On Local Asymptotics for Orthonormal Polynomials

Eli Levin and D. S. Lubinsky

1 Introduction: Compact Support in $\mathbb{R}$

The idea of "local limits" is most easily understood in the context of measures on a compact interval. (Those familiar with the topic, may skip to Sect. 4 to see the new results.) Let μ be a finite positive Borel measure on the real line with compact support, and with infinitely many points in its support. Then we may define orthonormal polynomials

$$p_n(\mu, x) = \gamma_n x^n + \ldots, \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int p_n(\mu, x)\, p_m(\mu, x)\, d\mu(x) = \delta_{mn}.$$

We denote their zeros by

$$x_{nn} < x_{n-1,n} < \ldots < x_{1n} < \infty.$$

E. Levin
Mathematics Department, The Open University of Israel, Raanana, Israel

D. S. Lubinsky (✉)
School of Mathematics, Georgia Institute of Technology, Atlanta, GA, USA
e-mail: lubinsky@math.gatech.edu

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The reproducing kernel

$$K_n(\mu, x, y) = \sum_{k=0}^{n-1} p_k(\mu, x)\, p_k(\mu, y)$$

and its normalized cousin

$$\widetilde{K}_n(\mu, x, y) = \mu'(x)^{1/2}\, \mu'(y)^{1/2}\, K_n(\mu, x, y)$$

play an important role in analyzing orthogonal polynomials.

The behavior of p_n as $n \to \infty$, is a central topic in orthogonal polynomials. Essentially for z away from the support, $p_n(z)$ exhibits geometric growth. Inside the interval of orthogonality, there is oscillatory behavior. As an example, let the support be $[-1, 1]$ and let μ' satisfy the Szegő condition

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty.$$

Then for $z \in \mathbb{C} \setminus [-1, 1]$,

$$\lim_{n\to\infty} p_n(\mu, z) / \left(z + \sqrt{z^2-1}\right)^n = G(z),$$

where G is a function analytic in $\mathbb{C} \setminus [-1, 1]$, involving the "Szegő function" for μ' [7, 26, 27, 29]. Here $z + \sqrt{z^2-1}$ is the conformal map of the exterior of $[-1, 1]$ onto the exterior of the unit ball. Under additional conditions on μ', such as a local Dini condition for the modulus of continuity of μ', there are asymptotics that reflect the oscillatory behavior for $x = \cos\theta \in (-1, 1)$:

$$p_n(\mu, x)\, \mu'(x)^{1/2} \left(1 - x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} \cos(n\theta + g(\theta)) + o(1),$$

with an appropriately defined function g [2, 7, 29]. The behavior near the endpoints of the interval is more delicate [1]. There is a very extensive literature on asymptotics of varying strengths and generality. See for example [22, 23, 26–28, 32].

There is a gap between the exterior asymptotics and those inside the support: one needs to stay a positive distance from the support to have the former. In a recent paper [13], the second author used universality limits to prove a "local limit", a type of ratio limit that holds in the complex plane close to the support. Here is a typical example:

Theorem 1.1 *Assume that μ is a regular measure with compact support* $\operatorname{supp}[\mu]$. *Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of I^o and $y_{jn} \in J$ be* *a zero of* p_n'. *Then*

$$\lim_{n\to\infty} \frac{p_n\left(\mu, y_{jn} + \frac{z}{n\omega(y_{jn})}\right)}{p_n\left(\mu, y_{jn}\right)} = \cos \pi z \tag{1.1}$$

uniformly for $y_{jn} \in J$ *and* z *in compact subsets of* $\mathbb{C}$. *Here* ω *is the equilibrium density, in the sense of the potential theory, for the support of* μ.

Let us expand on these hypotheses. We say that μ is *regular* (in the sense of Stahl, Totik, and Ullmann) if for every sequence of polynomials $\{P_n\}$ with degree P_n at most n,

$$\limsup_{n\to\infty} \left(\frac{|P_n(x)|}{\left(\int |P_n|^2 \, d\mu\right)^{1/2}} \right)^{1/n} \leq 1$$

for quasi-every $x \in \operatorname{supp}[\mu]$ (that is except in a set of logarithmic capacity 0). An equivalent formulation involves the leading coefficients $\{\gamma_n\}$ of the orthonormal polynomials for μ :

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap\,(\operatorname{supp}[\mu])},$$

where cap denotes logarithmic capacity. If the support consists of finitely many intervals, a sufficient condition for regularity is that $\mu' > 0$ a.e. in each subinterval. However much less is needed, and there are pure jump measures and singularly continuous measures that are regular [28].

The equilibrium measure for the compact set supp[μ] is the probability measure that minimizes the energy integral

$$\int\int \log \frac{1}{|x-y|} d\nu(x)\, d\nu(y)$$

amongst all probability measures ν supported on supp[μ]. If I is an interval contained in supp[μ], then the equilibrium measure is absolutely continuous in I, and moreover its density, which we denoted above by ω, is continuous in the interior I^o of I [25, p.216, Thm. IV.2.5].

An alternative formulation of (1.1) is

$$\lim_{n\to\infty} \frac{p_n\left(\mu, y_{jn} + \frac{z}{\tilde{K}_n(\mu, y_{jn}, y_{jn})}\right)}{p_n\left(\mu, y_{jn}\right)} = \cos \pi z \tag{1.2}$$

The essential feature of (1.1) is that it holds for z in a small complex neighborhood of y_{jn}, hence the name local limit. The proof of the limit involves normal families of analytic functions, and universality limits.

There is a close relationship between this asymptotic and "clock spacing" of zeros of orthogonal polynomials. From (1.1) and Hurwitz' Theorem, it is clear that near y_{jn}, the spacing between successive zeros is $\frac{1+o(1)}{n\omega(y_{jn})}$. In the case of $(-1, 1)$, this spacing becomes $\frac{\pi\sqrt{1-y_{jn}^2}}{n}\left(1 + o\left(1\right)\right)$. After the transformation $x_{jn} = \cos\theta_{jn}$, the spacing between successive θ_{jn} becomes $\frac{\pi}{n}\left(1 + o\left(1\right)\right)$. When projected onto $e^{i\theta_{jn}}$, this yields equispaced points around the unit circle. Hence the name "clock spacing". This has been studied intensively by many researchers in orthogonal polynomials [5, 27, 30, 31] again under varying hypotheses, and with varying levels of generality. The limit (1.1) may be thought of as emphasizing "clock spacing".

