)\, d\mu(x),$$

and the form of our measure in $[1-\eta, 1]$, we have for large enough n,

$$\begin{aligned} \int_{1-\varepsilon n^{-2}}^{1} P^2(x)\, d\mu(x) &\leq \left(\int_{1-\varepsilon n^{-2}}^{1} \lambda_n^{-1}(x)\, h(x)(1-x)^{\alpha}\, dx\right) \int_a^1 P^2(x)\, d\mu(x) \\ &\leq Cn\left(\int_{1-\varepsilon n^{-2}}^{1} (1-x)^{-\frac{1}{2}}\, dx\right) \int_a^1 P^2(x)\, d\mu(x) \\ &\leq C\varepsilon^{\frac{1}{2}} \int_a^1 P^2(x)\, d\mu(x), \end{aligned}$$

by Lemma 3.1, where C is independent of ε. Choosing ε small enough gives the result. ■

Proof of Theorem 1.1
Theorem A and (1.6) give for $a, b \in (0, \infty)$,

$$\lim_{n\to\infty} \frac{1}{(2n^2)^{1+\alpha}} K_n\left(1 - \frac{a^2}{2n^2}, 1 - \frac{b^2}{2n^2}\right) = \mathbb{J}_{\alpha}^{*}(a, b).$$

Next, using Lemma 3.1(a),

$$\eta_n = \left(\frac{\mathbb{J}_{\alpha}^{*}(0,0)}{K_n(1,1)}\right)^{1/(\alpha+1)} = \frac{1}{2n^2}(1 + o(1))$$

so the uniform convergence in Theorem B gives

$$\lim_{n\to\infty} \frac{K_n\left(1 - \frac{a^2}{2n^2}, 1 - \frac{b^2}{2n^2}\right)}{K_n(1,1)} = \frac{\mathbb{J}_{\alpha}^{*}(a^2, b^2)}{\mathbb{J}_{\alpha}^{*}(0,0)},$$

uniformly for a, b in compact subsets of $\mathbb{C}$. The result follows from Theorem 1.3 if we can show that

$$\sup_n \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{1-x_{jn}} < \infty. \tag{3.3}$$

First we use the extremal property of the largest zero, which implies that

$$1 - x_{1n} = \inf_{\deg(P) \le n-1} \frac{\int_a^1 (1-x) P^2(x) \, d\mu(x)}{\int_a^1 P^2(x) \, d\mu(x)}.$$

By Lemma 3.2, for such polynomials P,

$$\begin{aligned} \int_a^1 P^2(x) \, d\mu(x) &= \left(\int_a^{1-\varepsilon n^{-2}} + \int_{1-\varepsilon n^{-2}}^1 \right) P^2(x) \, d\mu(x) \\ &\le \int_a^{1-\varepsilon n^{-2}} P^2(x) \, d\mu(x) + \frac{1}{2} \int_a^1 P^2(x) \, d\mu(x) \end{aligned}$$

so

$$\int_a^1 P^2(x) \, d\mu(x) \le 2 \int_a^{1-\varepsilon n^{-2}} P^2(x) \, d\mu(x).$$

Hence

$$1 - x_{1n} \ge \inf_{\deg(P) \le n-1} \frac{\int_a^{1-\varepsilon n^{-2}} (1-x) P^2(x) \, d\mu(x)}{2 \int_a^{1-\varepsilon n^{-2}} P^2(x) \, d\mu(x)} = \frac{\varepsilon n^{-2}}{2}. \tag{3.4}$$

One can use a similar variational argument for other zeros, but we instead use the Markov-Stieltjes inequalities [2, p. 33] in the form

$$\lambda_n(x_{jn}) \le \int_{x_{j+1,n}}^{x_{j-1,n}} d\mu(t).$$

If $x_{jn} \in J$, this gives

$$\begin{aligned} \lambda_n(x_{jn}) &\le \int_{x_{j+1,n}}^{x_{j-1,n}} d\mu(t) \le (x_{j-1,n} - x_{j+1,n}) \sup_{[x_{j+1,n}, x_{j-1,n}]} \mu'(t) \\ &\le C (x_{j-1,n} - x_{j+1,n}) \sup_{t \in [x_{j+1,n}, x_{j-1,n}]} (1-t)^{\alpha}. \end{aligned} \tag{3.5}$$

