

Convergence of Product Integration Rules for Weights on the Whole Real Line II

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Abstract

We continue our investigation of product integration rules associated with weights on the whole real line, such as $\exp(-|x|^\beta)$, $\beta > 1$. In an earlier paper, we considered interpolatory integration rules whose abscissas are the zeros of an orthogonal polynomial associated with the weight. In this paper, we show the advantage of adding two extra points to the zeros, following an idea of J. Szabados. This allows convergence for a larger class of functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

The *product integration* approach for approximating integrals involves splitting the integrand into a “difficult” but specified function $k(x)$, and a relatively smooth, but initially unspecified function $f(x)$. Thus we seek to approximate

$$I[k; f] := \int_{-\infty}^{\infty} f(x)k(x)dx. \quad (1)$$

The function $k(x)$ is absorbed into the weights in the integration rule, which we denote by

$$I_n^*[k; f] := \sum_{j=0}^{n+1} w_{jn}^* f(x_{jn}^*). \quad (2)$$

Thus our integration rule involves $n + 2$ distinct points $\{x_{jn}^*\}_{j=0}^{n+1}$. At this stage we do not order these abscissas in any particular fashion.

We restrict ourselves to *interpolatory rules*, that is,

$$I_n^*[k; P] = I[k; P] = \int_{-\infty}^{\infty} k(x)P(x)dx, \quad P \in \mathcal{P}_{n+1}, \quad (3)$$

where \mathcal{P}_{n+1} denotes the set of all polynomials of degree $\leq n + 1$. If $L_n^*[f] \in \mathcal{P}_{n+1}$ denotes the Lagrange interpolation polynomial to f at $\{x_{jn}^*\}_{j=0}^{n+1}$, then

$$I_n^*[k; f] = \int_{-\infty}^{\infty} L_n^*[f](x)k(x)dx. \quad (4)$$

The convergence of interpolatory integration rules has been widely studied for weights on finite and infinite intervals [2], [9], [12], [17-19]. In an earlier paper, we investigated rules whose abscissas are zeros of orthogonal polynomials associated with a given weight. In this paper, we propose to show the advantage of adding two extra abscissa to the zeros of the orthogonal polynomials, at least from the point of view of convergence theory. This idea was first used by J. Szabados to reduce the size of Lebesgue constants in Lagrange interpolation [20].

The weights we consider are *Freud weights*, that is have the form $W := e^{-Q}$, where Q is even, and of smooth polynomial growth at ∞ . The archetypal example is

$$W_\beta(x) := \exp(-|x|^\beta), \quad \beta > 1. \quad (5)$$

See [8], [15] for surveys on weighted approximation and orthogonal polynomials. Let $p_n(x) := p_n(W^2, x) \in \mathcal{P}_n$ denote the n th orthonormal polynomial for the weight W^2 , so that

$$\int_{-\infty}^{\infty} p_n(W^2, x) p_m(W^2, x) W^2(x) dx = \delta_{mn}. \quad (6)$$

The zeros of p_n are denoted by

$$-\infty < x_{nn} < x_{n-1,n} < \dots < x_{1n} < \infty. \quad (7)$$

Let $\pm\xi_n$ denote the points where $|p_n W|$ attains its maximum, that is

$$|p_n W|(\pm\xi_n) = \|p_n W\|_{L_\infty(\mathbb{R})}. \quad (8)$$

It can be shown (see below) that ξ_n is close to x_{1n} but there is not a general result, stating, for example, that $\xi_n > x_{1n}$. We use as our $n + 2$ interpolation points

$$\{x_{n+1,n}^*, x_{nn}^*, \dots, x_{1n}^*, x_{0n}^*\} := \{-\xi_n, x_{nn}, x_{n-1,n}, \dots, x_{1n}, \xi_n\}. \quad (9)$$

We note that the points x_{jn}^* decrease in size as j increases, except possibly for $j = 0, n + 1$.

Some measure of the size of the weights w_{jn} in $I_n^*[k; f]$ is provided by the behaviour as $n \rightarrow \infty$ of the companion rule

$$I_n^{*c}[k; f] := \sum_{j=0}^{n+1} |w_{jn}^*| f(x_{jn}^*). \quad (10)$$

Our main result is:

Theorem 1. *Let $W := e^{-Q}$, where $Q : \mathbb{R} \rightarrow \mathbb{R}$ is even and continuous in \mathbb{R} , Q' is continuous in $(0, \infty)$ and $Q' > 0$ in $(0, \infty)$ while for some $A, B > 1$,*

$$A \leq \frac{\frac{d}{dx}(xQ'(x))}{Q'(x)} \leq B, \quad x \in (0, \infty). \quad (11)$$