A perhaps more impressive application of local limits is to asymptotics at the endpoints of the interval of orthogonality. It is more difficult to establish asymptotics of orthogonal polynomials at endpoints [1], and they are generally available under quite restrictive hypotheses. They involve J_α, the usual Bessel function of the first kind and order α,

$$J_\alpha\left(z\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(z/2\right)^{2n+\alpha}}{n!\Gamma\left(\alpha+n+1\right)}. \tag{1.3}$$

and its normalized form,

$$J_\alpha^*\left(z\right) = J_\alpha\left(z\right)/z^\alpha = 2^{-\alpha}\sum_{n=0}^{\infty} \frac{(-1)^n\left(z/2\right)^{2n}}{n!\Gamma\left(\alpha+n+1\right)} \tag{1.4}$$

The second author proved [14] a local limit of the form:

Theorem 1.2 *Assume that μ is a regular measure with support $[-1, 1]$. Assume that for some $\rho > 0$, μ is absolutely continuous in $J = [1-\rho, 1]$, and in J, its absolutely continuous component has the form $w(x) = h\left(x\right)\left(1-x\right)^\alpha$, where $\alpha > -1$, and h has a positive limit at 1. Then uniformly for z in compact subsets of $\mathbb{C}$, we have*

$$\lim_{n\to\infty} \frac{p_n\left(\mu, 1-\frac{z^2}{2n^2}\right)}{p_n\left(\mu, 1\right)} = \frac{J_\alpha^*\left(z\right)}{J_\alpha^*\left(0\right)}. \tag{1.5}$$

While this is a ratio asymptotic, and the behavior of $p_n\left(1\right)$ is not specified, it is still impressive because full asymptotics of orthogonal polynomials at 1 require either specific asymptotics about recurrence coefficients, or substantial local and global smoothness of μ'.

2 The Unit Circle

Let μ be a finite positive Borel measure on $[-\pi, \pi)$ (or equivalently on the unit circle) with infinitely many points in its support. Then we may define orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(z) \overline{\varphi_m(z)} d\mu(\theta) = \delta_{mn},$$

where $z = e^{i\theta}$. For such measures, the notion of regularity becomes

$$\lim_{n \to \infty} \kappa_n^{1/n} = 1,$$

since the unit circle has logarithmic capacity 1. This is true if for example $\mu' > 0$ a.e. in $[-\pi, \pi)$, but there are pure jump and pure singularly continuous measures that are regular. We denote the zeros of φ_n by $\left\{z_{jn}\right\}_{j=1}^{n}$. They lie inside the unit circle, and may not be distinct.

The nth reproducing kernel for μ is

$$K_n(z, u) = \sum_{j=0}^{n-1} \varphi_j(z) \overline{\varphi_j(u)}.$$

Local limits for the unit circle turn out to be more difficult, because there is no obvious analogue of the point 1 at the endpoint of $[-1, 1]$, or the local maximum point y_{jn} of $|p_n|$ inside the support. The derivative φ_n' of the orthonormal polynomial φ_n has all its zeros inside the unit circle. Moreover, $\left|\varphi_n\left(e^{i\theta}\right)\right|$ might have only a few local maxima for $\theta \in [-\pi, \pi]$. In [15], we proved:

Theorem 2.1 *Assume that μ is a regular measure with the unit circle as support. Assume that J is a closed subarc of the unit circle such that μ is absolutely continuous and μ' is positive and continuous in J. Let J_1 be a subarc of the (relative) interior of J. Let $\{z_n\}_{n \geq 1}$ be a sequence in J_1. For $n \geq 1$, we can choose at least one of $\zeta_n = z_n$* or *$\zeta_n = z_n e^{i\pi/n}$ such that from any infinite sequence of positive integers, we can extract a further subsequence $\mathcal{S}$ such that uniformly for u in compact subsets of $\mathbb{C}$,*

$$\lim_{n \to \infty, n \in \mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1 + \frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)} = e^u + C(e^u - 1) \tag{2.1}$$

where

$$C = \lim_{n\to\infty, n\in\mathcal{S}} \left(\frac{\zeta_n}{n} \frac{\varphi_n' (\zeta_n)}{\varphi_n (\zeta_n)} - 1 \right). \tag{2.2}$$

Moreover, $|C| \le 1$.

We note that there was a (fixable) mistake in the proof of Theorem 2.1 in [15]. The mistake was corrected in [16]. It is obviously of interest to consider the case $C = 0$. This turns out to require much more on the orthonormal polynomials than in the case of measures on $[-1, 1]$. In [17, Theorem 2.1], we established:

Theorem 2.2 *Assume that μ is a regular measure with the unit circle as support. Assume that J is a closed subarc of the unit circle such that μ is absolutely continuous and μ' is positive and continuous in J. The following are equivalent:*

(I) Uniformly for z in proper subarcs of J, and for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{\varphi_n \left(z \left(1 + \frac{u}{n} \right) \right)}{\varphi_n (z)} = e^u. \tag{2.3}$$

(II) Uniformly for z in proper subarcs of J,

$$\lim_{n\to\infty} \frac{\varphi_n \left(z e^{\pm i\pi/n} \right)}{\varphi_n (z)} = -1. \tag{2.4}$$

(III) Uniformly for z in proper subarcs of J,

$$\lim_{n\to\infty} |\varphi_n (z)|^2 \mu' (z) = 1. \tag{2.5}$$

Thus a full local limit such as (2.3) requires the pointwise asymptotics (2.5) of the absolute values of the orthogonal polynomials.

3 Local Limits for Varying Exponential Weights

The Hermite weight $W(x) = \exp\left(-\frac{1}{2}x^2\right)$, $x \in \mathbb{R}$, is probably the first exponential weight whose orthogonal polynomials were thoroughly investigated. Studies of the moment problem and weighted approximation led to consideration of more general exponential weights. It was Geza Freud and later Paul Nevai that began a systematic study of the orthogonal polynomials for exponential weights such as $W(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, in the 1970s. The introduction of potential theory, and later Riemann-Hilbert techniques permitted a dramatic expansion of the precision of analysis, and the classes of weights. See [3, 4, 9, 20, 23] for some historical perspectives and references.

Some of the interest in exponential weights arises from random matrix theory. There, rather than considering a fixed exponential weight, one considers a sequence of weights, such as $e^{-nQ(x)}$ or even $e^{-nQ_n(x)}$ [3]. These are called *varying exponential weights*. As with measures with compact support, potential theory plays a key descriptive role, though here it involves what are called external fields [25].

Let Σ be a closed set on the real line, and e^{-Q} be a continuous function on Σ. If Σ is unbounded, we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} (Q(x) - \log|x|) = \infty.$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu} \left(\int\int \log \frac{1}{|x-t|} d\nu(x)\, d\nu(t) + 2\int Q\, d\nu \right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ω_Q, characterized by the following conditions: let

$$V^{\omega_Q}(z) = \int \log \frac{1}{|z-t|} d\omega_Q(t)$$

denote the potential for ω_Q. Then

$$\begin{aligned} V^{\omega_Q} + Q &\geq F_Q \text{ on } \Sigma; \\ V^{\omega_Q} + Q &= F_Q \text{ in supp}\left[\omega_Q\right]. \end{aligned}$$

Here the number F_Q is a constant. We let $\sigma_Q(x) = \frac{d\omega_Q}{dx}$. See [25] for a comprehensive treatment of potential theory for external fields.

