EXACT INTERPOLATION, SPURIOUS POLES, AND UNIFORM CONVERGENCE OF MULTIPOINT PADÉ APPROXIMANTS

D. S. LUBINSKY

Abstract. We introduce the concept of an exact interpolation index n associated with a function f and open set $\mathcal{L}$: all rational interpolants $R = p/q$ of type (n, n) to f with interpolation points in $\mathcal{L}$, interpolate exactly in the sense that $fq - p$ has exactly $2n+1$ zeros in $\mathcal{L}$. We show that in the absence of exact interpolation, there are interpolants with interpolation points in $\mathcal{L}$ and spurious poles. Conversely, for sequences of integers that are associated with exact interpolation to an entire function, there is at least a subsequence with no spurious poles, and consequently, there is uniform convergence.

Padé approximation, Multipoint Padé approximants, spurious poles. 41A21, 41A20, 30E10. Dedicated to the 150th anniversary of Math. Sbornik

1. Introduction[1]

Let $\mathcal{D}$ be an open connected subset of $\mathbb{C}$, and $f : \mathcal{D} \to \mathbb{C}$ be analytic. Given $n \geq 1$ and not necessarily distinct points $\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1}$ in $\mathcal{D}$, and

$$\omega_n(z) = \omega_n(\Lambda_n, z) = \prod_{j=1}^{2n+1} (z - z_{jn}),$$

the multipoint Padé approximant to f with interpolation set Λ_n is a rational function

$$R_n(\Lambda_n, z) = \frac{p_n(\Lambda_n, z)}{q_n(\Lambda_n, z)},$$

or more simply,

$$R_n(z) = \frac{p_n(z)}{q_n(z)}$$

Date: May 15, 2017.

[1]Research supported by NSF grant DMS1362208

where p_n and q_n are polynomials of degree $\leq n$ with q_n not identically zero, and

$$\frac{e_n(z)}{\omega_n(z)} = \frac{e_n(\Lambda_n, z)}{\omega_n(\Lambda_n, z)} = \frac{f(z)\, q_n(z) - p_n(z)}{\omega_n(z)}$$

is analytic in $\mathcal{D}$. The special case where all $z_{jn} = 0$, gives the Padé approximant $[n/n](z)$. It is well known that R_n exists and is unique, though p_n and q_n are not separately unique. Moreover, it is possible that in order to satisfy the interpolation conditions, p_n and q_n may need to include some common factors $z - z_{jn}$ with zeros at the interpolation points $\{z_{jn}\}$.

The convergence of sequences of rational interpolants, and especially Padé approximants, is a complex and much studied subject. Many of the beautiful results from the Russian school headed by A. Gončar, have appeared in this journal. One of the unfortunate properties of such interpolants is the appearance of *spurious poles*: R_n may have poles that have no relation to singularities of the underlying function f. These are typically close to *spurious zeros*, that also have little relation to zeros of f. See [1], [3], [4], [5], [6], [8], [9], [10], [13], [14], [15], [16], [17], [19], [20], [22] for some references and surveys of the convergence theory, that bear on the issue of spurious poles. Of course this is not a precisely defined concept, at least for just one rational interpolant. It is best considered for sequences of interpolants, whose limit points of poles do not approach singularities of the underlying function.

Spurious poles are also known to be related in some sense to the appearance of extra zeros of $e_n(z)$, that is zeros other than $\{z_{jn}\}_{j=1}^{2n+1}$ [2], [20]. Especially for algebraic and elliptic functions, this has been established in a fairly precise sense, in particular for diagonal Padé approximants $\{[n/n]\}_{n\geq 1}$. For polynomial interpolation, "overinterpolation" has been investigated in [7]. The goal of this paper is to further explore this relationship, by considering all interpolants with interpolation points in a given set. This is a new idea to the best of our knowledge, and as we shall see, has several advantages.

Definition 1.1
Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, and $f : \mathcal{D} \to \mathbb{C}$ be analytic. Let $\mathcal{L} \subset \mathcal{D}$ be open and $n \geq 1$. We say n is an **exact interpolation index for f and $\mathcal{L}$** *if for every set of $2n + 1$ not necessarily distinct*

interpolation points $\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1}$ *in* $\mathcal{L}$*, and every corresponding interpolant* $R_n(\Lambda_n, z) = p_n(z)/q_n(z)$,

$$\frac{e_n(\Lambda_n, z)}{\omega_n(z)} = \frac{f(z)\,q_n(z) - p_n(z)}{\omega_n(z)}$$

has no zeros in $\mathcal{L}$.

Note that the condition forces at least one of p_n and q_n to have degree n. Otherwise we can add an extra zero c at any point in $\mathcal{L}$, since $p_n(z)(z-c)$ and $q_n(z)(z-c)$ will have degree at most n, while

$$\frac{f(z)\,q_n(z)(z-c) - p_n(z)(z-c)}{\omega_n(z)}$$

will have the extra zero c. The property that at least one of p_n, q_n have full degree is typically described as $R_n(\Lambda_n, z)$ having defect 0.

The relevance of exact interpolation to spurious poles is clear from the following simple:

Proposition 1.2
Let $\mathcal{D} \subset \mathbb{C}$ *be a connected open set, and* $f : \mathcal{D} \to \mathbb{C}$ *be analytic. Let* $n \geq 1$ *and* $\mathcal{L}$ *and* $\mathcal{B}$ *be open subsets of* $\mathcal{D}$*. Assume that whenever we are given a set of* $2n+3$ *not necessarily distinct points* $\Lambda_{n+1} \subset \mathcal{L} \cup \mathcal{B}$, $R_{n+1}(\Lambda_{n+1}, z)$ *does not have poles in* $\mathcal{B}$*. Then* n *is an exact interpolation index for* f *and* $\mathcal{L}$.

We shall prove this simple proposition in Section 2. Note that the pole free interpolant $R_{n+1}(\Lambda_{n+1}, z)$ has type $(n+1, n+1)$, not (n, n). We shall also prove a much deeper partial converse of Proposition 1.2, that exact interpolation to entire functions forces the absence of spurious poles, at least for a subsequence. Throughout this paper,

$$B_r = \{z : |z| < r\}, r > 0.$$

Theorem 1.3
Let f *be entire. Let* $\{n_k\}_{k\geq 1}$ *be an increasing sequence of positive integers such that for* $k \geq 1$*, and for some integer* $L > 1$,

$$\frac{n_{k+1}}{n_k} \leq L. \tag{1.1}$$

Assume that there is an increasing sequence $\{r_k\}_{k\geq 1}$ *of positive numbers with*

$$\lim_{k\to\infty} r_k = \infty \tag{1.2}$$

and for $k \geq 1$, $n_k - 1$ is an exact interpolation index for f and the ball B_{r_k}. Then there exists a subsequence $\left\{n_{k_j}\right\}_{j\geq 1}$ of $\{n_k\}_{k\geq 1}$ with the following property: let $r, s > 0$, and for $j \geq 1$, choose interpolation sets $\Lambda_{n_{k_j}}$ in B_r. Then for large enough j, $R_{n_{k_j}}\left(\Lambda_{n_{k_j}}, z\right)$ is analytic in B_s. Consequently, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{j\to\infty} R_{n_{k_j}}\left(\Lambda_{n_{k_j}}, z\right) = f(z).$$

We emphasize that the same subsequence $\left\{n_{k_j}\right\}_{j\geq 1}$ works for all sets of interpolation points in B_r, and for all r.