Let $1 < p < \infty$, $q := p/(p-1)$, $\Delta \in \mathbb{R}$, $\alpha > 0$. Then for

$$\lim_{n \rightarrow \infty} I_n^*[k; f] = I[k; f] \quad (12)$$

to hold for every $f : \mathbb{R} \rightarrow \mathbb{R}$ that is Riemann integrable in each finite interval and satisfies

$$\lim_{|x| \rightarrow \infty} |f(x)| W(x) (1 + |x|)^\alpha = 0, \quad (13)$$

and for every measurable function $k : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\left\| (kW^{-1})(x) (1 + |x|)^\Delta \right\|_{L_q(\mathbb{R})} < \infty, \quad (14)$$

it is necessary and sufficient that

$$\Delta > \frac{1}{p} - \min\{1, \alpha\}. \quad (15)$$

Moreover (15) guarantees that

$$\lim_{n \rightarrow \infty} I_n^{*c}[k; f] = I[k; f] \quad (16)$$

under the above conditions on f and k .

The condition (15) is far simpler than that in [9] where product integration rules based on $\{x_{jn}\}_{j=1}^n$ were studied. For the special case of the weights W_β , in [9] instead of (15), the necessary and sufficient condition turned out to be:

$$\begin{aligned} \Delta &> \frac{1}{p} - \min\{1, \alpha\}, & p \leq 4 \\ \Delta &> \frac{1}{p} - \min\{1, \alpha\} + \frac{\beta}{6} \left(1 - \frac{4}{p}\right), & p > 4 \text{ and } \alpha = 1; \\ \Delta &\geq \frac{1}{p} - \min\{1, \alpha\} + \frac{\beta}{6} \left(1 - \frac{4}{p}\right), & p > 4 \text{ and } \alpha > 1. \end{aligned}$$

(For the general weights treated above, the condition is more complicated to state and involved the behaviour of the Mhaskar–Rahmanov–Saff number). Thus for $p > 4$, basing rules solely on the zeros of the orthonormal polynomials results in convergence for a smaller class of functions f . The reason for this is the growth of the orthonormal polynomials p_n near x_{1n} and x_{nn} . Insertion of the extra abscissas $\pm \xi_n$ damps this growth. It seems certain from the proofs in [20] that we could replace ξ_n above by any $\zeta_n > 0$ satisfying

$$|p_n W|(\zeta_n) \geq \rho \|p_n W\|_{L_\infty(\mathbb{R})}.$$

Here $0 < \rho < 1$ is fixed independent of n .

It is obviously of interest to say something about the behaviour of the weights w_{jn}^* . To expand on this, we shall need more notation. Throughout, C, C_1, C_2, \dots denote positive constants independent of $n, P \in \mathcal{P}_n$, and $x \in \mathbb{R}$. We sometimes write $C \neq C(k)$ to emphasize that C is independent of k . The same symbol does not necessarily denote the same constant in different occurrences. For sequences of real numbers $(c_n), (d_n)$, we write

$$c_n \sim d_n$$

if there exist $C_1, C_2 > 0$ such that for $n \geq 1$,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

For Q as in Theorem 1, we let $a_n = a_n(Q)$ denote the n th Mhaskar–Rahmanov–Saff number for Q , that is a_n is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt, \quad n \geq 1. \quad (17)$$

It follows from the convexity of Q that a_n is well defined. One of its properties is [13, 14, 16]:

$$\|PW\|_{L_\infty(R)} = \|PW\|_{L_\infty[-a_n, a_n]}, \quad P \in \mathcal{P}_n. \quad (18)$$

As an example, if $Q(x) = |x|^\beta$, then

$$a_n = Cn^{1/\beta}, \quad n \geq 1,$$

where the constant C is explicitly given in terms of the gamma function. In the general case above,

$$a_n = O(n^{1/A}) = o(n). \quad (19)$$

We also need the relationship between the quadrature rule $I_n^*[k; \cdot]$ and the related quadrature rule $I_n[k; \cdot]$ based on the zeros of p_n and studied in [9]. Let $\{\ell_{jn}\}_{j=1}^n$ denote the fundamental polynomials of Lagrange interpolation for the zeros $\{x_{jn}\}_{j=1}^n$ of p_n and $\{\ell_{jn}^*\}_{j=0}^{n+1}$ denote the fundamental polynomials for $\{x_{jn}^*\}_{j=0}^{n+1}$. Let

$$I_n[k; f] := \sum_{j=1}^n w_{jn}[k] f(x_{jn}) = \int_{-\infty}^{\infty} L_n[f](x) k(x) dx \quad (20)$$

denote the interpolatory rule at the zeros $\{x_{jn}\}_{j=1}^n$ of p_n and let

$$G_n[f] := \sum_{j=1}^n \lambda_{jn} f(x_{jn}) \quad (21)$$

denote the Gauss quadrature rule for W^2 . Note that

$$G_n[\cdot] = I_n[W^2; \cdot]. \quad (22)$$

We again emphasize that I_n^*, L_n^* involve $n + 2$ points, whereas I_n, L_n involve n points, but no confusion should arise.

