# CORRECTION TO "LOCAL ASYMPTOTICS FOR ORTHONORMAL POLYNOMIALS ON THE UNIT CIRCLE VIA UNIVERSALITY"

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There is a mistake in the proof of Lemma 4.2(a) in [1], namely there $\overline{t_{jn}} z_{jn} = 1$, so that the denominator in $K_n(z_{jn}, t_{jn})$ is 0. This causes a gap in the proofs of Lemma 4.3(d), and Theorems 1.1 and 1.2 (but not Theorems 1.3 and 1.4). Below we give replacements for Lemma 4.2(a), Lemma 4.3(d) and revised proofs of Theorems 1.1 and 1.2. Note that the rest of those lemmas, including the hypotheses, remain the same.

**Revised Lemma 4.2(a)**

*Let $\rho > 0$ and $\mathcal{L}_n = \left\{ re^{i\theta} : 1 - \frac{\rho}{n} \leq r \leq 1 \text{ and } \theta \in J \right\}$. There exist $C_0, n_0$ such that for $n \geq n_0$ and $z_{jn}, z_{kn} \in \mathcal{L}_n$ with $j \neq k$, we have*

$$|z_{jn} - z_{kn}| \geq C_0/n. \tag{4.4}$$

*In particular all zeros of $\varphi_n$ in $\mathcal{L}_n$ are simple.*

**Proof**

If the result is false, we can find a subsequence of integers $\mathcal{S}$ and for $n \in \mathcal{S}$, $z_{jn}, z_{kn} \in \mathcal{L}_n$ with $j \neq k$, $j = j(n), k = k(n)$, such that $|z_{jn} - z_{kn}| = o\left(\frac{1}{n}\right)$. Suppose first $z_{jn} \neq z_{kn}$. Let $\tau_n = z_{jn}/|z_{jn}|$. Write $z_{jn} = \tau_n(1 + 2\pi i \alpha_n/n)$ and $z_{kn} = \tau_n(1 + 2\pi i \beta_n/n)$, so that $|\alpha_n - \beta_n| \to 0$, $n \to \infty, n \in \mathcal{S}$. From the Christoffel-Darboux formula (2.1) and the uniform universality limit (1.1),

$$0 = \frac{K_n\left(z_{jn}, \frac{1}{\overline{z_{kn}}}\right)}{K_n(\tau_n, \tau_n)} = e^{\iota\pi(\alpha_n - \beta_n(1+o(1)))} \mathbb{S}(\alpha_n - \beta_n(1 + o(1))) + o(1) = 1 + o(1).$$

Thus we have a contradiction. Next, if $z_{jn} = z_{kn}$, then $\varphi_n$ has at least a double zero at $z_{jn}$. Then $\overline{\varphi_n^*\left(\frac{1}{\bar{z}_{jn}} + 2\pi i \tau_n \frac{\bar{v}}{n}\right)}$ has at least a double zero at $v = 0$, so

$$\frac{K_n\left(z_{jn}, \frac{1}{\bar{z}_{jn}} + 2\pi i \tau_n \frac{\bar{v}}{n}\right)}{K_n(\tau_n, \tau_n)} = \frac{\overline{\varphi_n^*\left(\frac{1}{\bar{z}_{jn}} + 2\pi i \tau_n \frac{\bar{v}}{n}\right)} \varphi_n^*(z_{jn})}{2\pi i \tau_n \frac{v}{n} z_{jn}}$$

has at least a simple zero at $v = 0$. However, this contradicts the universality limit, which shows

$$\frac{K_n\left(z_{jn}, \frac{1}{\bar{z}_{jn}} + 2\pi i \tau_n \frac{\bar{v}}{n}\right)}{K_n(\tau_n, \tau_n)} = e^{-\iota\pi v} \mathbb{S}(v) + o(1).$$

■

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**Revised Lemma 4.3(d)**
*Let $A > 0$. There exist $n_0$, $C > 0$ such that for $n \geq n_0$ and $\zeta_n \in J_1$, with $|\varphi_n(\zeta_n)| \geq A$,*

$$\frac{1}{n^2} \sum_{j=1, j \neq j_1}^{n} \frac{1}{|\zeta_n - z_{jn}|^2} \leq C. \tag{4.15}$$