When we have mild regularity of errors of best rational approximation, we can establish uniform convergence of full sequences. Let K be a compact set and $f : K \to \mathbb{C}$ be continuous. We let

$$E_n(f, K) = \inf\{||f - \frac{p}{q}||_{L_\infty(K)} : p, q \text{ have degree } \leq n \text{ and } q \neq 0 \text{ in } K\}.$$

A best approximant of type (n, n), $R_n^*(f, K) = \frac{p_n^*}{q_n^*}$, is a rational function of type (n, n) satisfying

$$\|f - R_n^*(f, K)\|_{L_\infty(K)} = E_n(f, K).$$

We also let

$$\eta_n(f, K) = E_n(f, K)^{1/n}, \ n \geq 1.$$

Theorem 1.4
Let f be entire. Let $\{n_k\}_{k\geq 1}$ be a strictly increasing sequence of positive integers. Assume that there is an increasing sequence $\{r_k\}_{k\geq 1}$ of positive numbers satisfying (1.2), such that for $k \geq 1$, $n_k - 1$ is an exact interpolation index for f and the ball B_{r_k}. Assume in addition either that
(a) for some $\tau > 0, \delta \in (0, 1)$, integer $M > 1$ and large enough k,

$$E_{n_k}\left(f, \overline{B_\tau}\right) > E_{Mn_k}\left(f, \overline{B_\tau}\right)^{1-\delta}; \tag{1.3}$$

or
(b) for some $T > 1$ and an unbounded sequence of values of r,

$$\limsup_{k\to\infty}\left(E_{n_k}\left(f, \overline{B_r}\right)/E_{n_k}\left(f, \overline{B_{r/4}}\right)\right)^{1/n_k} < T. \tag{1.4}$$

Then given any $r, s > 0$, and for $k \geq 1$, interpolation sets Λ_{n_k} in B_r, *then for large enough k, $R_{n_k}(\Lambda_{n_k}, z)$ is analytic in B_s. Consequently, uniformly for z in compact subsets of $\mathbb{C}$,*

$$\lim_{k\to\infty} R_{n_k}(\Lambda_{n_k}, z) = f(z).$$

Remarks

(a) We note that the regularity condition (1.3) is a weak one. Indeed, we can reformulate it as

$$\eta_{n_k}\left(f, \overline{B_\tau}\right) > \eta_{Mn_k}\left(f, \overline{B_\tau}\right)^{(1-\delta)M}$$

and since $(1-\delta)M$ may be very large, while

$$\lim_{k\to\infty} \eta_{n_k}\left(f, \overline{B_\tau}\right) = 0,$$

certainly this is true for regularly behaved errors of approximation. For example, if for some ℓ, $\left\{\eta_n\left(f, \overline{B_\tau}\right)\right\}_{n\geq\ell}$ is decreasing, then (1.3) is true for $n_k = k$, $k \geq 1$. Note too that $\overline{B_\tau}$ can be replaced in (1.3) by any set of positive logarithmic capacity.

(b) Similarly, for regularly behaved functions, and for all $r > 0$

$$\lim_{n\to\infty} \left(E_n\left(f, \overline{B_r}\right)/E_n\left(f, \overline{B_{r/4}}\right)\right)^{1/(2n)} = 4,$$

[15], so (1.4) is not a severe condition. On the other hand, it is easy to construct entire functions with lacunary Maclaurin series for which (1.4) fails for a subsequence of integers.

(c) This circle of ideas may be extended to non-diagonal sequences of interpolants, and probably to functions meromorphic in the plane.

(d) The biggest question that arises from this paper is the existence of sequences of exact indices of interpolation. If for example, f has a normal Padé approximant at 0, so $[n/n] = p_n/q_n$ where p_n and q_n have full degree n, and

$$(fq_n - p_n)(z) = cz^{2n+1} + \ldots$$

with $c \neq 0$, then from classical continuity results for interpolation, there exists $\varepsilon > 0$ such that n is an exact index for f and B_ε. Indeed, this is an easy consequence of the explicit formulas for rational interpolants in terms of divided differences [1, pp. 338 ff.], which show that the interpolants vary continuously (and even analytically) in the interpolation points. However, the ε of course depends on n. To be useful, one needs a sequence of indices exact on balls that are independent of n. Such results follow for e^z from Proposition 1.2 and the fact that diagonal multipoint Padé approximants with interpolation points in any compact set have been shown to converge [21], but are worth exploring in a more general setting. Certainly Proposition 1.2 shows

that in the absence of exact interpolation indices, we cannot have uniform convergence of every sequence of interpolants with interpolation points in a compact ball.
(e) For rational interpolation to be regarded as "stable" or "robust", one would ideally prefer that when the interpolation points are shifted slightly, new spurious poles do not suddenly arise. Propositions 1.2, Theorems 1.3 and 1.4 suggest that such stability is associated with sequences of exact interpolation indices.
(f) The main tool in proving Theorems 1.3 and 1.4 is Theorem 3.1, which establishes a certain dichotomy. Roughly speaking, this asserts that when there are spurious poles for a sequence of interpolants, then either preceding indices are not exact interpolation indices, or we have smaller than expected errors of best rational approximation.

The paper is organized as follows: we prove Proposition 1.2 in Section 2. We establish a basic alternative in Section 3, and prove Theorems 1.3 and 1.4 in Section 4.