Finally, we need the weighted error in best approximation

$$E_n[f] := \inf_{P \in \mathcal{P}_n} \|(f - P)W\|_{L_\infty(\mathbb{R})}. \quad (23)$$

The following result shows that the w_{jn}^* are close to the w_{jn} :

Theorem 2. *Assume that W is as in Theorem 1, and that $k : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $(kW^{-1})(x)x^j \in L_1(\mathbb{R})$, $j = 0, 1, 2, \dots$.*

(a) *There exist $C_1 \neq C_1(n, k)$ such that*

$$\sum_{j=1}^n |w_{jn}^*[k] - w_{jn}[k]| \frac{W^{-1}(x_{jn})}{\sqrt{1 + |x_{jn}|}} \leq C_1 T_n, \quad (24)$$

where

$$T_n := \left[\left| \int_{\mathbb{R}} (kp_n)(x) dx \right| + \left| \int_{\mathbb{R}} (kp_n)(x) \frac{x}{a_n} dx \right| \right]. \quad (25)$$

(b) *Given $0 < \sigma < 1$, there exist $C_j \neq C_j(n, k)$, $j = 2, 3$ such that for $n \geq C_2$,*

$$\max_{|x_{jn}| \leq \sigma a_n} |w_{jn}^*[k] - w_{jn}[k]| W^{-1}(x_{jn}) \leq \frac{C_3 \sqrt{a_n}}{n} T_n. \quad (26)$$

(c) *There exist $C_4 \neq C_4(n, k)$ such that*

$$\left(|w_{0n}^*[k]| + |w_{n+1,n}^*[k]| \right) / W^{-1}(\xi_n) \leq \frac{C_4 \sqrt{a_n}}{n^{1/6}} T_n. \quad (27)$$

Corollary 3. *If*

$$\|kW^{-1}\|_{L_\infty(\mathbb{R})} < \infty,$$

then

$$T_n \leq C_1 a_n^{1/2} E_{n-1}[kW^{-2}] \leq C_2 a_n^{1/2} \quad (28)$$

and so as $n \rightarrow \infty$,

$$\max_{|x_{jn}| \leq \sigma a_n} |w_{jn}^*[k] - w_{jn}[k]| W^{-1}(x_{jn}) = O\left(\frac{a_n}{n}\right) = o(1). \quad (29)$$

If also as $n \rightarrow \infty$,

$$E_n[kW^{-2}] = o\left(\frac{1}{\log n}\right), \quad (30)$$

then

$$\max_{|x_{jn}| \leq \sigma a_n} |w_{jn}^*[k] - \lambda_{jn} k(x_{jn})| W^{-1}(x_{jn}) = o\left(\frac{a_n}{n}\right) = o(1). \quad (31)$$

We note that explicit asymptotics for the Christoffel numbers λ_{jn} are available under more assumptions on Q , see [7].

The results of this paper are proved in the next two sections.

2. PROOF OF THEOREM 1

We begin with some notation. Throughout, we assume that $W := e^{-Q}$ is as in Theorem 1, and that $1 < p < \infty$, $q := p/(p-1)$, $\alpha > 0$, $\Delta \in \mathbb{R}$. To emphasize the dependence of w_{jn} on k we write $w_{jn}[k]$ below. Note that w_{jn} is linear in k .

If $J : X \rightarrow Y$ is a linear operator between the normed spaces X, Y over the reals, we write

$$\|J\|_{X \rightarrow Y} := \sup \{ \|J[f]\|_Y : \|f\|_X \leq 1 \},$$

where $\|\cdot\|_X$ is the norm on X and so on. We let X^* denote the dual of X , that is the space of bounded linear functionals from X to \mathbb{R} , with norm $\|\cdot\|_{X \rightarrow \mathbb{R}}$.

The proof of the following lemma is essentially the same as that of Lemma 2.1 in [9] and the main ideas appeared earlier in [17] but we provide the details.