By Lemma 3.1(b),

$$\lambda_n(x_{jn}) \ge \frac{C}{n} (1 - x_{jn})^{\alpha + \frac{1}{2}}, \; x_{jn} \in [1 - \eta', 1].$$

Then

$$x_{j-1,n} - x_{j+1,n} \ge \frac{C}{n} (1 - x_{jn})^{\frac{1}{2}} \inf_{t \in [x_{j+1,n}, x_{j-1,n}]} \left(\frac{1 - x_{jn}}{1-t} \right)^{\alpha}.$$

If first for $t \in [x_{j+1,n}, x_{j-1,n}]$,

$$2 \ge \frac{1 - x_{jn}}{1-t} \ge \frac{1}{2}, \tag{3.6}$$

then

$$x_{j-1,n} - x_{j+1,n} \ge \frac{C}{n 2^{|\alpha|}} (1 - x_{jn})^{\frac{1}{2}} \ge \frac{C}{n} \max_{t \in [x_{j+1,n}, x_{j-1,n}]} (1-t)^{1/2} \tag{3.7}$$

and

$$\begin{aligned}\frac{1}{1-x_{jn}} &\leq \frac{C}{\max_{t\in[x_{j+1,n},x_{j-1,n}]}(1-t)} \\ &\leq \frac{Cn\left(x_{j-1,n}-x_{j+1,n}\right)}{\max_{t\in[x_{j+1,n},x_{j-1,n}]}(1-t)^{3/2}} \leq Cn\int_{x_{j+1,n}}^{x_{j-1,n}}\frac{dt}{(1-t)^{3/2}}. \qquad (3.8)\end{aligned}$$

If (3.6) fails, then either

$$\frac{1-x_{jn}}{1-x_{j-1,n}} > 2 \text{ or } \frac{1-x_{jn}}{1-x_{j+1,n}} < \frac{1}{2}.$$

In the first case,

$$\begin{aligned}x_{j-1,n}-x_{jn} &= (1-x_{jn})-(1-x_{j-1,n}) \\ &\geq (1-x_{jn})-\frac{1}{2}(1-x_{jn}) \\ &= \frac{1}{2}(1-x_{jn}) \geq \frac{C}{n}(1-x_{jn})^{1/2},\end{aligned}$$

in view of (3.3). Then

$$\frac{1}{1-x_{jn}} \leq \frac{Cn\left(x_{j-1,n}-x_{jn}\right)}{(1-x_{jn})^{3/2}} \leq Cn\int_{x_{jn}}^{x_{j-1,n}}\frac{dt}{(1-t)^{3/2}}. \qquad (3.9)$$

In the second case,

$$\begin{aligned}x_{jn}-x_{j+1,n} &= (1-x_{j+1,n})-(1-x_{jn}) \\ &\geq \frac{1}{2}(1-x_{j+1,n}) \geq \frac{1}{2}(1-x_{jn}),\end{aligned}$$

so

$$\begin{aligned}\frac{1}{1-x_{jn}} &\leq \frac{C}{(1-x_{jn})^{1/2}}\int_{x_{jn}-\frac{1}{2}(1-x_{jn})}^{x_{jn}}\frac{1}{(1-t)^{3/2}}dt \\ &\leq Cn\int_{x_{j+1,n}}^{x_{jn}}\frac{1}{(1-t)^{3/2}}dt. \qquad (3.10)\end{aligned}$$

Considering all the above cases, and adding over j with $x_{jn}\in[1-\eta',1]$, gives

$$\begin{aligned}&\sum_{j\geq 2,x_{jn}\in[1-\eta',1]}\frac{1}{1-x_{jn}} \\ &\leq Cn\int_{1-\eta}^{x_{1n}}\frac{1}{(1-t)^{3/2}}dt \\ &\leq Cn(1-x_{1n})^{-1/2} \leq Cn^2. \qquad (3.11)\end{aligned}$$