2. Nonexact Interpolation Implies Spurious Poles

We begin by showing the very simple result that if n is not an exact interpolation index, then there are rational interpolants with interpolation points close to a given set of interpolation points, having spurious poles close to any other given point:

Proposition 2.1
Let $\mathcal{D} \subset \mathbb{C}$ be open, and $f : \mathcal{D} \to \mathbb{C}$ be analytic. Let $n \geq 1$ and let us be given $2n+1$ not necessarily distinct interpolation points $\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1} \subset \mathcal{D}$. Assume that

$$\frac{e_n(\Lambda_n, z)}{\omega_n(\Lambda_n, z)} = \frac{f(z)\, q_n(z) - p_n(z)}{\omega_n(z)}$$

has a zero b in $\mathcal{D}$. Let $\varepsilon > 0$ and $c \in \mathcal{D}$. Then we can find an interpolation set of $2n+3$ points

$$\Lambda_{n+1} = \left\{z'_{j\ n+1}\right\}_{j=1}^{2n+1} \cup \{b', c'\} \tag{2.1}$$

with

$$\max_j \left|z_{jn} - z'_{j\ n+1}\right| < \varepsilon,\ 1 \leq j \leq 2n+1,\ \textit{and}\ |b - b'| < \varepsilon\ \textit{and}\ |c - c'| < \varepsilon, \tag{2.2}$$

such that

$$R_{n+1}(\Lambda_{n+1}, z) = \frac{p_{n+1}(\Lambda_{n+1}, z)}{q_{n+1}(\Lambda_{n+1}, z)}$$

has a pole and a zero less than an ε distance from c.

Proof

Choose sequences $\{a_m\}$ and $\{b_m\}$ with $a_m \neq b_m$ for all $m \geq 1$, and

$$\lim_{m\to\infty} a_m = c = \lim_{m\to\infty} b_m.$$

Assume in addition that $q_n(b_m) \neq 0$ and $p_n(a_m) \neq 0$ for all $m \geq 1$. Consider the functions

$$g_m(z) = f(z)\, q_n(z)(z-a_m) - p_n(z)(z-b_m),$$

$m \geq 1$. We see that uniformly for z in compact subsets of D,

$$\lim_{m\to\infty} g_m(z) = (f(z)\, q_n(z) - p_n(z))(z-c).$$

The right-hand side has zeros at the $2n+3$ zeros (counting multiplicity) of $\omega_n(z)(z-b)(z-c)$, by our hypothesis. By Hurwitz' Theorem, for large enough m, g_m has zeros of total multiplicity $2n+3$ that approach $\{z_j\}_{j=1}^{2n+1} \cup \{b,c\}$ as $m \to \infty$. It follows that for large enough m, we can choose a set Λ_{n+1} satisfying (2.1) and (2.2), and such that

$$R_{n+1}(\Lambda_{n+1}, z) = \frac{p_{n+1}(\Lambda_{n+1}, z)}{q_{n+1}(\Lambda_{n+1}, z)} = \frac{p_n(z)(z-b_m)}{q_n(z)(z-a_m)}$$

and in particular, this rational interpolant has a pole at a_m and a zero at b_m, arbitrarily close to c. ■

As a consequence:

Proof of Proposition 1.2

If n is not exact for f and $\mathcal{L}$, we can find Λ_n in $\mathcal{L}$ for which

$$\frac{f(z)\, q_n(\Lambda_n, z) - p_n(\Lambda_n, z)}{\omega_n(z)}$$

has a zero in $\mathcal{L}$. Then the construction of Proposition 2.1 shows that we can find $R_{n+1}(\Lambda_{n+1}, z)$ with $2n+2$ of its $2n+3$ interpolation points in $\mathcal{L}$ and one in $\mathcal{B}$ such that $R_{n+1}(\Lambda_{n+1}, z)$ has poles in $\mathcal{B}$, a contradiction. ■

3. The Basic Alternative

Recall the definition of the Gončar-Walsh class $\mathcal{R}_0(K)$: let K be a compact set, and f be analytic at each point of K. We write $f \in \mathcal{R}_0(K)$ if

$$\lim_{n\to\infty} E_n(f,K)^{1/n} = 0.$$

The main result of this section shows that spurious poles lead either to nonexact interpolation, or "smaller than expected" errors of best rational approximation for functions in the Gončar-Walsh class. We let

$$\|f\|_r = \sup\{|f(z)| : |z| = r\}.$$

Theorem 3.1
Let f be analytic in a neighborhood of $\overline{B_1}$ and belong to the Gončar-Walsh class there. Let $L > 1$ be an integer. Assume that $\delta \in (0,1)$, and $\varepsilon \in \left(0, \frac{1}{8}\right)$ is so small that

$$3\,(8\varepsilon)^{\frac{4}{\pi}\arcsin\left(\frac{1}{3}\right)}\,21^L < 1. \tag{3.1}$$

Assume that we are given an infinite sequence $\mathcal{S}$ of positive integers, and for each $n \in \mathcal{S}$, we are given $m = m(n)$ (not necessarily in $\mathcal{S}$) such that

$$1 \le m/n \le L.$$

Suppose also that for each $n \in \mathcal{S}$, there exist $2n+1$ not necessarily distinct interpolation points Λ_n in B_ε, such that $R_n(\Lambda_n, z)$ has a pole in B_ε. Then for large enough $n \in \mathcal{S}$, either
(I) There is a set Λ_{n-1} of $2n-1$ interpolation points in B_1, such that if $R_{n-1}(\Lambda_{n-1}, z) = \frac{p_{n-1}(z)}{q_{n-1}(z)}$, then $e_{n-1} = fq_{n-1} - p_{n-1}$ has at least $2n$ zeros in B_1, counting multiplicity,
or
(II)

$$E_n\left(f, \overline{B_1}\right) \le E_m\left(f, \overline{B_1}\right)^{1-\delta}. \tag{3.2}$$

We begin with a more technical form of Theorem 3.1. Then we present a series of lemmas, and finally prove Theorem 3.1.

Lemma 3.2
Let $\sigma \ge 1$ and f be analytic in $\overline{B_\sigma}$. Let $m, n \ge 1, \varepsilon \in (0,1)$. Assume that we are given
(i) $2n+1$ not necessarily distinct interpolation points Λ_n in B_ε;
(ii) $e_n = fq_n - p_n$ has zeros of total multiplicity $N(\ge 2n+1)$ in B_ε;
(iii) Suppose also that $R_n(\Lambda_n, z) = \frac{p_n(z)}{q_n(z)}$ has a pole $a \in B_\varepsilon$. Then either
(I) There is a set Λ_{n-1} of $2n-1$ interpolation points in B_1, such that if $R_{n-1}(\Lambda_{n-1}, z) = \frac{p_{n-1}(z)}{q_{n-1}(z)}$, then $e_{n-1} = fq_{n-1} - p_{n-1}$ has at least $N - 1 \ge 2n$ zeros in B_1, counting multiplicity,
or

(II) If $\varepsilon < r < \rho < 1$, *and*

$$\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)}\left(\frac{\rho}{r}\right)^{m+n}\left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m \le \frac{1}{2} \tag{3.3}$$

then

(3.4)

$$E_n\left(f,\overline{B_\rho}\right) \le 28n^2\left(\frac{\rho}{r}\right)^{m+n}\left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m \frac{\|q_n\|_\rho}{\min_{|t|=\rho}|q_n(t)|}E_m\left(f,\overline{B_\sigma}\right).$$

Proof

First observe that since $q_n(a) = 0$, and as a is a pole of $R_n(\Lambda_n, z)$,

$$e_n(a) = -p_n(a) \neq 0,$$

so

$$\begin{aligned} e_n(z) - e_n(a) &= f(z)\,q_n(z) - (p_n(z) - p_n(a)) \\ &= (z-a)\left(fq_{n-1}(z) - p_{n-1}(z)\right), \end{aligned}$$

where p_{n-1} and q_{n-1} have degree at most $n-1$.