Lemma 4. *Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be an even function that is decreasing in $(0, \infty)$ and satisfies*

$$\lim_{x \rightarrow \infty} \eta(x) = 0 \quad (32)$$

and

$$1 \geq \eta(x) \geq (1 + x^2)^{-1}, \quad x \in [0, \infty). \quad (33)$$

Define spaces X, Y, Z as follows: X is the space of continuous $g : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\|g\|_X := \|g(x)W(x)(1 + |x|)^\alpha \eta(x)^{-1}\|_{L_\infty(\mathbb{R})} < \infty; \quad (34)$$

Y is the space of measurable $k : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\|k\|_Y := \left\| k(x)W(x)^{-1}(1 + |x|)^\Delta \right\|_{L_q(\mathbb{R})} < \infty; \quad (35)$$

Z is the space of measurable $h : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\|h\|_Z := \left\| h(x)W(x)(1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} < \infty. \quad (36)$$

(i) Then

$$\|I_n^*[k; \cdot]\|_{X \rightarrow \mathbb{R}} = \sum_{j=0}^{n+1} |w_{jn}[k]| W^{-1}(x_{jn}^*) (1 + |x_{jn}^*|)^{-\alpha} \eta(x_{jn}^*), \quad (37)$$

and

$$\|I_n^*[\cdot; \cdot]\|_{Y \rightarrow X^*} = \sup_{\|k\|_Y \leq 1} \|I_n^*[k; \cdot]\|_{X \rightarrow \mathbb{R}} = \|L_n^*\|_{X \rightarrow Z}. \quad (38)$$

(ii) Moreover,

$$\sup_{n \geq 1} |I_n^*[k; f]| < \infty \quad \forall f \in X, \quad \forall k \in Y \quad (39)$$

iff

$$B := \sup_{n \geq 1} \|L_n^*\|_{X \rightarrow Z} < \infty. \quad (40)$$

In this case,

$$\|I_n^*[k; \cdot]\|_{X \rightarrow \mathbb{R}} \leq B \|k\|_Y \quad \forall k \in Y. \quad (41)$$

Proof.

(i) Firstly (37) is an immediate consequence of the definition of $\|I_n^*[k; \cdot]\|_{X \rightarrow \mathbb{R}}$. Next, from (4) and by duality of $L_p(\mathbb{R})$ and $L_q(\mathbb{R})$,

$$\begin{aligned} \sup_{\|k\|_Y \leq 1} |I_n^*[k; f]| &= \sup_{\|k\|_Y \leq 1} \left| \int_{-\infty}^{\infty} L_n^*[f](x) k(x) dx \right| \\ &= \sup_{\|h\|_{L_q(\mathbb{R})} \leq 1} \left| \int_{-\infty}^{\infty} L_n^*[f](x) W(x) (1 + |x|)^{-\Delta} h(x) dx \right| \\ &= \left\| L_n^*[f](x) W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} = \|L_n^*[f]\|_Z. \end{aligned}$$

So

$$\sup_{\|f\|_X \leq 1} \sup_{\|k\|_Y \leq 1} |I_n^*[k; f]| = \sup_{\|f\|_X \leq 1} \|L_n^*[f]\|_Z = \|L_n^*\|_{X \rightarrow Z}.$$

Interchanging the sup's on the left-hand side gives (38).

(ii) Note that $I_n^*[k; f]$ is linear in both f and k . Firstly if B in (40) is finite,

$$\begin{aligned} |I_n^*[k; f]| &\leq \|I_n^*[k; \cdot]\|_{X \rightarrow \mathbb{R}} \|f\|_X \\ &\leq \|I_n^*[\cdot; \cdot]\|_{Y \rightarrow X^*} \|k\|_Y \|f\|_X \leq B \|k\|_Y \|f\|_X, \end{aligned}$$

by (38) and (40). Then (39) follows. Conversely suppose that (39) is true for all $f \in X$ and $k \in Y$. Note that X and Y are Banach spaces. Then the uniform boundedness principle gives for each fixed $k \in Y$,

$$\sup_{n \geq 1} \|I_n^*[k; \cdot]\|_{X \rightarrow \mathbb{R}} < \infty.$$

But as the map that sends $k \in Y$ to $I_n[k; \cdot] \in X^*$ is linear in k , the uniform boundedness principle gives

$$B := \sup_{n \geq 1} \|I_n^*[\cdot; \cdot]\|_{Y \rightarrow X^*} < \infty.$$

Then (38) gives (40). \square

The following lemma relates the rules I_n^* , I_n , and G_n :

Lemma 5.

(a) For $1 \leq j \leq n$,

$$\begin{aligned} w_{jn}^*[k] - w_{jn}[k] &= \frac{-1}{p_n'(x_{jn}) (\xi_n^2 - x_{jn}^2)} \int_{-\infty}^{\infty} (kp_n)(x) (x_{jn} + x) dx; \end{aligned} \quad (42)$$

$$w_{0n}^*[k] = \frac{1}{2p_n(\xi_n)\xi_n} \int_{-\infty}^{\infty} (kp_n)(x) (\xi_n + x) dx; \quad (43)$$

$$w_{n+1,n}^*[k] = \frac{1}{2p_n(-\xi_n)\xi_n} \int_{-\infty}^{\infty} (kp_n)(x) (\xi_n - x) dx. \quad (44)$$

(b) If $k = SW^2$, where S is a polynomial of degree $\leq n - 2$, then

$$w_{jn}^*[k] = w_{jn}[k], \quad 1 \leq j \leq n; \quad (45)$$

$$w_{0n}^*[k] = 0 = w_{n+1,n}^*[k]; \quad (46)$$

$$I_n^*[k; f] = I_n[k; f] = G_n[Sf]; \quad (47)$$

$$I_n^{*c}[k; f] = G_n[|S|f]. \quad (48)$$

Proof.

- (a) This follows easily from the identities relating the fundamental polynomials ℓ_{jn} and ℓ_{jn}^* . Indeed, for $1 \leq j \leq n$,

$$\ell_{jn}^*(x) = \ell_{jn}(x) \left(\frac{x^2 - \xi_n^2}{x_{jn}^2 - \xi_n^2} \right),$$

and

$$\ell_{0n}^*(x) = \frac{p_n(x)}{2p_n(\xi_n)\xi_n}(\xi_n + x); \quad \ell_{n+1,n}^*(x) = \frac{p_n(x)}{2p_n(-\xi_n)\xi_n}(\xi_n - x).$$

Then if $1 \leq j \leq n$,

$$\begin{aligned} w_{jn}^*[k] - w_{jn}[k] &= \int_{-\infty}^{\infty} k(x) [\ell_{jn}^*(x) - \ell_{jn}(x)] dx \\ &= \int_{-\infty}^{\infty} k(x) \left[\ell_{jn}(x) \frac{x^2 - x_{jn}^2}{x_{jn}^2 - \xi_n^2} \right] dx. \end{aligned}$$

As

$$\ell_{jn}(x) = \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})},$$

(42) follows, and (43) is easier.

- (b) Firstly (45) and (46) follow immediately from the orthogonality of $p_n W^2$ to polynomials of degree $< n$. Next, the fact that G_n is exact for polynomials of degree $\leq 2n - 1$ shows that for P of degree $\leq n + 1$,

$$G_n[SP] = \int_{-\infty}^{\infty} PSW^2 = \int_{-\infty}^{\infty} Pk = I[k; P]$$

so by uniqueness of the interpolatory quadrature rule, we obtain (47). Finally then

$$w_{jn}[k] = \lambda_{jn} S(x_{jn})$$

so

$$I_n^{*c}[k; f] = \sum_{j=1}^n |w_{jn}[k]| f(x_{jn}) = \sum_{j=1}^n \lambda_{jn} |S(x_{jn})| f(x_{jn}) = G_n[|S|f].$$

□

Proof of the Sufficiency Part of Theorem 1.

Let X, Y, Z be the spaces defined in Lemma 4. Fix a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is Riemann integrable in each finite interval and that satisfies (13). Note that we may find a function η satisfying (32) and (33) such that $\|f\|_X$ is finite. We may use the notation $\|f\|_X$ even though f need not belong to X , since the norm is well defined and finite. Now Theorem 1.3 in [9] implies that

$$\lim_{n \rightarrow \infty} \|L_n^*[g] - g\|_Z = 0 \forall g \in X.$$

Then the uniform boundedness principle shows that (40) holds. (We note that $X \subset Z$, so that the identity operator from X to Z is bounded.) We shall use (40) and convergence theorems for Gauss quadrature to prove (12) and (16) for the given f . First let S be a polynomial of degree $\leq m$ say, and let

$$k_1 := SW^2.$$

For $n \geq m + 2$, we have (47) and (48). Now choose an entire function

$$G(x) := \sum_{j=0}^{\infty} g_{2j} x^{2j}, \quad g_{2j} \geq 0, \quad j \geq 0$$

such that

$$C_1 \leq G(x)W^{3/2}(x) \leq C_2, \quad x \in \mathbb{R}.$$

Such functions were constructed for example in Chapter 6 of [7]. Then

$$|Sf|(x)/G(x) \leq C_1^{-1} |Sf|(x)W^{3/2}(x) \rightarrow 0, \quad |x| \rightarrow \infty$$

in view of (13) and the fact that W decays faster than any polynomial. Also

$$\int_{-\infty}^{\infty} G(x)W^2(x)dx \leq C_2 \int_{-\infty}^{\infty} W^{1/2}(x)dx < \infty.$$

A classical convergence theorem on Gauss quadrature [3,p.94] implies that

$$\lim_{n \rightarrow \infty} I_n^{*c}[k_1; f] = \lim_{n \rightarrow \infty} G_n[|Sf|] = \int_{-\infty}^{\infty} |Sf|W^2 = I[|k_1|; f]. \quad (49)$$

Next consider an arbitrary $k \in Y$. We have

$$\begin{aligned} & \left| I_n^{*c}[k; f] - I[|k|; f] \right| \\ & \leq \left| I_n^{*c}[k; f] - I_n^{*c}[k_1; f] \right| \\ & \quad + \left| I_n^{*c}[k_1; f] - I[|k_1|; f] \right| + \left| I[|k_1|; f] - I[|k|; f] \right| \\ & =: \tau_{1n} + \tau_{2n} + \tau_3. \end{aligned}$$

Now

$$\begin{aligned}
\tau_{1n} &= \left| \sum_{j=0}^{n+1} \left(|w_{jn}^*[k]| - |w_{jn}^*[k_1]| \right) f(x_{jn}^*) \right| \\
&\leq \|f\|_X \sum_{j=0}^{n+1} \left| |w_{jn}^*[k]| - |w_{jn}^*[k_1]| \right| W^{-1}(x_{jn}^*) (1 + |x_{jn}^*|)^{-\alpha} \eta(x_{jn}^*) \\
&\leq \|f\|_X \sum_{j=0}^{n+1} |w_{jn}^*[k] - w_{jn}^*[k_1]| W^{-1}(x_{jn}^*) (1 + |x_{jn}^*|)^{-\alpha} \eta(x_{jn}^*) \\
&= \|f\|_X \sum_{j=0}^{n+1} |w_{jn}^*[k - k_1]| W^{-1}(x_{jn}^*) (1 + |x_{jn}^*|)^{-\alpha} \eta(x_{jn}^*) \\
&= \|f\|_X \|I_n^*[k - k_1; \cdot]\|_{X \rightarrow \mathbb{R}} \leq B \|f\|_X \|k - k_1\|_Y
\end{aligned}$$

by (37) and (41). Next we showed at (49) that

$$\lim_{n \rightarrow \infty} \tau_{2n} = 0.$$

Finally,

$$\begin{aligned}
\tau_3 : &= \left| I[|k_1|; f] - I[|k|; f] \right| = \left| \int_{-\infty}^{\infty} [|k_1| - |k|] f \right| \\
&\leq \|f\|_X \int_{-\infty}^{\infty} |k_1 - k| (x) W^{-1}(x) (1 + |x|)^{-\alpha} \eta(x) dx \\
&\leq \|f\|_X \|k - k_1\|_Y \left\| (1 + |x|)^{-\alpha - \Delta} \eta(x) \right\|_{L_p(\mathbb{R})},
\end{aligned}$$

by Hölder's inequality. Since $\eta \leq 1$, and $\alpha + \Delta > \frac{1}{p}$ by our hypothesis (15), the estimates for $\tau_{1n}, \tau_{2n}, \tau_3$ yield

$$\limsup_{n \rightarrow \infty} \left| I_n^{*c}[k; f] - I[|k|; f] \right| \leq C \|f\|_X \|k - k_1\|_Y$$

where C is independent of f, k, k_1 . Next by our hypothesis (14) on k ,

$$\|k - k_1\|_Y = \left\| (kW^{-2} - S)(x)W(x)(1 + |x|)^\Delta \right\|_{L_q(\mathbb{R})} < \infty$$

and $W(x)(1 + |x|)^\Delta$ decays faster at ∞ than $\exp(-|x|)$, so we can find a polynomial $S(x)$ for which the last term is as small as we please (see [4] or [6]). Then (16) follows.

We turn to the proof of (12). Let f be as above and P be a polynomial of degree $\leq m$. We have for $n \geq m - 1$,

$$\begin{aligned} \left| I_n^*[k; f] - I[k; f] \right| &= \left| I_n^*[k; f - P] - I[k; f - P] \right| \\ &\leq \sum_{j=0}^{n+1} |w_{j_n}^*[k]| |f - P|(x_{j_n}^*) + \int_{-\infty}^{\infty} |k| |f - P| \\ &\rightarrow 2 \int_{-\infty}^{\infty} |k| |f - P|, \quad n \rightarrow \infty, \end{aligned}$$

by the above convergence of the companion rule (The argument we applied to f applies equally well to $|f - P|$). In turn, we can bound this via Hölder's inequality, by

$$\leq 2 \left\| k(x)W(x)^{-1} (1 + |x|)^\Delta \right\|_{L_q(\mathbb{R})} \left\| (f - P)(x)W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})}.$$

Here

$$\left\| f(x)W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} \leq \|f\|_X \left\| (1 + |x|)^{-\alpha - \Delta} \eta(x) \right\|_{L_p(\mathbb{R})} < \infty$$

and $W(x) (1 + |x|)^{-\Delta}$ decays at ∞ faster than $\exp(-|x|)$ at ∞ , so given $\varepsilon > 0$, we can find a polynomial P such that

$$\left\| (f - P)(x)W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} < \varepsilon.$$

Then (12) follows. \square

Proof of the Necessity Part of Theorem 1.

Let $\eta(x) := (\log(2 + |x|))^{-1/(2p)}$, $x \in \mathbb{R}$. We assume that (12) holds for every continuous f satisfying (13) and every measurable k satisfying (14). Then (39) follows and by Lemma 4(ii), we have (40). That is $\forall f \in X$ and $n \geq 1$, we have

$$\begin{aligned} \left\| L_n^*[f](x)W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} &= \|L_n^*[f]\|_Z \\ &\leq B \|f\|_X = B \left\| f(x)W(x) (1 + |x|)^\alpha \eta(x)^{-1} \right\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

This is precisely the first step in the necessity part of Theorem 1.4 in [10] (see (49) there). Then exactly as there, we deduce (15). \square

3. PROOF OF THEOREM 2 AND COROLLARY 3

We first need some estimates on $p'_n(x_{jn})$ and so on. Throughout, we let

$$\phi_n(x) := \left| 1 - \frac{x}{a_n} \right| + n^{-2/3}, \quad x \in \mathbb{R}, \quad n \geq 1.$$

Lemma 6.

(a) For $n \geq 1$, $j = 0, 1$

$$\left| 1 - \frac{x_{jn}^*}{a_n} \right| \leq Cn^{-2/3}; \quad (50)$$

and for $1 \leq j \leq n+1$,

$$|x_{jn}^* - x_{j-1,n}^*| \sim \frac{a_n}{n} \phi_n^{-1/2}(x_{jn}^*). \quad (51)$$

(b)

$$\sup_{x \in \mathbb{R}} |p_n W|(x) \sim a_n^{-1/2} n^{1/6}. \quad (52)$$

(c) Let $0 < p \leq \infty$, $L > 0$. There exists $C > 0$ such that for $n \geq (2L)^{3/2}$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(\mathbb{R})} \leq C \|PW\|_{L_p(|x| \leq a_n(1-Ln^{-2/3}))}. \quad (53)$$

In particular, given $k \geq 1$, we have for $n > k$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(\mathbb{R})} \leq C \|PW\|_{L_p[-a_{n-k}, a_{n-k}]}. \quad (54)$$

(d) For $0 \leq j \leq n$, $n \geq 1$,

$$1 + |x_{jn}^*| \sim 1 + |x_{j+1,n}^*|. \quad (55)$$

(e) For $1 \leq j \leq n$ and $n \geq 1$,

$$|p'_n(W^2, x_{jn})| W(x_{jn}) \sim \frac{n}{a_n^{3/2}} \phi_n(x_{jn})^{1/4}. \quad (56)$$

(f) For $n \geq 1$,

$$a_1 n^{1/B} \leq a_n \leq a_1 n^{1/A}. \quad (57)$$

(g) For $n \geq 1$ and $0 \leq j \leq n$,

$$\phi_n(x) \sim \phi_n(x_{jn}^*) \sim \phi_n(x_{j+1,n}^*), \quad x \in [x_{j+1,n}^*, x_{j-1,n}^*]. \quad (58)$$

(h)

$$\int_{-\infty}^{\infty} |p_n W| \sim a_n^{1/2}. \quad (59)$$

Proof.

(a) For $j = 1$ (50) is Corollary 1.2(a) in [5,p.466] and for $j = 0$, it was proved by J. Szabados [20,p.104, Lemma 2]. For $2 \leq j \leq n$, (51) was proved in [1] (a weaker form appeared in [5]). J. Szabados [20,p.105, Lemma 4] proved that

$$|x_{1n}^* - x_{0n}^*| \sim a_n n^{-2/3} \sim \frac{a_n}{n} \phi_n^{-1/2}(x_{1n})$$

so (51) is true also for $j = 0$. The case $j = n + 1$ is similar.

(b) is part of Corollary 1.4 in [5,p.467].

(c) Firstly (53) is a special case of Theorem 1.8 in [5,p.469]; (54) follows from (53) together with the fact that for k fixed, $n \rightarrow \infty$,

$$\frac{a_n}{a_{n-k}} = 1 + O\left(\frac{1}{n}\right).$$

(See Lemma 5.2(c) in [5,p.478]).

(d) This is an easy consequence of the spacing (51).

(e) This is part of Corollary 1.3 in [5,p.467].

(f) This is Lemma 5.2(b) in [5,p.478].

(g) This follows from Lemma 5.2(c) in [11,p.47].

(h) This is the case $p = 1$ of Theorem 1 in [11,p.44]. \square

We turn to

The proof of Theorem 2.

- (a) Recall first that $x_{jn}^* = x_{jn}$ for $1 \leq j \leq n$. Assume that we fix j with $x_{jn} \geq 0$. Now from (42) and (50), with T_n given by (25),

$$\begin{aligned} |w_{jn}^*[k] - w_{jn}[k]| &\leq \frac{1}{|p'_n(x_{jn})(\xi_n^2 - x_{jn}^2)|} \left[\left| \int_{-\infty}^{\infty} (kp_n)(x)xdx \right| \right. \\ &\quad \left. + Ca_n \left| \int_{-\infty}^{\infty} (kp_n)(x)dx \right| \right] \\ &\leq \frac{Ca_n^{3/2} |x_{j-1,n}^* - x_{jn}^*| \phi_n(x_{jn}^*)^{1/4} W(x_{jn})}{|\xi_n^2 - x_{jn}^2|} T_n. \end{aligned}$$

In the last line, we used (51) and (56). Next, from (50) and (51),

$$|\xi_n - x_{jn}| = |x_{0n} - x_{jn}| \sim a_n \left(\left| 1 - \frac{x_{jn}}{a_n} \right| + n^{-2/3} \right) = a_n \phi_n(x_{jn})$$

so we deduce that

$$\begin{aligned} |w_{jn}^*[k] - w_{jn}[k]| / W(x_{jn}) \\ \leq Ca_n^{-1/2} |x_{j-1,n}^* - x_{jn}^*| \phi_n(x_{jn})^{-3/4} T_n. \end{aligned} \quad (60)$$

A similar estimate holds for $x_{jn} < 0$. Adding over j and using (55) and (58) gives

$$\begin{aligned} \sum_{j=1}^n |w_{jn}^*[k] - w_{jn}[k]| \frac{W^{-1}(x_{jn})}{\sqrt{1 + |x_{jn}|}} &\leq C_1 a_n^{-1/2} T_n \int_{-a_n}^{a_n} \frac{\phi_n^{-3/4}(t)}{\sqrt{1 + |t|}} dt \\ &\leq C_1 a_n^{-1/2} T_n \left(\int_0^{\frac{1}{2}a_n} \frac{dt}{\sqrt{1 + t}} + a_n^{-1/2} \int_{\frac{1}{2}a_n}^{a_n} \phi_n^{-3/4}(t) dt \right) \\ &\leq C_1 T_n \left(1 + \int_{\frac{1}{2}}^1 (|1 - |u|| + n^{-2/3})^{-3/4} du \right) \leq C_2 a_n^{-1/2} T_n. \end{aligned}$$

- (b) For $|x_{jn}| \leq \sigma a_n$, we have $\phi_n(x_{jn}) \sim 1$ uniformly in j and n . Then (60) and (51) give uniformly in j and n ,

$$|w_{jn}^*[k] - w_{jn}[k]| / W(x_{jn}) \leq C \frac{\sqrt{a_n}}{n} T_n.$$

(c) From (43)

$$\begin{aligned} |w_{0n}[k]/W(\xi_n)| &= \frac{1}{2|p_n W|(\xi_n)} \left| \int_{-\infty}^{\infty} (kp_n)(x) \left(1 + \frac{x}{\xi_n}\right) dx \right| \\ &\leq C a_n^{1/2} n^{-1/6} \left[\left| \int_{-\infty}^{\infty} (kp_n)(x) dx \right| + \frac{a_n}{\xi_n} \left| \int_{-\infty}^{\infty} (kp_n)(x) \frac{x}{a_n} dx \right| \right] \end{aligned}$$

by (52) and then (27) follows. \square

Proof of Corollary 3.

Now by orthogonality, if S is a polynomial of degree $\leq n-1$,

$$\begin{aligned} \left| \int_{\mathbb{R}} (kp_n)(x) \frac{x}{a_n} dx \right| &= \left| \int_{\mathbb{R}} (kW^{-2} - S)(x) (p_n W^2)(x) \frac{x}{a_n} dx \right| \\ &\leq \| (kW^{-2} - S) W \|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |p_n W| (x) \left| \frac{x}{a_n} \right| dx \\ &\leq C \| (kW^{-2} - S) W \|_{L^\infty(\mathbb{R})} a_n^{1/2} \end{aligned}$$

by (54) and then (59). Taking the inf' s over S gives

$$\left| \int_{\mathbb{R}} (kp_n)(x) \frac{x}{a_n} dx \right| \leq C E_{n-1}[kW^{-2}] a_n^{1/2} \leq C \|kW^{-1}\|_{L^\infty(\mathbb{R})} a_n^{1/2}.$$

The other term in T_n admits a similar estimate and we deduce (29). Finally, if the estimate (30) holds, then in Theorem 1.3 in [9], we proved that

$$\max_{|x_{jn}| \leq \sigma a_n} |w_{jn}[k] - \lambda_{jn} k(x_{jn})| W^{-1}(x_{jn}) = o\left(\frac{a_n}{n}\right) = o(1).$$

\square

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