Next,

$$\sum_{j\geq 2,x_{jn}\leq 1-\eta'}\frac{1}{1-x_{jn}} \leq n/\eta'.$$

Together with (3.4) and (3.10), this gives (3.3). ■

Proof of Corollary 1.2

Because of the uniform convergence, we can differentiate the asymptotic (1.11): uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{z p_n'\left(1-\frac{z^2}{2n^2}\right)}{n^2 p_n(1)} = -\frac{J_\alpha^{*\prime}(z)}{J_\alpha^*(0)},$$

so dividing by z, and recalling that $J_\alpha^{*\prime}(0) = 0$,

$$\lim_{n\to\infty} \frac{p_n'(1)}{n^2 p_n(1)} = -\frac{J_\alpha^{*\prime\prime}(0)}{J_\alpha^*(0)} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha+2)} = \frac{1}{2\alpha+2},$$

which gives the result. ■

4. **Proof of Theorem 1.4**

I could not find the following result, though am sure it is well known:

Lemma 4.1
Assume that μ is supported on $[-1,1]$ and lies in $\mathcal{M}$. Then

$$\lim_{n\to\infty} \frac{p_{n-1}(1)}{p_n(1)} = 1.$$

Proof
We first note that $\frac{p_{n-1}(x)}{p_n(x)}$ is decreasing in $(1,\infty)$. Indeed this follows from the following identity, a consequence of the Lagrange interpolation formula and the confluent form of the Christoffel-Darboux formula:

$$\frac{p_{n-1}(x)}{p_n(x)} = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=1}^{n} \frac{\lambda_n(x_{jn})\, p_{n-1}^2(x_{jn})}{x - x_{jn}}.$$

Let $\varphi(x) = x + \sqrt{x^2-1}, x \in (1,\infty)$. It is known [7, p. 33] that for $x \in (1,\infty)$,

$$\lim_{n\to\infty} \frac{p_{n-1}(x)}{p_n(x)} = \varphi(x)^{-1}.$$

Then for $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{p_{n-1}(1)}{p_n(1)} \geq \liminf_{n\to\infty} \frac{p_{n-1}(1+\varepsilon)}{p_n(1+\varepsilon)} = \varphi(1+\varepsilon)^{-1}.$$

Letting $\varepsilon \to 0+$, gives

$$\liminf_{n\to\infty} \frac{p_{n-1}(1)}{p_n(1)} \geq 1.$$

Next, let

$$\tau := \limsup_{n\to\infty} \frac{p_{n-1}(1)}{p_n(1)}.$$

We use the recurrence relation in the form

$$p_n(1)(1-b_n) = a_n p_{n+1}(1) + a_{n-1} p_{n-1}(1)$$

so since $a_n \to \frac{1}{2}$ and $b_n \to 0$ as $n \to \infty$,

$$\begin{aligned} 1 + o(1) &= \left(\frac{1}{2} + o(1)\right) \frac{p_{n+1}(1)}{p_n(1)} + \left(\frac{1}{2} + o(1)\right) \frac{p_{n-1}(1)}{p_n(1)} \\ &\geq \left(\frac{1}{2} + o(1)\right) (\tau + o(1))^{-1} + \left(\frac{1}{2} + o(1)\right) \frac{p_{n-1}(1)}{p_n(1)}. \end{aligned}$$

Letting $n \to \infty$ through an appropriate sequence of integers gives

$$1 \geq \frac{1}{2}\left(\tau^{-1}+\tau\right) \Rightarrow \tau = 1.$$

Thus

$$1 = \limsup_{n\to\infty} \frac{p_{n-1}(1)}{p_n(1)} \geq \liminf_{n\to\infty} \frac{p_{n-1}(1)}{p_n(1)} \geq 1.$$

■

Proof of Theorem 1.4
Fix $r \in (0,1)$. Let

$$A = (2\alpha+2)\, 2^{1+\alpha} \mathbb{J}^*_\alpha(0,0) = \frac{1}{2^\alpha \Gamma(\alpha+1)^2},$$

see [5, p. 4, (1.10)] and

$$c_k = \left(\frac{p_k(1)}{k^{\alpha+\frac{1}{2}}}\right)\frac{1}{\sqrt{A}},\ k \geq 1.$$

We use the confluent Christoffel-Darboux formula in the form

$$\frac{p_k'(1)}{p_k(1)} - \frac{p_{k-1}'(1)}{p_{k-1}(1)} = \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^{-1} \frac{K_k(1,1)}{p_k(1)\,p_{k-1}(1)}.$$

Then adding for $k = [nr]$,$[nr]+1, ..., n$, gives

$$\frac{p_n'(1)}{p_n(1)} - \frac{p_{[nr]}'(1)}{p_{[nr]}(1)} = \sum_{k=[nr]+1}^{n} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^{-1} \frac{K_k(1,1)}{p_k(1)\,p_{k-1}(1)}.$$

Applying Corollary 1.2, the previous lemma, our asymptotic for Christoffel functions at 1, and the fact that μ lies in $\mathcal{M}$, gives

$$\frac{n^2}{2\alpha+2}\left(1-r^2\right)(1+o(1)) = \sum_{k=[nr]+1}^{n} \frac{2^{2+\alpha}\mathbb{J}^*_\alpha(0,0)\,k^{2+2\alpha}}{p_k^2(1)(1+o(1))}$$

so that

$$\frac{1-r^2}{2}(1+o(1)) = \frac{1}{n}\sum_{k=[nr]+1}^{n} \frac{1}{c_k^2}\frac{k}{n}. \tag{4.1}$$

Next, we use

$$K_{n+1}(1,1) - K_{[nr]}(1,1) = \sum_{k=[nr]+1}^{n} p_k^2(1)$$

and our asymptotics to obtain

$$2^{1+\alpha}\mathbb{J}^*_\alpha(0,0)\,n^{2+2\alpha}\left(1-r^{2+2\alpha}\right)(1+o(1)) = \sum_{k=[nr]+1}^{n} p_k^2(1)$$

$$\Rightarrow \frac{1-r^{2+2\alpha}}{2+2\alpha}(1+o(1)) = \frac{1}{n}\sum_{k=[nr]+1}^{n} c_k^2\left(\frac{k}{n}\right)^{2\alpha+1}.$$

This and (4.1) give

$$\begin{aligned}
&\frac{1}{n(1-r)}\sum_{k=[nr]+1}^{n}\left(\frac{1}{c_k}\left(\frac{k}{n}\right)^{1/2}-c_k\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right)^2\\
=\;&\frac{1}{n(1-r)}\sum_{k=[nr]+1}^{n}\frac{1}{c_k^2}\frac{k}{n}+\frac{1}{n(1-r)}\sum_{k=[nr]+1}^{n}c_k^2\left(\frac{k}{n}\right)^{2\alpha+1}-\frac{2}{n(1-r)}\sum_{k=[nr]+1}^{n}\left(\frac{k}{n}\right)^{\alpha+1}\\
=\;&\frac{1+r}{2}(1+o(1))+\frac{1-r^{2+2\alpha}}{1-r}\left(\frac{1+o(1)}{2+2\alpha}\right)-\frac{2}{1-r}\int_r^1 x^{\alpha+1}dx\,(1+o(1))\\
=\;&\frac{1+r}{2}(1+o(1))+\frac{1-r^{2+2\alpha}}{1-r}\left(\frac{1+o(1)}{2+2\alpha}\right)-2\frac{1-r^{\alpha+2}}{1-r}\left(\frac{1+o(1)}{2+\alpha}\right)
\end{aligned}$$

so

$$\lim_{r\to1-}\left(\limsup_{n\to\infty}\frac{1}{n(1-r)}\sum_{k=[nr]+1}^{n}\left(\frac{1}{c_k}\left(\frac{k}{n}\right)^{1/2}-c_k\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right)^2\right)=0.$$

Then also

$$\lim_{r\to1-}\left(\limsup_{n\to\infty}\left(\inf_{[nr]+1\le k\le n}\left(\frac{1}{c_k}\left(\frac{k}{n}\right)^{1/2}-c_k\left(\frac{k}{n}\right)^{\alpha+\frac{1}{2}}\right)^2\right)\right)=0.$$

Since $r\le\frac{k}{n}\le1$ for $[nr]+1\le k\le n$, and r is close to 1, we also obtain

$$\lim_{r\to1-}\left(\limsup_{n\to\infty}\left(\inf_{[nr]+1\le k\le n}\left(\frac{1}{c_k}-c_k\right)^2\right)\right)=0.$$

Using the inequality

$$\left(\frac{1}{x}-x\right)^2=(1-x)^2\left(\frac{1+x}{x}\right)^2\ge(1-x)^2,$$

we deduce

$$\lim_{r\to1-}\left(\limsup_{n\to\infty}\left(\inf_{[nr]+1\le k\le n}(1-c_k)^2\right)\right)=0.$$

This is equivalent to the conclusion of Theorem 1.4. The assertion (1.19) about lim inf's also follows. ■

Remark

The circle of ideas of this paper are also useful inside the support of the measure [6].

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