(I) Suppose for some $s \in [\varepsilon, 1]$,

$$\min_{|z|=s}|e_n(z)| > |e_n(a)|.$$

Then by Rouché's Theorem, $e_n(z) - e_n(a)$ has the same multiplicity of zeros in B_s as e_n, and in particular, at least N. Then $e_{n-1} = fq_{n-1} - p_{n-1}$ has at least $N - 1 \ge 2n$ zeros inside $\{z : |z| = s\}$, and this gives us Λ_{n-1}. In fact, as we can omit one zero of e_{n-1} from Λ_{n-1}, there might be multiple choices for Λ_{n-1}. So we have (I).

If the hypothesis of (I) fails, then

(II) For all $s \in (\varepsilon, 1]$,

$$\min_{|z|=s}|e_n(z)| \le |e_n(a)| \le \|e_n\|_\varepsilon. \tag{3.5}$$

We apply the Beurling-Nevanlinna Theorem [18, p. 120, Thm. 4.5.6]. Let $\varepsilon < \rho \le 1$, and

$$u(z) = \frac{\log\left(|e_n(\rho z)| / \|e_n\|_\rho\right)}{\left|\log\left(\|e_n\|_\varepsilon / \|e_n\|_\rho\right)\right|}, |z| < 1.$$

Then u is subharmonic in $|z| < 1$, and clearly $u \le 0$ in $|z| < 1$, while for $\frac{\varepsilon}{\rho} \le r < 1$, our hypothesis (3.5) shows that

$$\inf_{|z|=r} u(z) \le \frac{\log\left(\|e_n\|_\varepsilon / \|e_n\|_\rho\right)}{\left|\log\left(\|e_n\|_\varepsilon / \|e_n\|_\rho\right)\right|} = -1.$$

On the other hand, for $0 \leq r \leq \frac{\varepsilon}{\rho}$, the maximum modulus principle shows that even

$$\sup_{|z|=r} u(z) \leq \frac{\log\left(\|e_n\|_\varepsilon / \|e_n\|_\rho\right)}{\left|\log\left(\|e_n\|_\varepsilon / \|e_n\|_\rho\right)\right|} = -1.$$

In summary, we have shown that u is subharmonic in $|z| < 1$, that $u(z) \leq 0$ there, and for all $0 \leq r < 1$,

$$\inf_{|z|=r} u(z) \leq -1.$$

Then the Beurling-Nevanlinna Theorem [18, p. 120, Thm. 4.5.6] shows that for all $|z| \leq 1$,

$$u(z) \leq -\frac{2}{\pi} \arcsin\left(\frac{1-|z|}{1+|z|}\right)$$

which can be reformulated as

$$\frac{|e_n(\rho z)|}{\|e_n\|_\rho} \leq \left(\frac{\|e_n\|_\varepsilon}{\|e_n\|_\rho}\right)^{\frac{2}{\pi}\arcsin\left(\frac{1-|z|}{1+|z|}\right)}.$$

Now considering $\rho|z| = r$ gives

$$\frac{\|e_n\|_r}{\|e_n\|_\rho} \leq \left(\frac{\|e_n\|_\varepsilon}{\|e_n\|_\rho}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)}, \ 0 < r < \rho < 1.$$

Next, the maximum modulus principle shows that

$$\left\|\frac{e_n}{\omega_n}\right\|_\varepsilon \leq \left\|\frac{e_n}{\omega_n}\right\|_\rho,$$

so

$$\frac{\|e_n\|_\varepsilon}{\|e_n\|_\rho} \leq \frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho} |\omega_n(t)|}.$$

Thus also

$$\frac{\|e_n\|_r}{\|e_n\|_\rho} \leq \left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)}, 0 < r < \rho < 1. \tag{3.6}$$

Next, write $R_m^*\left(f, \overline{B_\sigma}\right) = p_m^*/q_m^*$, and observe that if $e_m^* = f q_m^* - p_m^*$, then

$$e_n q_m^* - e_m^* q_n = p_m^* q_n - p_n q_m^*.$$

By Bernstein's growth inequality [18, p. 156] applied to the right-hand side, which is a polynomial of degree $\leq m+n$,

$$\|e_n q_m^* - e_m^* q_n\|_\rho \leq \left(\frac{\rho}{r}\right)^{m+n} \|e_n q_m^* - e_m^* q_n\|_r$$

$$\Rightarrow \quad \|e_n q_m^*\|_\rho - \|e_m^* q_n\|_\rho \le \left(\frac{\rho}{r}\right)^{m+n} \left(\|e_n q_m^*\|_r + \|e_m^* q_n\|_r\right)$$

$$\Rightarrow \quad \|e_n\|_\rho \min_{|t|=\rho} |q_m^*(t)| - \|e_m^*\|_\rho \|q_n\|_\rho \le \left(\frac{\rho}{r}\right)^{m+n} \left(\|e_n\|_r \|q_m^*\|_r + \|e_m^*\|_\rho \|q_n\|_\rho\right)$$

$$\Rightarrow \quad \|e_n\|_\rho \left\{1 - \frac{\|e_n\|_r}{\|e_n\|_\rho}\left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_m^*\|_r}{\min_{|t|=\rho}|q_m^*(t)|}\right\} \le 2\|e_m^*\|_\rho \left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_n\|_\rho}{\min_{|t|=\rho}|q_m^*(t)|}, \tag{3.7}$$

recall that $r < \rho$. Next, as q_m^* has no zeros in $\overline{B_\sigma}$,

$$\frac{\|q_m^*\|_r}{\min_{|t|=\rho}|q_m^*(t)|} \le \frac{\|q_m^*\|_\rho}{\min_{|t|=\rho}|q_m^*(t)|} \le \left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m. \tag{3.8}$$

Then using (3.6),

$$\frac{\|e_n\|_r}{\|e_n\|_\rho}\left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_m^*\|_r}{\min_{|t|=\rho}|q_m^*(t)|}$$
$$\le \left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)} \left(\frac{\rho}{r}\right)^{m+n} \left(\frac{1+\rho/\sigma}{1-\rho/\sigma}\right)^m \le \frac{1}{2},$$

by our hypothesis (3.3). So (3.7) gives

$$\|e_n\|_\rho \le 4\|e_m^*\|_\rho \left(\frac{\rho}{r}\right)^{m+n} \frac{\|q_n\|_\rho}{\min_{|t|=\rho}|q_m^*(t)|}. \tag{3.9}$$