*Here $z_{j_1 n}$ is the closest zero of $\varphi_n$ to $\zeta_n$.*

**Proof**

Let $\mathcal{L}_n$ be as in Lemma 4.2(a). Represent the zeros of $\varphi_n$ in $\mathcal{L}_n$ in terms of increasing distance to $\zeta_n$, say, as $z_{j_k n}$, $1 \leq k \leq k_0$. Here $k_0$ depends on $n$. In view of Lemma 4.2(a) and the fact that $\mathcal{L}_n$ has width $\frac{\rho}{n}$, with $\rho$ fixed, any sector $\left\{ re^{i\theta} : 1 - \frac{\rho}{n} \leq r \leq 1 \text{ and } \theta \in [\alpha, \beta] \right\}$ contained in $\mathcal{L}_n$ can contain at most $\left( \frac{2\rho}{C_0} + 1 \right) \left( \frac{2(\beta - \alpha)n}{C_0} + 1 \right)$ zeros of $\varphi_n$. It follows that there exists $C_1 > 0$ depending on $\rho$ and on $C_0$ in Lemma 4.2(a), but not on $n, \zeta_n$, such that

$$|\zeta_n - z_{j_k n}| \geq C_1 \frac{k}{n}, 2 \leq k \leq k_0.$$

Then

$$\frac{1}{n^2} \sum_{k=2}^{k_0} \frac{1}{|\zeta_n - z_{j_k n}|^2} \leq \frac{1}{C_1^2} \sum_{k=2}^{\infty} \frac{1}{k^2}.$$

Next we deal with the zeros $z_{jn} = |z_{jn}| e^{i\theta_{jn}}$ of $\varphi_n$ with $|z_{jn}| < 1 - \frac{\rho}{n}$ and $e^{i\theta_{jn}} \in J_1$. Summing over these zeros, we have

$$\frac{1}{n^2} \sum \frac{1}{|\zeta_n - z_{jn}|^2} \leq \frac{1}{\rho n} R_n(\zeta_n) \leq C,$$

by (4.8) and as $|\varphi_n(\zeta_n)| \geq A$. For the remaining zeros, their distance to $\zeta_n$ is bounded below by $C_2 > 0$, independent of $n$. Summing over such zeros, we obtain

$$\frac{1}{n^2} \sum \frac{1}{|\zeta_n - z_{jn}|^2} \leq \frac{n}{n^2 C_2^2} = o(1).$$

Adding the three sums gives the result. ■

**Revised Proof of Theorem 1.1**

We need to show that the conditions (1.10) of Theorem 1.3 are satisfied for large $n$ and for $\zeta_n = z_n$ or $\zeta_n = z_n e^{i\pi/n}$. We do this below. Then by Theorem 1.3, from any subsequence of integers, we can extract another subsequence $\mathcal{S}$ for which (1.11) holds. Moreover, from Lemma 4.3(a),

$$|C| = \lim_{n \to \infty, n \in \mathcal{S}} \left| \frac{\zeta_n \varphi_n'(\zeta_n)}{n \varphi_n(\zeta_n)} - 1 \right| \leq 1, \tag{*}$$

provided $\left| \varphi_n\left(\zeta_n e^{\pm i\pi/n}\right) / \varphi_n(\zeta_n) \right| \leq 1$ in the right-hand side of (4.12). We turn to the proof of the latter and (1.10). Let

$$\zeta_n = \begin{cases} z_n, & \text{if } |\varphi_n(z_n)| \geq \left| \varphi_n\left(z_n e^{i\pi/n}\right) \right| \\ z_n e^{i\pi/n}, & \text{otherwise} \end{cases}.$$

By (4.12), with appropriate choice of $z$, we have (*), and then the first condition in (1.10) follows. Next (4.7) shows

$$|\varphi_n(\zeta_n)|^2 \geq \left|\varphi_n(z_n)\varphi_n\left(z_n e^{i\pi/n}\right)\right| \geq \frac{1+o(1)}{\mu'(z_n)}.$$