In [12, Theorem 2.1, p. 4] we proved:

Theorem 3.1 *Let e^{-Q} be a continuous non-negative function on the set Σ, which is assumed to consist of at most finitely many intervals. If Σ is unbounded, we assume also*

$$\lim_{|x|\to\infty, x\in\Sigma} (Q(x) - \log|x|) = \infty.$$

Let h be a bounded positive continuous function on Σ, and for $n \geq 1$, let

$$d\mu_n(x) = h(x)\, e^{-2nQ(x)} dx. \tag{3.1}$$

Let J be a closed interval lying in the interior of $\operatorname{supp}\left[\omega_Q\right]$*, where ω_Q denotes the equilibrium measure for Q. Assume that ω_Q is absolutely continuous in a neighborhood of J, and that σ_Q and Q' are continuous in that neighborhood, while $\sigma_Q > 0$ there. Then uniformly for $y_{jn} \in J$ that is a local maximum of $|p_n(\mu_n, \cdot)| e^{-nQ(\cdot)}$ and for z in compact subsets of the plane, we have*

$$\lim_{n\to\infty} \frac{p_n\left(\mu_n, y_{jn} + \frac{z}{\tilde{K}_n(\mu_n, y_{jn}, y_{jn})}\right)}{p_n\left(\mu_n, y_{jn}\right)} e^{-\frac{nQ'(y_{jn})}{\tilde{K}_n(\mu_n, y_{jn}, y_{jn})}z} = \cos \pi z. \tag{3.2}$$

We also proved [12, Theorem 2.2, p. 4] a result allows that varying Q_n based on asymptotics from [11]:

Theorem 3.2 *For $n \geq 1$, let $I_n = (c_n, d_n)$, where $-\infty \leq c_n < d_n \leq \infty$. Assume that for some $r^* > 1$, $\left[-r^*, r^*\right] \subset I_n$, for all $n \geq 1$. Assume that*

$$\mu_n'(x) = e^{-2nQ_n(x)}, x \in I_n, \tag{3.3}$$

where

(i) $Q_n(x) / \log(2 + |x|)$ *has limit ∞ at c_n+ and d_n-.*
(ii) Q_n' *is strictly increasing and continuous in I_n.*
(iii) *There exists $\alpha \in (0, 1)$, $C > 0$ such that for $n \geq 1$ and $x, y \in \left[-r^*, r^*\right]$,*

$$\left|Q_n'(x) - Q_n'(y)\right| \leq C|x - y|^{\alpha}. \tag{3.4}$$

(iv) *There exists $\alpha_1 \in \left(\frac{1}{2}, 1\right)$, $C_1 > 0$, and an open neighborhood I_0 of 1 and -1, such that for $n \geq 1$ and $x, y \in I_n \cap I_0$,*

$$\left|Q_n'(x) - Q_n'(y)\right| \leq C_1|x - y|^{\alpha_1}. \tag{3.5}$$

(v) $[-1, 1]$ *is the support of the equilibrium distribution ω_{Q_n} for Q_n.*
Let $J \subset (-1, 1)$ be compact. Then uniformly for y_{jn} in J, that is a local maximum of $|p_n(\mu_n, \cdot)| e^{-nQ_n(\cdot)}$ and for z in compact subsets of the plane, we have

$$\lim_{n\to\infty} \frac{p_n\left(\mu_n, y_{jn} + \frac{z}{\tilde{K}_n(\mu_n, y_{jn}, y_{jn})}\right)}{p_n\left(\mu_n, y_{jn}\right)} e^{-\frac{nQ_n'(y_{jn})}{\tilde{K}_n(\mu_n, y_{jn}, y_{jn})}z} = \cos \pi z. \tag{3.6}$$

We deduced Theorems 3.1 and 3.2 from a general proposition for a sequence of measures $\{\mu_n\}$ [12, Theorem 2.3, p. 5]. It involves the sinc kernel

$$\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}.$$

Theorem 3.3 *Assume that for $n \geq 1$ we have a measure μ_n supported on the real line with infinitely many points in its support, and all finite power moments. Let $\{\xi_n\}$ be a bounded sequence of real numbers, and $\{\tau_n\}$ be a sequence of positive numbers that is bounded above and below by positive constants, while $\{\Psi_n\}$ is a sequence of real numbers. Assume that uniformly for a, b in compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} \frac{K_n\left(\mu_n, \xi_n + \frac{a\tau_n}{n}, \xi_n + \frac{b\tau_n}{n}\right)}{K_n(\mu_n, \xi_n, \xi_n)} e^{\Psi_n(a+b)} = \mathbb{S}(a-b). \tag{3.7}$$

Let us be given some infinite sequence of integers $\mathcal{T}$. The following are equivalent:

(I)

$$\sup_{n\in\mathcal{T}} \left| \frac{\tau_n}{n} \sum_{j=1}^{n} \frac{1}{\xi_n - x_{jn}} + \Psi_n \right| < \infty \text{ and } \sup_{n\in\mathcal{T}} \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{\left(\xi_n - x_{jn}\right)^2} < \infty. \tag{3.8}$$

(II) For each $R > 0$, there exists C_R such that

$$\sup_{n\in\mathcal{T}} \sup_{|z|\leq R} \left| \frac{p_n\left(\mu_n, \xi_n + \frac{\tau_n z}{n}\right)}{p_n(\mu_n, \xi_n)} e^{\Psi_n z} \right| \leq C_R. \tag{3.9}$$

(III) From every subsequence of $\mathcal{T}$, there is a further subsequence $\mathcal{S}$ such that

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n\left(\mu_n, \xi_n + \frac{z\tau_n}{n}\right)}{p_n(\mu_n, \xi_n)} e^{\Psi_n z} = \cos(\pi z) + \frac{\alpha}{\pi} \sin \pi z, \tag{3.10}$$

uniformly for z in compact subsets of $\mathbb{C}$, where

$$\alpha = \lim_{n\to\infty, n\in\mathcal{S}} \left[\frac{\tau_n}{n} \frac{p_n'(\mu_n, \xi_n)}{p_n(\mu_n, \xi_n)} + \Psi_n \right] \tag{3.11}$$

and α is bounded independently of $\mathcal{S}$.

This equivalence is an exponential weight analogue of similar results in [13]. In [12], we also considered local limits at the soft edge. These involve the Airy function and kernel, defined as follows:

$$\mathbb{A}i\left(a,b\right)=\begin{cases}\frac{Ai(a)Ai'(b)-Ai'(a)Ai(b)}{a-b}, & a\neq b,\\ Ai'\left(a\right)^{2}-aAi\left(a\right)^{2}, & a=b.\end{cases} \tag{3.12}$$

Ai is the Airy function, defined on the real line by Vallee and Soares [33]

$$Ai\left(x\right)=\frac{1}{\pi}\int_{0}^{\infty}\cos\left(\frac{1}{3}t^{3}+xt\right)dt. \tag{3.13}$$

The Airy function satisfies the differential equation

$$Ai''\left(z\right)-zAi\left(z\right)=0. \tag{3.14}$$

We proved the following edge analogue of Theorem 3.3:

Theorem 3.4 *Assume that for $n\geq 1$ we have a measure μ_n supported on the real line with infinitely many points in its support, and all finite power moments. Let $\{\rho_n\}$ be a sequence of positive numbers with limit 0, while $\{\Phi_n\}$ is a sequence of real numbers, such that uniformly for u, v in compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty}\frac{K_n\left(1+\rho_n u,1+\rho_n v\right)}{K_n\left(1,1\right)}e^{-\Phi_n(u+v)}=\frac{\mathbb{A}i\left(u,v\right)}{\mathbb{A}i\left(0,0\right)}. \tag{3.15}$$

Let us be given some infinite sequence of integers $\mathcal{T}$. The following are equivalent:

(I)

$$\sup_{n\in\mathcal{T}}\left|\rho_n\sum_{j=1}^{n}\frac{1}{1-x_{jn}}+\Phi_n\right|<\infty \text{ and } \sup_{n\in\mathcal{T}}\rho_n^2\sum_{j=1}^{n}\frac{1}{\left(1-x_{jn}\right)^2}<\infty. \tag{3.16}$$

(II) For each $R>0$, there exists C_R such that

$$\sup_{n\in\mathcal{T}}\sup_{|z|\leq R}\left|\frac{p_n\left(1+\rho_n z\right)}{p_n\left(1\right)}e^{\Phi_n z}\right|\leq C_R. \tag{3.17}$$

(III) From every subsequence of $\mathcal{T}$, there is a further subsequence $\mathcal{S}$ such that

$$\lim_{n\to\infty,n\in\mathcal{S}}\frac{p_n\left(1+\rho_n z\right)}{p_n\left(1\right)}e^{\Phi_n z}=\frac{Ai'\left(z\right)}{Ai'\left(0\right)}+c_0\left\{Ai\left(z\right)Ai'\left(0\right)-Ai'\left(z\right)Ai\left(0\right)\right\}, \tag{3.18}$$

uniformly for z in compact subsets of $\mathbb{C}$, where

$$c_0 = \frac{1}{Ai'(0)^2} \lim_{n\to\infty, n\in\mathcal{S}} \left\{ \rho_n \frac{p_n'(1)}{p_n(1)} + \Phi_n \right\} \tag{3.19}$$

and c_0 is bounded independently of $\mathcal{S}$.

Unfortunately, the universality limit at the soft edge, namely (3.15), seems to require far more restrictive conditions on the weight than those inside the "bulk" of the support. The most general results have been established using deep Riemann-Hilbert techniques [8, 18, 19].

4 Fixed Exponential Weights

As noted above, it was Freud who first began to consider quantitative features of orthogonal polynomials for exponential weights $W^2 = e^{-2Q}$. He usually considerd Q of polynomial growth at ∞, the prime example being $Q(x) = |x|^\alpha$, $\alpha > 0$. Weights e^{-2Q} where Q grows faster than any polynomial, are often called Erdős weights, due to Erdős' paper on them [6]. Subsequently there were efforts to investigate exponential weights on both finite and infinite intervals, and of Q of all rates of growth. Here is one such class of weights from [9, p. 7]:

Definition 4.1 Let $I = (c, d)$ be an open interval, bounded or unbounded, containing 0. Let $W = e^{-Q}$, where $Q : I \to [0, \infty)$ satisfies the following conditions:

(a) Q' is continuous in I and $Q(0) = 0$.
(b) Q'' exists and is positive in $I \setminus \{0\}$;
(c)

$$\lim_{|t|\to\infty} Q(t) = \infty.$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)}, \ t \neq 0,$$

is quasi-increasing in $(0, d)$, in the sense that for some $C > 0$,

$$0 < x < y < d \Rightarrow T(x) \leq CT(y).$$

We assume, with an analogous definition, that T is quasi-decreasing in $(c, 0)$. In addition, we assume that for some $\Lambda > 1$,

$$T(t) \geq \Lambda \text{ in } I \backslash \{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{Q'(x)}{Q(x)} \text{ a.e. } x \in I \backslash \{0\}.$$

Then we write $W \in \mathcal{F}\left(C^2\right)$.

Examples of weights in this class are $W = \exp(-Q)$, where

$$Q(x) = \begin{cases} Ax^{\alpha}, & x \in [0, \infty) \\ B|x|^{\beta}, & x \in (-\infty, 0) \end{cases},$$

where $\alpha, \beta > 1$ and $A, B > 0$. More generally, if $\exp_k = \exp(\exp(\ldots\exp()))$ denotes the kth iterated exponential, we may take

$$Q(x) = \begin{cases} \exp_k(Ax^{\alpha}) - \exp_k(0), & x \in [0, \infty) \\ \exp_{\ell}\left(B|x|^{\beta}\right) - \exp_{\ell}(0), & x \in (-\infty, 0) \end{cases}$$

where $k, \ell \geq 1$, $\alpha, \beta > 1$. An example on $I = (-1, 1)$ is

$$Q(x) = \begin{cases} \exp_k\left(\left(1 - x^2\right)^{-\alpha}\right) - \exp_k(1), & x \in [0, 1) \\ \exp_{\ell}\left(\left(1 - x^2\right)^{-\beta}\right) - \exp_{\ell}(1), & x \in (-1, 0) \end{cases},$$

where $\alpha, \beta > 0$.

A key descriptive role is played by the Mhaskar-Rakhmanov-Saff numbers

$$a_{-n} < 0 < a_n,$$

defined for $n \geq 1$ by the equations

$$n = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} dx;$$
$$0 = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} dx.$$

In the case where Q is even, $a_{-n} = -a_n$. The existence and uniqueness of these numbers is established in the monographs [9, 25], but goes back to earlier work of Mhaskar, Rakhmanov, and Saff [20, 21, 24].

We also define,

$$\beta_n = \frac{1}{2}(a_n + a_{-n}) \text{ and } \delta_n = \frac{1}{2}(a_n + |a_{-n}|),$$

which are respectively the center, and half-length of the Mhaskar-Rakhmanov-Saff interval

$$\Delta_n = [a_{-n}, a_n].$$

The linear transformation

$$L_n(x) = \frac{x - \beta_n}{\delta_n}$$

maps Δ_n onto $[-1, 1]$. Its inverse

$$L_n^{[-1]}(u) = \beta_n + u\delta_n$$

maps $[-1, 1]$ onto Δ_n. For $0 < \varepsilon < 1$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1 + \varepsilon, 1 - \varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n].$$

We let $p_n\left(W^2, x\right)$ denote the nth orthonormal polynomial for W^2, so that

$$\int_I p_n\left(W^2, x\right) p_m\left(W^2, x\right) W^2(x)\, dx = \delta_{mn}.$$

Moreover, we let

$$K_n\left(W^2, x, t\right) = \sum_{j=0}^{n-1} p_j\left(W^2, x\right) p_j\left(W^2, t\right)$$

and

$$\tilde{K}_n\left(W^2, x, t\right) = W(x)\, W(t)\, K_n\left(W^2, x, t\right).$$

The new result of this paper is:

Theorem 4.2 *Let* $W = \exp(-Q) \in \mathcal{F}\left(C^2\right)$. *Let* $0 < \varepsilon < 1$. *Then uniformly for* z *in compact subsets of the plane, and uniformly for* $y_{kn} \in J_n(\varepsilon)$, *we have*

$$\lim_{n\to\infty} \frac{p_n\left(W^2, y_{kn} + \frac{z}{\tilde{K}_n\left(W^2, y_{kn}, y_{kn}\right)}\right)}{p_n(y_{kn})} e^{-\frac{Q'(y_{kn})z}{\tilde{K}_n\left(W^2, y_{kn}, y_{kn}\right)}} = \cos \pi z. \tag{4.1}$$

In particular, if W is even, this holds uniformly for $|y_{kn}| \leq (1-\varepsilon)\, a_n$.

To prove this result, we apply Theorem 3.3 with

$$Q_n(x) = \frac{1}{n} Q\left(L_n^{[-1]}(x)\right), \; x \in L_n(I); \tag{4.2}$$

$$W_n(x) = \exp\left(-Q_n(x)\right), \; x \in L_n(I); \tag{4.3}$$

$$d\mu_n(x) = W_n^{2n}(x)\, dx. \tag{4.4}$$

Observe that with the definition of L_n above

$$W_n^{2n} = W^2 \circ L_n^{[-1]}. \tag{4.5}$$

The orthogonal polynomials $p_m(\mu_n, x)$ are related to those for W^2 by the identity

$$p_m(\mu_n, x) = \delta_n^{1/2} p_m\left(W^2, L_n^{[-1]}(x)\right). \tag{4.6}$$

This is easily established by a substitution in the orthonormality relations for $\left\{p_n\left(W^2, x\right)\right\}$. The reproducing kernel $K_n(\mu_n, x, t)$ for W_n^{2n} is related to the reproducing kernel $K_n\left(W^2, x, t\right)$ for W^2 by the identity

$$K_n(\mu_n, x, t) = \delta_n K_n\left(W^2, L_n^{[-1]}(x), L_n^{[-1]}(t)\right). \tag{4.7}$$

In what follows, we shall denote the zeros of $p_n\left(W^2, x\right)$ by

$$x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n},$$

and the zeros of $p_n(\mu_n, x)$ by

$$\hat{x}_{nn} < \hat{x}_{n-1,n} < \ldots < \hat{x}_{2n} < \hat{x}_{1n}.$$

From (4.6),

$$\hat{x}_{jn} = L_n\left(x_{jn}\right). \tag{4.8}$$

We denote the zeros of $\left(p_n\left(W^2, x\right) W(x)\right)'$ by y_{jn}, so that

$$y_{jn} \in \left(x_{j+1,n}, x_{jn}\right), 1 \leq j \leq n-1.$$

We denote the zeros of $\left(p_n(\mu_n, x)\, e^{-nQ_n(x)}\right)'$ by $\hat{y}_{jn}$ so that

$$\hat{y}_{jn} = L_n\left(y_{jn}\right) \in \left(\hat{x}_{j+1,n}, \hat{x}_{jn}\right). \tag{4.9}$$

Because of the linear nature of L_n, we see that

$$\hat{x}_{jn} - \hat{x}_{kn} = \frac{x_{jn} - x_{kn}}{\delta_n}; \tag{4.10}$$

$$\hat{x}_{jn} - \hat{y}_{kn} = \frac{x_{jn} - y_{kn}}{\delta_n}. \tag{4.11}$$

We need some technical lemmas. For sequences $\{c_n\}$, $\{d_n\}$ of non-0 real numbers, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 such that for $n \geq 1$,

$$C_1 \leq c_n / d_n \leq C_2.$$

Similar notation is used for sequences of functions. In addition, $C, C_1, C_2, \ldots$ denote constants independent of n, x, t that may be different in different occurrences.

Lemma 4.3 *Let* $W \in \mathcal{F}\left(C^2\right)$. *Let* $0 < \varepsilon < 1$. *There exists* n_0 *such that uniformly for* $n \geq n_0$ *and* $\hat{x}_{jn} \in [-1 + \varepsilon, 1 - \varepsilon]$,

$$\hat{x}_{jn} - \hat{x}_{j+1,n} \sim \frac{1}{n}. \tag{4.12}$$

Proof It is shown in [9, Corollary 13.4, p. 361] that uniformly for $n \geq 1$, $1 \leq j \leq n - 1$,

$$x_{jn} - x_{j+1,n} \sim \varphi_n\left(x_{jn}\right) \tag{4.13}$$

where [9, p. 19] for $x \in [a_{-n}, a_n]$,

$$\varphi_n(x) = \frac{|x - a_{-2n}|\,|x - a_{2n}|}{n\sqrt{[|x - a_{-n}| + |a_{-n}\eta_{-n}|]\,[|x - a_n| + |a_n \eta_n|]}}.$$

Moreover, $\varphi_n(x) = \varphi_n(a_n)$ for $x > a_n$ and $\varphi_n(x) = \varphi_n(a_{-n})$, for $x < a_{-n}$. Here [9, p. 15]

$$\eta_{\pm n} = \left[nT\left(a_{\pm n}\right) \sqrt{\frac{|a_{\pm n}|}{\delta_n}} \right]^{-2/3} = o(1)$$

(See [9, Lemma 3.7, p. 76] for the $o(1)$ relation). The class of weights in Corollary 13.4 in [9] was $\mathcal{F}\left(lip\frac{1}{2}+\right)$. As noted in [9, p. 14], this class contains $\mathcal{F}\left(C^2\right)$. We now estimate φ_n for $x \in L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [-a_n + \varepsilon\delta_n, a_n - \varepsilon\delta_n]$. Since $a_{\pm 2n} \sim a_{\pm n}$ [9, Lemma 3.5(a), pp. 71–72] we see that for such x,

$$\varphi_n(x) \sim \frac{\delta_n^2}{n\sqrt{\delta_n^2}} = \frac{\delta_n}{n}. \tag{4.14}$$

Thus uniformly for $x_{jn} \in L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon]$, (4.13) gives

$$x_{jn} - x_{j+1,n} \sim \frac{\delta_n}{n}.$$

Then (4.12) follows from (4.10). ■

Next we need an estimate for the distance between critical points and zeros:

Lemma 4.4 *Let $W \in \mathcal{F}(C^2)$. Let $0 < \varepsilon < 1$. There exists n_0 such that uniformly for $n \geq n_0$ and $\hat{x}_{jn} \in [-1+\varepsilon, 1-\varepsilon]$,*

$$\hat{x}_{jn} - \hat{y}_{jn} \sim \hat{y}_{jn} - \hat{x}_{j+1,n} \sim \frac{1}{n}. \tag{4.15}$$

Proof We analyze the spacing between $x_{jn}, x_{j+1,n}$ and y_{jn}. Now

$$\begin{aligned} & p_n\left(W^2, y_{jn}\right) W\left(y_{jn}\right) \\ &= p_n\left(W^2, y_{jn}\right) W\left(y_{jn}\right) - p_n\left(W^2, x_{j+1,n}\right) W\left(x_{j+1,n}\right) \\ &= \left(p_n\left(W^2, x\right) W(x)\right)'_{|x=\zeta} \left(y_{jn} - x_{j+1,n}\right), \end{aligned} \tag{4.16}$$

for some ζ between y_{jn} and $x_{j+1,n}$. We find an upper bound for the derivative and a lower bound for the orthogonal polynomial at y_{jn}. Now $x_{j+1,n+1} \in \left(x_{j+1,n}, x_{jn}\right)$, and $\left|p_n\left(W^2, x\right) W^2(x)\right|$ has its maximum in $\left(x_{j+1,n}, x_{jn}\right)$ at y_{jn}, so

$$\begin{aligned} & \left|p_n\left(W^2, y_{jn}\right) W\left(y_{jn}\right)\right| \\ \geq & \left|p_n\left(W^2, x_{j+1,n+1}\right) W\left(x_{j+1,n+1}\right)\right| \\ \sim & \ \delta_{n+1}^{-1}\left[\left|x_{j+1,n+1} - a_{n+1}\left(1+\eta_{n+1}\right)\right| \left|x_{j+1,n+1} - a_{-n-1}\left(1+\eta_{-n-1}\right)\right|\right]^{1/4}, \end{aligned}$$

by Theorem 13.2 in [9, Theorem 13.2, p. 360], uniformly in $1 \leq j \leq n$. As above $\eta_{\pm(n+1)} = o(1)$ and for $x_{jn} \in L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon]$,

$$\begin{aligned} &\left|x_{j+1,n+1} - a_{\pm(n+1)}\left(1+\eta_{\pm(n+1)}\right)\right| \\ &\geq C\left|x_{jn} - a_{\pm n}\right| \geq C\delta_n, \end{aligned}$$

so that uniformly for $x_{jn} \in L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon]$

$$\left|p_n\left(W^2, y_{jn}\right) W\left(y_{jn}\right)\right| \geq C\delta_n^{-1/2}. \tag{4.17}$$

To estimate the derivative, we use [9, p. 22, Theorem 1.17]

$$\sup_{x\in I}\left|p_n\left(W^2, x\right) W(x)\right| |(x-a_{-n})(a_n-x)|^{1/4} \sim 1. \tag{4.18}$$

Let

$$R(x) = p_n\left(W^2, x\right)\frac{(x-a_{-n})(a_n-x)}{\delta_n^2},$$

a polynomial of degree $n+2$. From (4.18), we have for $x \in [a_{-n-2}, a_{n+2}]$, that

$$|R(x)|\, W(x) \leq C\frac{|(x-a_{-n})(a_n-x)|^{3/4}}{\delta_n^2} \leq C\delta_n^{-1/2}.$$

By a restricted range inequality in [9, Theorem 4.2(a), p. 96],

$$\sup_{x\in I}|R(x)|\, W(x) \leq C\delta_n^{-1/2}.$$

Then a Markov-Bernstein inequality in [9, Theorem 10.1, p. 293] yields

$$\sup_{x\in I}\left|(RW)'(x)\,\varphi_{n+2}(x)\right| \leq C\sup_{x\in I}|R(x)|\, W(x) \leq C\delta_n^{-1/2}.$$

Next, if $x \in L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon]$, this and (4.14) give

$$\begin{aligned} &\left|\left(p_n\left(W^2, x\right) W(x)\right)'\right| \frac{|(x-a_{-n})(a_n-x)|}{\delta_n^2} \\ \leq &\left|\left(p_n\left(W^2, x\right) W(x)\right)\right| \frac{\left|\frac{d}{dx}\left[(x-a_{-n})(a_n-x)\right]\right|}{\delta_n^2} + Cn\delta_n^{-3/2} \end{aligned}$$

$$\leq C\left|(x-a_{-n})(a_n-x)\right|^{-1/4}\frac{|2x-\beta_n|}{\delta_n^2}+Cn\delta_n^{-3/2}$$
$$\leq C\delta_n^{-1/2}+Cn\delta_n^{-3/2}\leq Cn\delta_n^{-3/2},$$

by (4.18). Then we obtain for all $x\in L_n^{[-1]}\left[-1+\varepsilon,1-\varepsilon\right]$,

$$\left|\left(p_n\left(W^2,x\right)W(x)\right)'\right|\leq Cn\delta_n^{-3/2}.$$

In particular, this is is true for $x=\zeta$ in (4.16), so that combining (4.16), (4.17),

$$C\delta_n^{-1/2}\leq Cn\delta_n^{-3/2}\left(y_{jn}-x_{j+1,n}\right)$$

so that

$$y_{jn}-x_{j+1,n}\geq C\frac{\delta_n}{n}.$$

By (4.11), we can reformulate this as

$$\hat{y}_{jn}-\hat{x}_{j+1,n}\geq\frac{C}{n}.$$

The corresponding upper bound follows from

$$\hat{y}_{jn}-\hat{x}_{j+1,n}\leq\hat{x}_{jn}-\hat{x}_{j+1,n}\leq\frac{C}{n},$$

by the previous lemma. So we have shown

$$\hat{y}_{jn}-\hat{x}_{j+1,n}\sim\frac{1}{n}.$$

The proof for $\hat{x}_{jn}-\hat{y}_{jn}$ is similar. ■

By combining the two previous lemmas, we obtain the main estimates we need to apply Theorem 3.3:

Lemma 4.5 *Let $W\in\mathcal{F}\left(C^2\right)$. Let $0<\varepsilon<1$.*

(a) There exists n_0 and C such that uniformly for $n\geq n_0$ and $\hat{y}_{kn}\in[-1+\varepsilon,1-\varepsilon]$,

$$\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left(\hat{y}_{kn}-\hat{x}_{jn}\right)^2}\leq C. \tag{4.19}$$

(b)

$$\sum_{j=1}^{n} \frac{1}{\hat{y}_{kn} - \hat{x}_{jn}} - nQ_n'\left(\hat{y}_{kn}\right) = 0. \tag{4.20}$$

Proof

(a) Let $0 < \varepsilon < \varepsilon' < 1$. Lemma 4.4 shows that

$$\left|\hat{y}_{kn} - \hat{x}_{kn}\right|, \left|\hat{y}_{kn} - \hat{x}_{k+1,n}\right| \geq \frac{C}{n}.$$

This and Lemma 4.3 shows that there exists $C > 0$ such that for $\hat{x}_{jn}, \hat{y}_{kn} \in \left[-1+\varepsilon', 1-\varepsilon'\right]$,

$$\left|\hat{y}_{kn} - \hat{x}_{jn}\right| \geq C\frac{1+|k-j|}{n}.$$

Thus

$$\frac{1}{n^2} \sum_{\hat{x}_{jn}\in[-1+\varepsilon',1-\varepsilon']} \frac{1}{\left(\hat{y}_{kn} - \hat{x}_{jn}\right)^2} \leq C \sum_{k=-\infty}^{\infty} \frac{1}{(1+|k-j|)^2} \leq C.$$

The remaining terms may be estimated for $\hat{y}_{kn} \in [-1+\varepsilon, 1-\varepsilon]$ by

$$\frac{1}{n^2} \sum_{\hat{x}_{jn}\notin[-1+\varepsilon',1-\varepsilon']} \frac{1}{\left(\hat{y}_{kn} - \hat{x}_{jn}\right)^2} \leq \frac{Cn}{n^2} \leq C.$$

(b) We have

$$\begin{aligned} 0 &= \frac{\left(p_n\left(W^2, x\right) W(x)\right)'_{|x=y_{kn}}}{p_n\left(W^2, y_{kn}\right) W(y_{kn})} \\ &= \sum_{j=1}^{n} \frac{1}{y_{kn} - x_{jn}} - Q'(y_{kn}). \end{aligned} \tag{4.21}$$

Next from (4.2),

$$Q_n'(x) = \frac{1}{n} Q'\left(L_n^{[-1]}(x)\right) \delta_n$$

so that

$$Q_n'\left(\hat{y}_{kn}\right) = \frac{1}{n} Q'\left(y_{kn}\right)\delta_n.$$

Using (4.11), we reformulate (4.21) as

$$\sum_{j=1}^{n} \frac{1}{\delta_n\left(\hat{y}_{kn} - \hat{x}_{jn}\right)} - \frac{nQ_n'\left(\hat{y}_{kn}\right)}{\delta_n} = 0,$$

giving (4.20). ■

Finally, we need universality limits. Recall that $\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}$:

Lemma 4.6 *Let* $0 < \varepsilon < 1$. *Uniformly for* $t \in [-1+\varepsilon, 1-\varepsilon]$, *and* a, b *in compact subsets of the plane,*

$$\lim_{n\to\infty} \frac{K_n\left(\mu_n, t + \frac{a}{\tilde{K}_n(\mu_n,t,t)}, t + \frac{b}{\tilde{K}_n(\mu_n,t,t)}\right)}{K_n\left(\mu_n, t, t\right)} e^{-\frac{nQ_n'(t)}{\tilde{K}_n(\mu_n,t,t)}(a+b)} = \mathbb{S}\left(a-b\right). \tag{4.22}$$

Proof It was proved in [10, Theorem 7.4, p. 771] for the bigger class of weights $\mathcal{F}(dini)$, that uniformly for $x \in J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon]$, and a, b in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{K_n\left(W^2, x + \frac{a}{\tilde{K}_n(W^2,x,x)}, x + \frac{b}{\tilde{K}_n(W^2,x,x)}\right)}{K_n\left(W^2, x, x\right)} = \mathbb{S}\left(a-b\right).$$

This was actually deduced by applying Theorem 1.2 there. With $\{\mu_n\}$ as above, Theorem 1.2 there first gave that for $t \in [-1+\varepsilon, 1-\varepsilon]$,

$$\lim_{n\to\infty} \frac{K_n\left(\mu_n, t + \frac{a}{\tilde{K}_n(\mu_n,t,t)}, t + \frac{b}{\tilde{K}_n(\mu_n,t,t)}\right)}{K_n\left(\mu_n, t, t\right)} = \mathbb{S}\left(a-b\right). \tag{4.23}$$

As noted at (1.13) in [10, p. 749, (1.13)], the proof of Theorem 1.2 showed that uniformly for $t \in [-1+\varepsilon, 1-\varepsilon]$, and a, b in compact subsets of the complex plane,

$$\lim_{n\to\infty} \frac{K_n\left(\mu_n, t + \frac{a}{\tilde{K}_n(\mu_n,t,t)}, t + \frac{b}{\tilde{K}_n(\mu_n,t,t)}\right)}{K_n\left(\mu_n, t, t\right)} e^{-\frac{nQ_n'(t)}{\tilde{K}_n(\mu_n,t,t)}(a+b)} = \mathbb{S}\left(a-b\right). \quad ■$$

Proof of Theorem 4.2 We apply Theorem 3.3 to the measures $\{\mu_n\}$. We choose in Theorem 3.3,

$$\xi_n = \hat{y}_{kn};$$
$$\tau_n = \frac{n}{\tilde{K}_n\left(\mu_n, \hat{y}_{kn}, \hat{y}_{kn}\right)};$$
$$\Psi_n = -\frac{nQ_n'\left(\hat{y}_{kn}\right)}{\tilde{K}_n\left(\mu_n, \hat{y}_{kn}, \hat{y}_{kn}\right)}.$$

First note that Lemma 4.6 gives the universality limit (3.7) in Theorem 3.3 with these choices of ξ_n, τ_n, and Ψ_n. Moreover, it follows from Lemma 7.7 in [10, pp. 775–776] that $\{\tau_n\}$ is bounded above and below. Next from Lemma 4.5(a), we have the second condition (3.8) with ξ_n replaced by $\hat{y}_{kn}$ and x_{jn} replaced by $\hat{x}_{jn}$. Next, Lemma 4.5(b) gives

$$\frac{1}{\tilde{K}_n\left(\mu_n, \hat{y}_{kn}, \hat{y}_{kn}\right)}\sum_{j=1}^{n}\frac{1}{\hat{y}_{kn}-\hat{x}_{jn}} - \frac{nQ_n'\left(\hat{y}_{kn}\right)}{\tilde{K}_n\left(\mu_n, \hat{y}_{kn}, \hat{y}_{kn}\right)} = 0,$$

which gives a much stronger form of the first condition in (3.8). Also $\alpha = 0$ in (3.11), as follows from this last relation. Thus (3.10) of Theorem 3.3 gives

$$\lim_{n\to\infty}\frac{p_n\left(\mu_n, \hat{y}_{kn} + \frac{z}{\tilde{K}_n\left(\mu_n, \hat{y}_{kn}, \hat{y}_{kn}\right)}\right)}{p_n\left(\mu_n, \hat{y}_{kn}\right)} e^{-\frac{nQ_n'\left(\hat{y}_{kn}\right)}{\tilde{K}_n\left(\mu_n, \hat{y}_{kn}, \hat{y}_{kn}\right)}z} = \cos\left(\pi z\right),$$

uniformly for z in compact subsets of the plane. Using (4.6) , (4.7), and (4.2), this is easily reformulated as (4.1). ■

References

1. A.I. Aptekarev, Asymptotics of orthogonal polynomials in a neighborhood of the endpoints on the interval of orthogonality. Math. Sb. (Russia) **76**, 35–50 (1993)
2. V.M. Badkov, The asymptotic behavior of orthogonal polynomials (Russian). Mat. Sb. (N.S.) **109**(151)(1), 46–59 (1979)
3. P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, vol. 3. Courant Institute Lecture Notes (New York University Pres, New York, 1999)
4. P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. Commun. Pure Appl. Math. **52**, 1335–1425 (1999)
5. B. Eichinger, M. Lukić, B. Simanek, An approach to universality using Weyl m-functions, manuscript.

6. P. Erdős, On the distribution of roots of orthogonal polynomials, in *Proceedings of the Conference on Constructive Theory of Functions*, ed. by G. Alexits et al. (Akademiai Koado, Budapest, 1972), pp. 145–150
7. G. Freud, *Orthogonal Polynomials* (Pergamon Press/Akademiai Kiado, Budapest, 1971)
8. T. Kriecherbauer, K.T.-R. McLaughlin, Strong asymptotics of polynomials orthogonal with respect to Freud weights. Int. Math. Res. Not. **1999**, 299–333 (1999)
9. E. Levin, D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights* (Springer, New York, 2001)
10. E. Levin, D.S. Lubinsky, Universality limits in the bulk for varying measures. Adv. Math. **219**, 743–779 (2008)
11. E. Levin, D.S. Lubinsky, *Bounds and Asymptotics for Orthogonal Polynomials for Varying Weights*. Springer Briefs in Mathematics (Springer, Cham, 2018)
12. E. Levin, D.S. Lubinsky, Local limits for orthogonal polynomials for varying weights via universality. J. Approx. Theory **254**, 105394 (2020)
13. D.S. Lubinsky, Local asymptotics for orthonormal polynomials in the interior of the support via universality. Proc. Am. Math. Soc. **147**, 3877–3886 (2019)
14. D.S. Lubinsky, Pointwise asymptotics for orthonormal polynomials at the endpoints of the interval via universality. Int. Maths Res. Not. **2020**, 961–982 (2020). https://doi.org/10.1093/imrn/rny042
15. D.S. Lubinsky, Local asymptotics for orthonormal polynomials on the unit circle via universality. J. Anal. Math. **141**, 285–304 (2020)
16. D.S. Lubinsky, Correction to Lemma 4.2(a) and 4.3(d). J. Anal. Math. **144**, 397–400 (2021)
17. D.S. Lubinsky, On zeros, bounds, and asymptotics for orthogonal polynomials on the unit circle. Math. Sbornik. **213**, 31–49 (2022)
18. K.T.-R. McLaughlin, P. Miller, The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights. Int. Math. Res. Pap., Article ID 48673, 1–78 (2006)
19. K.T.-R. McLaughlin, P. Miller, The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights. Int. Math. Res. Not., Art. ID rnn 075, 1–66 (2008)
20. H.N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation* (World Scientific, Singapore, 1996)
21. H.N. Mhaskar, E.B. Saff, Where does the sup norm of a weighted polynomial live? Constr. Approx. **1**, 71–91 (1985)
22. P. Nevai, *Orthogonal Polynomials*, Memoirs of the AMS no. 213 (1979)
23. P. Nevai, Geza Freud, orthogonal polynomials and Christoffel functions: a case study. J. Approx. Theory **48**, 3–167 (1986)
24. E.A. Rakhmanov, On asymptotic properties of polynomials orthogonal on the real axis. Math. USSR. Sbornik **47**, 155–193 (1984)
25. E.B. Saff, V. Totik, *Logarithmic Potentials with External Fields* (Springer, New York, 1997)
26. B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2 (American Mathematical Society, Providence, 2005)
27. B. Simon, *Szegö's Theorem and its Descendants* (Princeton University Press, Princeton, 2011)
28. H. Stahl, V. Totik, *General Orthogonal Polynomials* (Cambridge University Press, Cambridge, 1992)
29. G. Szegő, *Orthogonal Polynomials*. American Mathematical Society Colloquium Publications (American Mathematical Society, Providence, 1975)
30. V. Totik, Universality and fine zero spacing on general sets. Arkiv Mat. **47**, 361–391 (2009)
31. V. Totik, Universality under Szegő's condition. Canadian Math. Bull. **59**, 211–224 (2016)
32. V. Totik, Oscillatory behavior of orthogonal polynomials. Acta Math. Hungar. **160**, 453–467 (2020)
33. O. Vallee, M. Soares, *Airy Functions and Applications to Physics* (World Scientific, Singapore, 2004)

New Trends in Geometric Function Theory

Khalida Inayat Noor and Mohsan Raza

1 Introduction

Geometric function theory is a classical branch of mathematics which deals with the geometrical behaviour of analytic functions, and Riemann, Cauchy, Weierstrass, Koebe [40, 45, 81] being the pioneers in this field. Riemann mapping theorem [17] gave us the permission to use the open unit disc E instead of any arbitrary domain $D \subset \mathbb{C}$ with at least two boundary points. Koebe discovered that if analytic functions have an additional property of being univalent in $\mathbb{E}$, then in this case the conformality and the assertion of Riemann mapping theorem are confirmed. The set of functions f which are analytic and univalent in $\mathbb{E}$, and satisfying the normalization conditions $f(0) = 0 = \left\{f'(0) - 1\right\}$ was denoted by the class $\mathscr{S}$ and has been the fundamental component for this area. On making use of the geometry of the image domain, certain sub-classes of $\mathscr{S}$ were defined, and among these, the most significant are the classes $\mathscr{S}^*$ and $\mathscr{C}$ which, respectively, consist of starlike and convex univalent functions. Thus it can be seen that geometric function theory establishes a beautiful relation between geometry and analysis. Nevanlinna and many other notable researchers studied these classes, see [5, 26, 81, 98]. Kaplan [37] introduced the class $\mathscr{K} \subset \mathscr{S}$ of close-to-convex functions and provided the geometrical characterization of the functions in this class.

Apart from the geometry of the image domain, these functions have been discussed with respect to estimates and bounds for the coefficients of the series expansion. The first result was due to Bieberbach's estimates for the second

K. I. Noor
Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

M. Raza (✉)
Department of Mathematics, Government College University Faisalabad, Faisalabad, Pakistan

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