Here provided q_n has no zeros on the circle $|t| = \rho$,

$$\begin{aligned} \|e_n\|_\rho &\ge \min_{|t|=\rho}|q_n(t)| \left\|f - \frac{p_n}{q_n}\right\|_\rho \\ &\ge \min_{|t|=\rho}|q_n(t)|\, E_n(f, \{t : |t| = \rho\}) \ge \min_{|t|=\rho}|q_n(t)| \frac{1}{7n^2} E_n\left(f, \overline{B_\rho}\right), \end{aligned}$$

by a classical inequality of Gončar and Grigorjan for analytic parts of meromorphic functions, for the simply connected domain B_ρ [12, Corollary 1, p. 145], [11, Thm. 1, p. 571]. Moreover,

$$\|e_m^*\|_\rho \le E_m\left(f, \overline{B_\sigma}\right)\|q_m^*\|_\rho.$$

Combining the last two inequalities and (3.9) gives

$$\min_{|t|=\rho}|q_n(t)| \frac{1}{7n^2} E_n\left(f, \overline{B_\rho}\right) \le 4\frac{\|q_m^*\|_\rho}{\min_{|t|=\rho}|q_m^*(t)|}\left(\frac{\rho}{r}\right)^{m+n} \|q_n\|_\rho E_m\left(f, \overline{B_\sigma}\right).$$

Applying (3.8) once more, we obtain (3.4). ■

We also give an alternative form that involves errors of the same approximant on balls of different radii:

Lemma 3.3
Assume the hypotheses of Lemma 3.2. Then either we have (I) there, or
(II′) for $\varepsilon < r < \rho < 1$, and $\rho < s < \sigma$,

$$\frac{E_n\left(f,\overline{B_r}\right)}{E_n\left(f,\overline{B_s}\right)} \leq \frac{7n^2}{1-\rho/s}\left(\frac{2}{1-\rho/s}\right)^n\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)} \times\frac{\|\omega_n\|_\rho}{\min_{|t|=s}|\omega_n(t)|}\frac{\|q_n\|_s}{\min_{|t|=r}|q_n(t)|}.$$

Proof
(II′) We start with (3.6). As in the previous proof,

$$\|e_n\|_r \geq \frac{1}{7n^2}\min_{|t|=r}|q_n(t)|\,E_n\left(f,\overline{B_r}\right). \tag{3.10}$$

Also, if $\rho < s \leq \sigma$ and $R_n^*\left(f,\overline{B_s}\right) = p_n^*/q_n^*$, Cauchy's integral formula gives for $|z| < s$,

$$\begin{aligned}\frac{(fq_n-p_n)(z)\,q_n^*(z)}{\omega_n(z)} &= \frac{1}{2\pi i}\int_{|t|=s}\frac{(fq_n-p_n)(t)\,q_n^*(t)}{\omega_n(t)}\frac{dt}{t-z}\\ &= \frac{1}{2\pi i}\int_{|t|=s}\frac{(fq_n^*-p_n^*)(t)\,q_n(t)}{\omega_n(t)}\frac{dt}{t-z},\end{aligned}$$

(since $(p_n^*q_n-p_nq_n^*)(t)/(\omega_n(t)(t-z))$ is analytic outside this circle and $O(t^{-2})$ at ∞). We deduce that

$$\|e_n\|_\rho \leq \frac{\|\omega_n\|_\rho}{\min_{|t|=s}|\omega_n(t)|}E_n\left(f,\overline{B_s}\right)\frac{\|q_nq_n^*\|_s}{\min_{|t|=\rho}|q_n^*|(t)}\frac{1}{1-\rho/s}.$$

Combining this, (3.10), and (3.6) gives

$$\begin{aligned}&\frac{E_n\left(f,\overline{B_r}\right)}{E_n\left(f,\overline{B_s}\right)}\\ \leq\ &\frac{7n^2}{\min_{|t|=r}|q_n(t)|}\frac{\|e_n\|_r}{\|e_n\|_\rho}\frac{\|e_n\|_\rho}{E_n\left(f,\overline{B_s}\right)}\\ \leq\ &\frac{7n^2}{1-\rho/s}\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)}\\ &\times\frac{\|\omega_n\|_\rho}{\min_{|t|=s}|\omega_n(t)|}\frac{\|q_nq_n^*\|_s}{\min_{|t|=r}|q_n(t)|\min_{|t|=\rho}|q_n^*|(t)}.\end{aligned}$$

provided q_n has no zeros on the circle $|t| = r$. Finally as q_n^* has no zeros in $\overline{B_s}$,

$$\frac{\|q_n^*\|_s}{\min_{|t|=\rho} |q_n^*| (t)} \leq \left(\frac{2}{1-\rho/s} \right)^n .$$

■

Next, we apply Cartan's Lemma along standard lines:

Lemma 3.4
Let Q be a polynomial of degree $\leq n$ and $s \geq 1$. Let $\eta \in (0,1)$. There exists a set $\mathcal{E} \subset [0,s]$ of linear measure $\leq s\eta$ such that for $r \in [0,s] \backslash \mathcal{E}$, we have

$$\frac{\|Q\|_s}{\min_{|t|=r} |Q(t)|} \leq \left(\frac{12e}{\eta} \right)^n .$$

Proof
We may factorize Q as

$$Q(z) = \left(\prod_{|z_j|<2s} (z - z_j) \right) \left(\prod_{|z_j|\geq 2s} \left(1 - \frac{z}{z_j} \right) \right) .$$

Let k be the number of terms in the first product and ℓ be the number of terms in the second. Then for $r \leq s$,

$$\frac{\|Q\|_s}{\min_{|t|=r} |Q(t)|} \leq \frac{(3s)^k \, 3^\ell}{\min_{|t|=r} \left| \prod_{|z_j|<2s} (t - z_j) \right|} .$$

By Cartan's Lemma [1, p. 325, Thm. 6.6.7],

$$\left| \prod_{|z_j|<2s} (t - z_j) \right| \geq \varepsilon^k ,$$

outside a union of at most k circles, the sum of whose diameters is at most $4e\varepsilon$. Let $\varepsilon = \frac{s\eta}{4e}$ and $\mathcal{E}$ be the set of all $r \in [0,\infty)$ for which some z with $|z| = r$ lies in one of these circles. Then it is clear that $\mathcal{E}$ has linear measure at most $4e\varepsilon = s\eta$, and for $r \notin \mathcal{E}$,

$$\min_{|t|=r} \left| \prod_{|z_j|<2s} (t - z_j) \right| \geq \left(\frac{s\eta}{4e} \right)^k$$

so

$$\frac{\|Q\|_s}{\min_{|t|=r} |Q(t)|} \leq \frac{(3s)^k \, 3^\ell}{\left(\frac{s\eta}{4e} \right)^k} \leq \left(\frac{12e}{\eta} \right)^n .$$

■

The next lemma appears in [14, p. 514, Lemma 3.3], as a consequence of a more general result. However, for completeness, we give a simpler proof of this special case.