Then the second condition in (1.10) follows from the revised Lemma 4.3(d), provided

$$|z_{j_1 n} - \zeta_n| \geq C_2/n,$$

where $z_{j_1 n}$ is the closest zero of $\varphi_n$ to $\zeta_n$. Suppose this fails for a subsequence $\mathcal{S}$ of integers so that as $n \to \infty, n \in \mathcal{S}$, $|z_{j_1 n} - \zeta_n| = o(1/n)$ and $1 - |z_{j_1 n}| = o\left(\frac{1}{n}\right)$. Then the universality limit (1.1) gives for such $n$,

$$\frac{K_n(z_{j_1 n}, \zeta_n)}{K_n(\zeta_n, \zeta_n)} = 1 + o(1).$$

From the Christoffel-Darboux formula, and as $|\varphi_n^*(\zeta_n)| = |\varphi_n(\zeta_n)|$,

$$(\#) \qquad |\varphi_n^*(z_{j_1 n})\varphi_n(\zeta_n)| = \left|\frac{K_n(z_{j_1 n}, \zeta_n)}{K_n(\zeta_n, \zeta_n)}\right| K_n(\zeta_n, \zeta_n)\left|1 - \overline{\zeta_n} z_{j_1 n}\right| = o(1)$$

by (4.10). Next, by the universality limit, and the fact that it holds uniformly,

$$\left|\frac{K_n\left(z_{j_1 n}, \zeta_n e^{\pm i\pi/n}\right)}{K_n(\zeta_n, \zeta_n)}\right| = \left|\mathbb{S}\left(\frac{1}{2}\right)\right| + o(1) = \frac{2}{\pi} + o(1),$$

while

$$\left|\varphi_n^*(z_{j_1 n})\varphi_n\left(\zeta_n e^{\pm i\pi/n}\right)\right| = \left|\frac{K_n\left(z_{j_1 n}, \zeta_n e^{\pm i\pi/n}\right)}{K_n(\zeta_n, \zeta_n)}\right| K_n(\zeta_n, \zeta_n)\left|1 - \overline{\zeta_n e^{\pm i\pi/n}} z_{j_1 n}\right| \geq C.$$

This and (#) show

$$(\#\#) \qquad \left|\frac{\varphi_n(\zeta_n)}{\varphi_n\left(\zeta_n e^{\pm i\pi/n}\right)}\right| = o(1),$$

contradicting the choice of $\zeta_n$. ■

**Revised Proof of Theorem 1.2**

Only the proof of (IV)⇒(I) requires changes. From (4.8) and our hypothesis (1.8), we have (4.20), so from (4.9),

$$\lim_{n\to\infty, n\in\mathcal{S}} \operatorname{Re}\left(\frac{\zeta_n \varphi_n'(\zeta_n)}{n\varphi_n(\zeta_n)} - 1\right) = 0.$$

We also assumed (1.7), so that (4.19) follows from this last limit. Then we have the first condition in (1.10). Next, (4.20) shows that $|\varphi_n(\zeta_n)| \geq C_3$. Then the second condition in (1.10) follows from the revised Lemma 4.3(d), provided we can show that the closest zero $z_{j_1 n}$ of $\varphi_n$ to $\zeta_n$ satisfies $|z_{j_1 n} - \zeta_n| \geq \frac{C_4}{n}$. If this fails for a subsequence, then as in the proof of Theorem 1.1, (##) holds. This contradicts the consequence of (4.19) and (4.12) that

$$\frac{\varphi_n\left(\zeta_n e^{\pm i\pi/n}\right)}{\varphi_n(\zeta_n)} = -1 + o(1).$$

So, the conditions (1.10) of Theorem 1.3 are fulfilled. By Theorem 1.3, from every subsequence of $\mathcal{S}$, we can extract a further subsequence $\mathcal{S}_1$, for which

$$\lim_{n\to\infty, n\in\mathcal{S}_1} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)} = e^u, \tag{0.1}$$

recall that $C$ given by (4.19) is 0. As the limit is independent of the subsequence $\mathcal{S}_1$ of $\mathcal{S}$, we obtain (1.4). ■

## References

[1] D.S. Lubinsky, *Local Asymptotics for Orthonormal Polynomials on the Unit Circle via Universality*, Journal d'Analyse Math., 141(2020), 285-304.

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