Lemma 3.5
Let $\mathcal{D}$ be a bounded simply connected open set and let $f \in \mathcal{R}_0\left(\bar{\mathcal{D}}\right)$. Let T, K be compact subsets of $\mathcal{D}$ with T having positive logarithmic capacity. Let $\delta \in (0,1)$. Then for large enough n, we have

$$E_n(f,K) \leq E_n(f,T)^{1-\delta}.$$

Proof
Write $R_n^*(f,T) = p_n^*/q_n^*$. Let $\theta \in (0,1)$ and for k so large that $E_n(f,T) < 1$,

$$k = k(n) = \text{least integer } \geq \frac{\log E_n(f,T)}{\log \theta}. \tag{3.11}$$

We shall choose θ small enough later. Observe that $k \to \infty$ as $n \to \infty$, and

$$\theta^k \leq E_n(f,T). \tag{3.12}$$

Since $f \in R_0\left(\bar{\mathcal{D}}\right)$, we can find for large enough n, and $k = k(n)$, a rational function $p_k^{\#}/q_k^{\#}$ of type (k,k) such that

$$\left\| f - p_k^{\#}/q_k^{\#} \right\|_{L_\infty\left(\bar{\mathcal{D}}\right)} \leq \theta^k. \tag{3.13}$$

Then

$$\left\| p_n^*/q_n^* - p_k^{\#}/q_k^{\#} \right\|_{L_\infty(T)} \leq E_n(f,T) + \theta^k \leq 2E_n(f,T),$$

so

$$\left\| p_n^* q_k^{\#} - p_k^{\#} q_n^* \right\|_{L_\infty(T)} \leq 2E_n(f,T) \left\| q_n^* q_k^{\#} \right\|_{L_\infty(T)}.$$

Next, as T has positive logarithmic capacity, the Bernstein-Walsh inequality [18, p. 156] shows that there is a constant C_0 depending only on T and $\bar{\mathcal{D}}$ such that

$$\left\| p_n^* q_k^{\#} - p_k^{\#} q_n^* \right\|_{L_\infty\left(\bar{\mathcal{D}}\right)} \leq C_0^{n+k} \left\| p_n^* q_k^{\#} - p_k^{\#} q_n^* \right\|_{L_\infty(T)}$$

and hence

$$\left\| p_n^* q_k^{\#} - p_k^{\#} q_n^* \right\|_{L_\infty\left(\bar{\mathcal{D}}\right)} \leq 2C_0^{n+k} E_n(f,T) \left\| q_n^* q_k^{\#} \right\|_{L_\infty(T)}. \tag{3.14}$$

Then for $z \in \bar{\mathcal{D}}$,

$$\begin{aligned} \left| f - \frac{p_n^*}{q_n^*} \right| (z) &\leq \left| f - \frac{p_k^\#}{q_k^\#} \right| (z) + \frac{\left| p_k^\# q_n^* - p_n^* q_k^\# \right| (z)}{\left| q_n^* q_k^\# \right| (z)} \\ &\leq E_{nn}(f, T) \left\{ 1 + 2C_0^{n+k} \frac{\left\| q_n^* q_k^\# \right\|_{L_\infty(T)}}{\left| q_n^* q_k^\# \right| (z)} \right\}. \end{aligned}$$

Here we have used (3.12-14). Next, as in the proof of Lemma 3.4, given $\eta > 0$,

$$\frac{\left\| q_n^* q_k^\# \right\|_{L_\infty(T)}}{\left| q_n^* q_k^\# \right| (z)} \leq (C_1/\eta)^{n+k},$$

for $z \in \mathcal{D} \backslash \mathcal{E}$, where C_1 is a constant that depends only on the diameter of $\mathcal{D}$, and $\mathcal{E}$ is the union of at most $n+k$ open balls, the sum of whose diameters is at most η. Let us choose η as half the distance from K to the boundary of $\mathcal{D}$. This parameter is independent of f, n, k, θ. Then we can find a simple closed contour Γ in $\mathcal{D}$ that encloses K, but lies inside $\mathcal{D}$ that does not intersect any ball in $\mathcal{E}$. For example, we can take Γ to be $\{t : \operatorname{dist}(t, \partial\mathcal{D}) = \eta/3, \, t \text{ inside } \mathcal{D}\}$, but where this level curve intersects $\mathcal{E}$, we deform Γ to run along the boundary of $\mathcal{E}$. Thus

$$\sup_{z \in \Gamma} \left| f - \frac{p_n^*}{q_n^*} \right| (z) \leq 4E_n(f, T) \left(\frac{C_0 C_1}{\eta} \right)^{n+k}.$$

Next, as the interior of Γ is simply connected, the classical inequality of Gončar-Grigorjan shows that

$$\begin{aligned} E_n(f, K) &\leq 7n^2 E_n(f, \Gamma) \leq 7n^2 \sup_{z \in \Gamma} \left| f - \frac{p_n^*}{q_n^*} \right| (z) \\ &\leq 28n^2 E_n(f, T) \left(\frac{C_0 C_1}{\eta} \right)^{n+k}. \end{aligned}$$

Here, letting $B = \frac{C_0 C_1}{\eta}$, our choice (3.11) of k gives

$$\left(\frac{C_0 C_1}{\eta} \right)^{k-1} = \exp((k-1)\log B) \leq E_n(f, T)^{\frac{\log B}{\log \theta}}.$$

Thus

$$\begin{aligned} E_n(f, K) &\leq E_n(f, T) \left(28n^2 B^{n+1} E_n(f, T)^{\frac{\log B}{\log \theta}} \right) \\ &\leq E_n(f, T)^{1-\delta}, \end{aligned}$$

for n large enough, if we choose θ so small that $\left|\frac{\log B}{\log \theta}\right| \leq \delta/2$, and also use that

$$\lim_{n\to\infty} E_n\left(f, T\right)^{1/n} = 0.$$

■

Proof of Theorem 3.1

We simplify (3.3-4). We choose $\sigma = 1$ in Lemma 3.2, and $s = 1$ and $\eta = 1/5$ in Lemma 3.4. The latter lemma shows that there exists $\rho \in \left[\frac{1}{2}, \frac{3}{4}\right]$ with

$$\frac{\|q_n\|_\rho}{\min_{|t|=\rho} |q_n(t)|} \leq \frac{\|q_n\|_1}{\min_{|t|=\rho} |q_n(t)|} \leq (60e)^n.$$

Also, we choose $r = 1/4$. Then $\frac{\rho - r}{\rho + r} \geq \frac{1}{3}$. Also as $m \leq Ln$, and as all zeros of ω_n lie in B_ε,

$$\begin{aligned}
&\left(\frac{\|\omega_n\|_\varepsilon}{\min_{|t|=\rho}|\omega_n(t)|}\right)^{\frac{2}{\pi}\arcsin\left(\frac{\rho-r}{\rho+r}\right)} \left(\frac{\rho}{r}\right)^{m+n} \left(\frac{1+\rho}{1-\rho}\right)^m \\
\leq\ & \left(\frac{2\varepsilon}{\rho-\varepsilon}\right)^{\frac{2}{\pi}(2n+1)\arcsin\left(\frac{1}{3}\right)} 3^{n(1+L)} \left(\frac{1+3/4}{1-3/4}\right)^{Ln} \\
\leq\ & \left[(8\varepsilon)^{\frac{4}{\pi}\arcsin\left(\frac{1}{3}\right)} 3^{1+L} 7^L\right]^n < \frac{1}{2},
\end{aligned}$$

for large enough n, by (3.1). So (3.3) is satisfied. Next, we reformulate (3.4) as

$$E_n\left(f, \overline{B_\rho}\right) \leq 28n^2 3^{m+n} \left(\frac{1+3/4}{1-3/4}\right)^m (60e)^n E_m\left(f, \overline{B_1}\right).$$

Since $m \geq n$, and $f \in \mathcal{R}_0\left(\overline{B_s}\right)$, for some $s > 1$, this in turn implies that for large enough n,

$$E_n\left(f, \overline{B_\rho}\right) \leq E_m\left(f, \overline{B_1}\right)^{1-\delta/2}.$$

In view of Lemma 3.5, we can replace B_ρ by B_1 for large enough n. ■

As an immediate corollary of Theorem 3.1, we have:

Corollary 3.6

Assume the hypotheses of Theorem 3.1, except the hypothesis about the poles of $\{R_n(\Lambda_n, z)\}$. Assume also that for $n \in \mathcal{S}$, $n-1$ is an exact interpolation index for f and B_1, and (3.2) fails. Then for large enough $n \in \mathcal{S}$, and any $\Lambda_n \subset B_\varepsilon$, $R_n(\Lambda_n, z)$ has no poles in B_ε.

4. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3

Fix $A > 1$ and let $g(z) = f(Az)$. This is entire, and for large enough k, our hypothesis on f ensures that $n_k - 1$ is an exact interpolation index for g and B_1. For $k \geq 1$, define $k^* = k^*(k)$ by

$$n_{k^*} = \inf\{n_j : n_j \geq Ln_k\}.$$

This is well defined as $\{n_j\}$ is increasing and has limit ∞. Moreover,

$$n_{k^*-1} < Ln_k,$$

so using (1.1),

$$Ln_k \leq n_{k^*} \leq Ln_{k^*-1} < L^2 n_k.$$

Next, we have

$$\lim_{k\to\infty} \eta_{n_k}\left(g, \overline{B_1}\right) = 0$$

so we can choose a subsequence $\left\{n_{k_j}\right\}$ of $\{n_k\}$ with the property that

$$\eta_{n_{k_j}}\left(g, \overline{B_1}\right) > \eta_{n_\ell}\left(g, \overline{B_1}\right) \text{ whenever } \ell > k_j.$$

Observe that with k_j^* defined as above, we have

$$Ln_{k_j} \leq n_{k_j^*} < L^2 n_{k_j},$$

and by choice of k_j, for large enough j,

$$\begin{aligned} E_{n_{k_j}}\left(g, \overline{B_1}\right) &> \left(E_{n_{k_j^*}}\left(g, \overline{B_1}\right)\right)^{n_{k_j}/n_{k_j^*}} \\ &\geq E_{n_{k_j^*}}\left(g, \overline{B_1}\right)^{1/L} \\ &= E_{n_{k_j^*}}\left(g, \overline{B_1}\right)^{1-\delta}, \end{aligned}$$

where $\delta = 1 - 1/L > 0$. Thus with $n = n_{k_j}$ and $m = n_{k_j^*}$, (3.2) in Theorem 3.1 is not true. Assume now that ε satisfies (3.1) - it does not depend on A, g, f, but does depend on L. If for infinitely many k, and corresponding $\Lambda_{n_k} \subset B_{A\varepsilon}$, $R_{n_{k_j}}\left(\Lambda_{n_{k_j}}, \cdot\right)$ for f has a pole in $B_{A\varepsilon}$, then for the corresponding interpolant for g with points in B_ε, the interpolant has a pole in B_ε. In this case, we are fulfilling the initial hypotheses of Theorem 3.1, but neither of the alternative conclusions (I) or (II) hold, so we have a contradiction. Thus for large enough k, and any $\Lambda_{n_k} \subset B_{A\varepsilon}$, $R_{n_{k_j}}\left(\Lambda_{n_{k_j}}, \cdot\right)$ for f cannot have poles in $B_{A\varepsilon}$. Since A is arbitrary, and ε is independent of A, we are done. ■

Proof of Theorem 1.4 assuming (1.3)

Let $A > 1$. Choose $\delta' \in (\delta, 1)$. By Lemma 3.5, our hypothesis (1.3) gives for large enough k,

$$E_{n_k}\left(f, \overline{B_A}\right) > E_{Mn_k}\left(f, \overline{B_A}\right)^{1-\delta'}.$$

Applying this to $g(z) = f(Az)$ gives

$$E_{n_k}\left(g, \overline{B_1}\right) > E_{Mn_k}\left(g, \overline{B_1}\right)^{1-\delta'}.$$

Also for large enough k, $n_k - 1$ is an exact interpolation index for g and B_1. We can then apply Theorem 3.1 with $\sigma = 1, n = n_k$, $m = Mn_k$, and L replaced by M. Since both alternatives (I), (II) of Theorem 3.1 fail, it follows that for large enough $n = n_k$, $R_{n_k}(\Lambda_{n_k}, z)$ for g has no poles in B_ε, where ε satisfies

$$3(8\varepsilon)^{\frac{4}{\pi}\arcsin\left(\frac{1}{3}\right)} 21^M < 1.$$

Then for large enough k, $R_{n_k}(\Lambda_{n_k}, z)$ for f has no poles in $B_{A\varepsilon}$. As ε does not depend on A, we are done. ■

Proof of Theorem 1.4 assuming (1.4)

We apply Lemma 3.3. Let $g(z) = f(2rz)$ where r is one of the sequence of values r with the property (1.4). Assume that for infinitely many $n = n_k$, $R_{n_k}(\Lambda_{n_k}, z)$ for g has a pole in B_ε, where $\varepsilon \in \left(0, \frac{1}{8}\right)$. Assume that $\varepsilon < r < \rho < 1$ and $\rho < s < \sigma$. Our hypothesis on exact indices for f shows that alternative (I) in Lemma 3.3 is not possible for g. We now show that this leads to a contradiction in the alternative (II′) in Lemma 3.3. Combining Lemmas 3.3 and 3.4, we have for $n \in \{n_k\}$, such that $R_{n_k}(\Lambda_{n_k}, z)$ for g has a pole in B_ε,

$$\begin{aligned}\frac{E_n\left(g, \overline{B_r}\right)}{E_n\left(g, \overline{B_s}\right)} \;\leq\; & \frac{7n^2}{1-\rho/s}\left(\frac{2}{1-\rho/s}\right)^n \left(\frac{2\varepsilon}{\rho-\varepsilon}\right)^{\frac{2}{\pi}(2n+1)\arcsin\left(\frac{\rho-r}{\rho+r}\right)} \\ & \times \left(\frac{\rho+\varepsilon}{s-\varepsilon}\right)^{2n+1}\left(\frac{12e}{\eta}\right)^n.\end{aligned}$$

Here also by Lemma 3.4, we need $r \in [0,1] \backslash \mathcal{E}$, where meas$(\mathcal{E}) < \eta$. Choose $\eta = \frac{1}{5}, s = 1, \sigma = 2, \rho = \frac{7}{8}$, and some suitable $r \in \left[\frac{1}{2}, \frac{3}{4}\right]$. Then, using monotonicity of errors of rational approximation in the set, we obtain

$$\frac{E_n\left(g, \overline{B_{1/2}}\right)}{E_n\left(g, \overline{B_2}\right)} \leq 56n^2 16^n (4\varepsilon)^{\frac{2}{\pi}(2n+1)\arcsin\left(\frac{1}{13}\right)} \left(\frac{8}{7}\right)^{2n+1} (60e)^n.$$

For large enough n, our hypothesis (1.4) transferred from f to g, gives

$$\left(\frac{1}{T}\right)^{n} \leq 56n^{2}16^{n}\left(4\varepsilon\right)^{\frac{2}{\pi}(2n+1)\arcsin\left(\frac{1}{13}\right)}\left(\frac{8}{7}\right)^{2n+1}\left(60e\right)^{n}.$$

Let $T' > T$. Taking nth roots, for large enough $n \in \{n_k\}$,

$$\frac{1}{T'} \leq 16\left(4\varepsilon\right)^{\frac{4}{\pi}\arcsin\left(\frac{1}{13}\right)}\left(\frac{8}{7}\right)^{2} 60e.$$

Thus for large enough $n \in \{n_k\}$, any interpolant $R_{n_k}\left(\Lambda_{n_k}, z\right)$ for g with interpolation points in B_ε has no poles in B_ε if ε is so small that it violates this last bound. Hence also any interpolant $R_{n_k}\left(\Lambda_{n_k}, z\right)$ for f with points in $B_{A\varepsilon}$ has no poles in $B_{A\varepsilon}$. ■

References

[1] G.A. Baker, P. Graves-Morris, *Padé Approximants*, 2nd Edition, Cambridge University Press, Cambridge, 1996.

[2] A. I. Aptekarev, M. L. Yattselev, *Padé Approximants for Functions with Branch Points -Strong Asymptotics of Nuttall-Stahl Polynomials*, Acta Mathematica, 215(2015), 217–280.

[3] B. Beckermann, G. Labahn, A. C. Matos, *On Rational Functions without Froissart Doublets,* manuscript.

[4] V. I. Buslaev, *The Baker-Gammel-Wills Conjecture in the Theory of Padé Approximants*, Math. USSR. Sbornik, 193(2002), 811–823.

[5] V. I. Buslaev, *Convergence of the Rogers-Ramanujan Continued Fraction*, Math.USSR. Sbornik, 194(2003), 833–856.

[6] V. I. Buslaev, A. A. Gončar, and S. P. Suetin, *On Convergence of Subsequences of the mth Row of a Padé Table*, Math. USSR Sbornik, 48(1984), 535–540.

[7] D. Coman, E.A. Poletsky, *Overinterpolation*, J. Math. Anal. Appl., 335(2007), 184-197.

[8] J. Gilewicz, Y. Kryakin, *Froissart Doublets in Padé Approximation in the Case of Polynomial Noise,* J. Comp. Appl. Math., 153(2003), 235–242.

[9] A.A. Gončar, *Estimates of the Growth of Rational Functions and Some of Their Applications*, Math. USSR Sbornik, 1(1967), 445-456.

[10] A.A. Gončar, *On Uniform Convergence of Diagonal Padé Approximants*, Math. USSR. Sbornik, 46(1983), 539-559.

[11] A.A. Gončar and L.D. Grigorjan, *On Estimates of the Norm of the Holomorphic Component of a Meromorphic Function*, Math. USSR. Sbornik, 28(1976), 571-575.

[12] L.D. Grigorjan, *Estimates of the Norm of the Holomorphic Components of Functions Meromorphic in Domains with a Smooth Boundary*, Math. USSR. Sbornik, 29(1976), 139-146.

[13] D.V. Khristoforov, *On the Phenomenon of Spurious Interpolation of Elliptic Functions by Diagonal Padé Approximants*, Mathematical Notes, 87(2010), 564-574.

[14] D.S. Lubinsky, *Distribution of Poles of Diagonal Rational Approximants to Functions of Fast Rational Approximability*, Constructive Approximation, 7(1991), 501-519.

[15] D.S. Lubinsky, *Spurious Poles in Diagonal Rational Approximation*, (in) Progress in Approximation Theory (eds. A.A. Gončar and E.B. Saff), Springer-Verlag, Berlin, 1992, pp. 191-213.

[16] D.S. Lubinsky, *Rogers-Ramanujan and the Baker-Gammel-Wills (Padé) Conjecture*, Annals of Mathematics, 157(2003), 847-889.

[17] E. A. Rakhmanov, *On the Convergence of Padé Approximants in Classes of Holomorphic Functions*, Math. USSR Sbornik, 40(1981), 149–155.

[18] T.M. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[19] H. Stahl, *Spurious Poles in Padé Approximation*, J. Comp. Appl. Math., 9(1998), 511-527.

[20] S.P. Suetin, *Distribution of the Zeros of Padé polynomials and analytic continuation*, Russian Math. Surveys, 70(2015), 901-951.

[21] F. Wielonsky, *Riemann-Hilbert Analysis and Uniform Convergence of Rational Interpolants to the Exponential Function*, J. Approx. Theory, 131(2004), 100-148.

[22] M. Yattselev, *Meromorphic Approximation: Symmetric Contours and Wandering Poles*, manuscript.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu