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SHARP ESTIMATES FOR THE MAXIMUM OVER MINIMUM MODULUS OF RATIONAL FUNCTIONS

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ABSTRACT. Let m, $n \geq 0$, $\lambda > 1$, and R be a rational function with numerator, denominator of degree $\leq m, n$, respectively. In several applications, one needs to know the size of the set $\mathcal{S} \subset [0,1]$ such that for $r \in \mathcal{S}$,

$$\max_{|z|=r} |R(z)| \,/\, \min_{|z|=r} |R(z)| \leq \lambda^{m+n}.$$

In an earlier paper, we showed that

$$meas\,(\mathcal{S}) \geq \frac{1}{4} \exp\left(-\frac{13}{\log \lambda}\right),$$

where *meas* denotes linear Lebesgue measure. Here we obtain, for each λ, the sharp version of this inequality in terms of condenser capacity. In particular, we show that as $\lambda \to 1+$,

$$meas\,(\mathcal{S}) \geq 4 \exp\left(-\frac{\pi^2}{2 \log \lambda}\right)(1 + o(1)).$$

1. INTRODUCTION AND RESULTS

In applications including rational approximation, and the theory of meromorphic functions, one needs estimates for the ratio of the maximum and minimum modulus of a rational function [3]. The classical way to obtain such estimates involves Cartan's lemma on small values of polynomials. In [3], the author used a form of Cartan's lemma in a metric space setting to establish the following result, and hence to investigate convergence of diagonal Padé approximants:

Theorem 1. *Let $\lambda > 1$ and m, $n \geq 0$. Then for rational functions R with numerator, denominator of degree $\leq m$, n respectively,*

$$\max_{|z|=r} |R(z)| \,/\, \min_{|z|=r} |R(z)| \leq \lambda^{m+n}, \quad r \in \mathcal{S}, \tag{1.1}$$

where $\mathcal{S} \subset [0,1]$ has Lebesgue measure $meas\,(\mathcal{S})$ satisfying

$$meas\,(\mathcal{S}) \geq \frac{1}{4} \exp\left(-\frac{13}{\log \lambda}\right). \tag{1.2}$$

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This is sharp in form in the following sense: let $0 < \varepsilon < 1$. Then for λ close enough to 1 and m large enough, there exists a polynomial R of degree m for which the set $\mathcal{S} \subset [0,1]$ on which (1.2) holds satisfies

$$meas\,(\mathcal{S}) \leq \exp\left(-\frac{2-\varepsilon}{\log\lambda}\right). \tag{1.3}$$

In this paper, we use potential theory to close the gap between $2-\varepsilon$ and 13. Let us recall some potential theoretic notions [4]. Let

$$\mathcal{H} := \{z : \operatorname{Re} z > 0\}$$

denote the open right-half plane. Its boundary is the imaginary axis $\partial\mathcal{H} = i\mathbb{R}$. The Green's function for the right-half plane with pole at $\xi \in \mathcal{H}$ is

$$g\,(z,\xi) = \log\left|\frac{z+\overline{\xi}}{z-\xi}\right|.$$

Moreover, given compact $E \subset \mathcal{H}$, its Green energy is

$$V_E^{\mathcal{H}} := \inf_{\mu(E)=1} \int_E \int_E g\,(x,t)\,d\mu\,(x)\,d\mu\,(t)\,, \tag{1.4}$$

where the inf is taken over all non-negative Borel measures μ with support in E and with $\mu\,(E) = 1$. It is known that there is a unique measure $\mu_E^{\mathcal{H}}$, called the *Green equilibrium measure*, attaining the infimum. The *condenser capacity* of the pair $(E, \partial\mathcal{H}) = (E, i\mathbb{R})$ is defined to be

$$C\,(E, i\mathbb{R}) = 1/V_E^{\mathcal{H}}. \tag{1.5}$$

It is easily seen from (1.4) that $V_E^{\mathcal{H}}$ decreases as the set E increases, and hence $C\,(E, i\mathbb{R})$ is a monotone set function. For further orientation, see [4, p. 132 ff.]. We shall need to consider in detail the set $E = [b,1]$, and, for that purpose, we need some notation for elliptic integrals. Given $b \in (0,1)$, *the complete elliptic integrals of the first kind* are

$$K(b) := \int_0^1 \frac{dx}{\sqrt{(1-b^2x^2)\,(1-x^2)}}; \quad K'(b) := K\left(\sqrt{1-b^2}\right).$$

Theorem 2. *Let $0 < b < 1$.*

(a) *$\mu_{[b,1]}^{\mathcal{H}}$ is absolutely continuous w.r.t. Lebesgue measure on $[b,1]$ and*

$$\frac{d\mu_{[b,1]}^{\mathcal{H}}(x)}{dx} = \frac{\kappa(b)}{\sqrt{(x^2-b^2)\,(1-x^2)}}, \quad x \in (b,1)\,, \tag{1.6}$$

where

$$\kappa(b) = 1\Big/\int_b^1 \frac{dx}{\sqrt{(x^2-b^2)\,(1-x^2)}} = 1/K'(b). \tag{1.7}$$

(b) *Let*

$$F(b) := C\,([b,1], i\mathbb{R})\,, \quad 0 < b < 1. \tag{1.8}$$

Then

$$F(b) = \frac{K'(b)}{\pi K(b)}. \tag{1.9}$$

(c) *F is a strictly decreasing function of b, mapping $(0,1)$ onto $(0,\infty)$, and satisfying*

$$F(b) = \frac{2}{\pi^2}\left|\log\frac{b}{4}\right| + o\left(1\right), \quad b \to 0+; \tag{1.10}$$

$$F(b) = \frac{1}{\left|\log(1-b)\right|}\big(1+o(1)\big), \quad b \to 1- . \tag{1.11}$$

In the sequel, we let $F^{[-1]} : (0,\infty) \to (0,1)$ denote the inverse of F, so that

$$F\left(F^{[-1]}(x)\right) = x, \quad x \in (0,\infty).$$

Following is our main result:

Theorem 3. *Let $\lambda > 1$ and m, $n \geq 0$.*

(a) *Then for rational functions R with numerator, denominator of degree $\leq m$, n respectively,*

$$\max_{|z|=r}|R(z)| \Big/ \min_{|z|=r}|R(z)| \leq \lambda^{m+n}, \quad r \in \mathcal{S}, \tag{1.12}$$

where $\mathcal{S} \subset [0,1]$ satisfies

$$meas(\mathcal{S}) \geq F^{[-1]}\left(\frac{1}{\log\lambda}\right). \tag{1.13}$$

(b) *This is sharp in the sense that given $\varepsilon > 0$, there exists for large enough m, a polynomial R of degree m, such that (with $n = 0$)*

$$meas(\mathcal{S}) \leq F^{[-1]}\left(\frac{1}{\log\lambda}\right) + \varepsilon. \tag{1.14}$$

(c) *In particular,*

$$F^{[-1]}\left(\frac{1}{\log\lambda}\right) = 4\exp\left(-\frac{\pi^2}{2\log\lambda}\right)\big(1+o(1)\big), \quad \lambda \to 1+; \tag{1.15}$$

$$F^{[-1]}\left(\frac{1}{\log\lambda}\right) = 1 - \lambda^{-1+o(1)}, \quad \lambda \to \infty. \tag{1.16}$$

Remarks. (a) Let $\rho > 0$. By replacing $R(z)$ by $R\left(\rho z\right)$, we deduce that (1.12) holds on a set $\mathcal{S} \subset [0,\rho]$ with

$$meas(\mathcal{S}) \geq \rho F^{[-1]}\left(\frac{1}{\log\lambda}\right). \tag{1.17}$$

(b) One may formulate a generalisation of Theorem 3 for potentials (cf. [3, Theorem 6]).

(c) There is a (distant) connection between Theorem 3 and estimates for the minimum modulus of functions of slow growth [2, p. 376 ff.].

This paper is organised as follows: in Section 2 we prove Theorem 3(a), in Section 3 we prove Theorems 3(b), (c), and in Section 4 we establish Theorem 2.

2. The proof of Theorem 3(a)

We shall do this in five steps:

Step 1: Reduction to R with real poles and zeros. Note first that if a, $b \in \mathbb{C}$, then

$$\max_{|z|=r}\left|\frac{z-a}{z-b}\right| \Big/ \min_{|z|=r}\left|\frac{z-a}{z-b}\right| \leq \left|\left(\frac{r+|a|}{r-|a|}\right)\left(\frac{r+|b|}{r-|b|}\right)\right|.$$

It follows that it suffices to consider

$$\mathcal{S} := \left\{ r \in [0,1] : \left|\prod_{j=1}^{m+n}\left(\frac{r+\alpha_j}{r-\alpha_j}\right)\right| \leq \lambda^{m+n} \right\},$$

where all $\alpha_j > 0$. Indeed, this merely decreases the size of $\mathcal{S}$, and we are searching for a lower bound for that size. Next, note that we have also assumed that we have numerator and denominator of exact degree m and n respectively. This may be achieved by adding some $\alpha_j = 1$, which again reduces the size of $\mathcal{S}$. Finally, we note that we may assume that all $\alpha_j \leq 1$: again, replacing any $\alpha_j > 1$ by 1 reduces the size of $\mathcal{S}$. So, in the sequel, we assume that all $\alpha_j \in (0,1]$.

Let us set $\ell := m + n$ and

$$\mathcal{S}_0 : = \left\{ r \in [0,1] : \left|\prod_{j=1}^{\ell}\left(\frac{r+\alpha_j}{r-\alpha_j}\right)\right| < \lambda^{\ell} \right\}; \tag{2.1}$$

$$E : = \left\{ r \in [0,1] : \left|\prod_{j=1}^{\ell}\left(\frac{r+\alpha_j}{r-\alpha_j}\right)\right| \geq \lambda^{\ell} \right\}$$
$$= \left\{ r \in [0,1] : \left|\prod_{j=1}^{\ell}\left(\frac{r-\alpha_j}{r+\alpha_j}\right)\right| \leq \lambda^{-\ell} \right\}.$$

Since the equation $\left|\prod_{j=1}^{\ell}\left(\frac{r+\alpha_j}{r-\alpha_j}\right)\right| = \lambda^{\ell}$ has at most 2ℓ solutions in r, we see that

$$meas(\mathcal{S}) = meas\left(\mathcal{S}_0\right) = 1 - meas(E). \tag{2.2}$$

We must look for an upper bound for $meas(E)$. It is clear that $E \subset (0,1]$ and consists of finitely many intervals, some of which may degenerate to a single point.

Step 2: The basic inequality for E. We shall show that

$$C\left(E, i\mathbb{R}\right) \leq \frac{1}{\log \lambda}. \tag{2.3}$$

If firstly E consists of finitely many points, then $V_E^{\mathcal{H}} = \infty$ from (1.4), so $C\left(E, i\mathbb{R}\right) = 0$. Let us now assume that E contains at least one non-empty interval. Note that each $\alpha_j \neq 1$ lies inside such a non-empty interval; if $\alpha_j = 1$, it is the right-endpoint of a non-empty interval. Let $\mu_E^{\mathcal{H}}$ denote the Green equilibrium measure for E. We shall need a property of the Green equlibrium potential:

$$\int_E g\left(r, \alpha_j\right) d\mu_E^{\mathcal{H}}\left(r\right) = V_E^{\mathcal{H}}, \quad 1 \leq j \leq \ell. \tag{2.4}$$

In [4, Thm. 5.11, p.132], it is shown that if we replace α_j by x, this identity holds for "quasi-every" $x \in E$. But the Green potential is continuous on each of the non-empty intervals of E, since these are regular with respect to the Dirichlet problem in the plane. (See, for example, [4, pp. 54-55].) Since, as we have noted, each such α_j is contained in such an interval, we have (2.4) as stated.

Next, from (2.1),

$$\begin{aligned}\lambda^{-\ell} &\geq \int_E \left|\prod_{j=1}^{\ell} \frac{r-\alpha_j}{r+\alpha_j}\right| d\mu_E^{\mathcal{H}}(r) \\ &= \int_E \exp\left(-\sum_{j=1}^{\ell} g(r,\alpha_j)\right) d\mu_E^{\mathcal{H}}(r) \\ &\geq \exp\left(-\int_E \sum_{j=1}^{\ell} g(r,\alpha_j)\, d\mu_E^{\mathcal{H}}(r)\right) = \exp\left(-\ell V_E^{\mathcal{H}}\right).\end{aligned}$$

Here we have used (2.4) and the arithmetic-geometric mean inequality. This last inequality is easily reformulated as (2.3).

Step 3: Show that $meas(E)$ is maximal if E is of the form $[b,1]$. Set

$$b := F^{[-1]}\left(\frac{1}{\log\lambda}\right) \Leftrightarrow F(b) = \frac{1}{\log\lambda}.$$

The existence and uniqueness of b follows from Theorem 2(c). Then

$$V_{[b,1]}^{\mathcal{H}} = \frac{1}{C([b,1], i\mathbb{R})} = \log\lambda.$$

We shall assume that E of (2.1) satisfies

$$meas(E) > meas([b,1])$$

and derive a contradiction. Now

$$\lim_{y\to 1-} meas(E\cap[0,y]) = meas(E),$$

so we may choose $y_0 < 1$ such that

$$E_0 := E\cap[0,y_0] \text{ has } meas(E_0) = meas([b,1]).$$

We shall "shift left" the Green equilibrium measure from $[b,1]$ to E_0, and then derive a contradiction to (2.3). The basic idea is that

$$g(x+c, y+c) > g(x,y) \text{ if } x,y,c>0.$$

We may omit the discrete points from E_0 and assume that E_0 is a union of k disjoint intervals

$$E_0 = \bigcup_{j=1}^{k} I_j,$$

where

$$I_j = [\alpha_j, \beta_j] \text{ and each } \beta_j < \alpha_{j+1}.$$

Define a strictly increasing piecewise linear map h from E_0 onto $[b,1]$ by

$$h(x) := x + b - \alpha_j + \sum_{i=1}^{j-1} (\beta_i - \alpha_i) =: x + A_j, \quad x \in [\alpha_j, \beta_j],$$

$1 \leq j \leq k$. (The empty sum is interpreted as 0.) Now define an absolutely continuous measure ν on E_0 by

$$\nu'(x) := \big(\mu_{[b,1]}\big(h(x)\big)\big)' = \left(\mu^{\mathcal{H}}_{[b,1]}\right)' (h(x))\, h'(x), \quad x \in E_0.$$

Then ν has total mass 1. Next, as $[b,1]$ is regular with respect to the Dirichlet problem in the plane, we have

$$\int_b^1 g\,(x,t)\, d\mu^{\mathcal{H}}_{[b,1]}(t) = V^{\mathcal{H}}_{[b,1]} = \log \lambda, \quad x \in [b,1].$$

Hence

$$\int_{E_0} g\big(h(y), h(s)\big) d\nu\,(s) = \log \lambda, \quad y \in E_0.$$

We shall show that there exists $\eta > 0$ such that

$$g\big(h(y), h(s)\big) \geq g\,(y,s) + \eta \,\forall\, s, \quad y \in E_0, \tag{2.5}$$

and then

$$\begin{aligned} \log \lambda &\geq \int_{E_0} \int_{E_0} g\,(y,s)\, d\nu\,(s)\, d\nu\,(y) + \eta \\ &\geq V^{\mathcal{H}}_{E_0} + \eta. \end{aligned}$$

This implies that

$$C\,(E, i\mathbb{R}) \geq C\,(E_0, i\mathbb{R}) \geq \frac{1}{\log \lambda - \eta},$$

so we obtain the desired contradiction to (2.3).

Step 4: Proof of (2.5). Let us suppose that $y \in I_i, s \in I_j$, where, for example, $i \leq j$, so that

$$\begin{aligned} g\big(h(y), h(s)\big) &= \log \left| \frac{(y + A_i) + (s + A_j)}{(y + A_i) - (s + A_j)} \right| \\ &= \log \left| \frac{y+s}{y-s} \right| + \log \left| 1 + \frac{A_i + A_j}{y+s} \right| - \log \left| 1 - \frac{A_j - A_i}{y-s} \right|. \end{aligned} \tag{2.6}$$

Note that for each m,

$$A_m - A_{m-1} = \beta_{m-1} - \alpha_m < 0$$

so $A_j - A_i \leq 0$, while $y - s \leq 0$. Also

$$\frac{A_j - A_i}{y-s} \leq \frac{A_i - A_j}{\alpha_j - \beta_i} \leq 1.$$

Then as $A_k \leq A_i, A_j$,

$$g\big(h(y), h(s)\big) \geq g(y,s) + \log\,(1 + A_k) + 0,$$

so we may take $\eta := \log(1 + A_k)$. Note here that $h\,(\beta_k) = 1 \Rightarrow A_k = 1 - \beta_k > 0$.

Step 5: Completion of the proof. We have shown that

$$meas\,(E) \leq meas\,([b,1]) = 1-b,$$

so (2.2) gives

$$meas\,(\mathcal{S}) \geq b = F^{[-1]}\left(\frac{1}{\log\lambda}\right). \quad \square$$

3. The proof of Theorem 3(b), (c)

The proof of Theorem 3(b). We shall use a crude discretisation procedure, of the type used in the theory of orthogonal polynomials in the 1980's. The finer method of Totik [4] would yield sharper estimates, but those are not needed here. Fix $\lambda > 1$, $\varepsilon > 0$, and choose $\lambda' > \lambda$ such that

$$b' := F^{[-1]}\left(\frac{1}{\log\lambda'}\right) < F^{[-1]}\left(\frac{1}{\log\lambda}\right) + \frac{\varepsilon}{4}. \tag{3.1}$$

Recall that

$$\int_{b'}^{1} g\,(x,t)\, d\mu^{\mathcal{H}}_{[b',1]}\,(t) = V^{\mathcal{H}}_{[b',1]} = \log\lambda', \quad x\in[b',1]. \tag{3.2}$$

Let us choose

$$b' = t_0 < t_1 < t_2 < \cdots < t_m = 1$$

such that if $J_j := [t_j, t_{j+1})$, then

$$\int_{J_j} d\mu^{\mathcal{H}}_{[b',1]}\,(t) = \frac{1}{m}, \quad 0\leq j\leq m-1. \tag{3.3}$$

It is easily seen from the explicit formula (1.6) for $\mu^{\mathcal{H}}_{[b',1]}$ that $\exists C_i \neq C_i\,(m,j) > 0, i=1,2$, such that

$$t_{j+1} - t_j \geq \frac{C_1}{m} \text{ if } J_j \subset \left[b' + \frac{\varepsilon}{8}, 1-\frac{\varepsilon}{8}\right], \tag{3.4}$$

and

$$t_{j+1} - t_j \leq \frac{C_2}{m}, \quad 0\leq j\leq m-1. \tag{3.5}$$

As our polynomial, we choose

$$R(x) := \prod_{j=1}^{m}(x-t_j)$$

so that for $r\in[0,1]$,

$$\frac{\max_{|z|=r}|R\,(z)|}{\min_{|z|=r}|R\,(z)|} = \left|\prod_{j=1}^{m}\frac{r+t_j}{r-t_j}\right| =: |U\,(r)|\,, \tag{3.6}$$

say. Next, for $r\in[b',1]$, (3.2) implies that

$$\begin{aligned}\frac{1}{m}\log|U(r)| - \log\lambda' &= \sum_{j=0}^{m-1}\int_{J_j}\big[g(r,t_j) - g(r,t)\big]\, d\mu^{\mathcal{H}}_{[b',1]}\,(t) \\ &=: \sum_{j=0}^{m-1}\Delta_j,\end{aligned} \tag{3.7}$$

say. We shall find a lower bound for this difference for large enough m and all $r \in \left[b' + \frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}\right]$. For such an r, choose $k = k(r)$ such that $r \in J_k$. Since $g(r, t_j) \geq 0$, we see that for $|j - k| \leq 2$,

$$\begin{aligned}\Delta_j &\geq -\int_{J_j} g(r,t)\, d\mu^{\mathcal{H}}_{[b',1]}(t) \\ &\geq -C_3 \int_{J_j} \log\left|\frac{2}{r-t}\right| dt \\ &\geq -C_4 \frac{\log m}{m},\end{aligned}$$

where $C_3, C_4 > 0$ are independent of m, j, r. Here we have used the fact that $(\mu^{\mathcal{H}}_{[b',1]})'$ is bounded in $\left[b' + \frac{\varepsilon}{8}, 1 - \frac{\varepsilon}{8}\right]$, as well as (3.5). Next, if $|j - k| \geq 2$ and $t \in J_j$, we see that for some s between t_j and t,

$$\begin{aligned}|g(r,t_j) - g(r,t)| &= \left|\frac{\partial g}{\partial s}(x,s)(t - t_j)\right| \\ &\leq \frac{2}{|r-s|}(t_{j+1} - t_j) \\ &\leq \frac{C_5}{|r-t|}(t_{j+1} - t_j).\end{aligned}$$

Again, C_5 is independent of m, j, r, t, and we have used (3.4). Then, using (3.5), we obtain for some $C_6, \ldots, C_9 > 0$, independent of m, j, r,

$$\begin{aligned}\sum_{0 \leq j \leq m-1 : |j-k| > 2} |\Delta_j| &\leq \frac{C_6}{m} \int_{\left\{t \in \left[b'+\frac{\varepsilon}{8}, 1-\frac{\varepsilon}{8}\right] : |t-r| \geq C_7/m\right\}} \frac{dt}{|x-t|} + \frac{C_8}{m} \\ &\leq \frac{C_9 \log m}{m}.\end{aligned}$$

Thus for $r \in \left[b' + \frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}\right]$, (3.7) shows that

$$\log|U(r)| \geq m \log \lambda' - C_{10} \log m > m \log \lambda,$$

for m large enough. Then it follows from (3.6) that

$$\frac{\max_{|z|=r} |R(z)|}{\min_{|z|=r} |R(z)|} > \lambda^m, \quad r \in \left[b' + \frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}\right],$$

so the set $\mathcal{S} \subset [0,1]$ satisfying (1.12) (with $n = 0$) has

$$\begin{aligned}&\mathcal{S} \subset [0, b' + \frac{\varepsilon}{4}) \cup (1 - \frac{\varepsilon}{4}, 1] \\ &\quad \Rightarrow meas(\mathcal{S}) \leq b' + \frac{\varepsilon}{2} < F^{[-1]}\left(\frac{1}{\log \lambda}\right) + \varepsilon,\end{aligned}$$

by (3.1).

The proof of Theorem 3(c). We note that (1.15) and (1.16) follow easily from (1.10) and (1.11) by inverting the asymptotic relations. □

4. The proof of Theorem 2

The proof of Theorem 2(a), (b). We shall use a well-known example [4, p.133]: let $0 < a < 1$ and $\mathcal{G} := \{z : |z| < 1\}$. Then

$$\frac{d\mu_{[-a,a]}^{\mathcal{G}}(x)}{dx} = \frac{\tau}{\sqrt{(a^2 - x^2)(1 - a^2x^2)}}, \quad x \in [-a, a],$$

where $\tau > 0$ is chosen so that $\mu_{[-a,a]}^{\mathcal{G}}$ has total mass 1. The Green's function for $\mathcal{G}$ with pole at t is

$$g_{\mathcal{G}}(z, t) = \log \left| \frac{1 - \bar{t}z}{z - t} \right|.$$

The properties of the Green equilibrium potential then give [4, p. 132 ff.]

$$\int_{-a}^{a} g_{\mathcal{G}}(x, t)\, d\mu_{[-a,a]}^{\mathcal{G}}(t) = V_{[-a,a]}^{\mathcal{G}}, \quad x \in [-a, a]. \tag{4.1}$$

We now map $\mathcal{H}$ conformally onto $\mathcal{G}$ in such a way that $[b, 1]$ is mapped onto $[-a, a]$ for some $a > 0$. Let us set, for the given b,

$$\phi(z) := \frac{z - \sqrt{b}}{z + \sqrt{b}}; \quad a := \frac{1 - \sqrt{b}}{1 + \sqrt{b}}.$$

Then ϕ maps $\mathcal{H}$ conformally onto $\mathcal{G}$, with $\phi([b, 1]) = [-a, a]$. Now let us set

$$x = \phi(y); \quad t = \phi(s).$$

Straightforward (but lengthy) calculations show that

$$g_{\mathcal{G}}(x, t) = g_{\mathcal{G}}\big(\phi(y), \phi(s)\big) = \log \left| \frac{y + s}{y - s} \right| = g(y, s),$$

and for some constant $\kappa > 0$,

$$\left(\mu_{[-a,a]}^{\mathcal{G}}\right)' (\phi(s))\, \phi'(s) = \frac{\kappa}{\sqrt{(s^2 - b^2)(1 - s^2)}}, \quad s \in (b, 1).$$

Then (4.1) shows that for some constant A,

$$\int_{b}^{1} g(y, s) \left(\mu_{[-a,a]}^{\mathcal{G}}\right)' (\phi(s))\, \phi'(s) ds = A, \quad y \in [b, 1]. \tag{4.2}$$

The uniqueness property of the Green equilibrium potential [4, Thm. 5.12, p.132] then shows that

$$\left(\mu_{[b,1]}^{\mathcal{H}}\right)' (s) = \left(\mu_{[-a,a]}^{\mathcal{G}}\right)' \big(\phi(s)\big) \phi'(s), \quad s \in [b, 1];$$
$$A = V_{[b,1]}^{\mathcal{H}}.$$

We then obtain (1.6) and the first equality in (1.7). Next, the property (4.2) with $y = 1$ gives

(4.3)

$$F(b) = \frac{1}{V^{\mathcal{H}}_{[b,1]}} = \int_b^1 \frac{dx}{\sqrt{(x^2-b^2)(1-x^2)}} \Big/ \int_b^1 \log\left|\frac{1+x}{1-x}\right| \frac{dx}{\sqrt{(x^2-b^2)(1-x^2)}}.$$

Then (1.9) follows from [1, p.564, 4.297, no.9] and [1, p.246, 3.152, no.9]. This also gives the second equality in (1.7).

The proof of Theorem 2(c). We have already noted that $C(E, i\mathbb{R})$ increases as E increases, and hence $F(b)$ is a decreasing function. To show that it is strictly increasing one assumes that $F(b') = F(b)$, for some $b' < b$, and "shifts left" $\mu^{\mathcal{H}}_{[b,1]}$ to a unit measure on $[b', b'+1-b]$, thereby obtaining a contradiction as in Step 3 in the proof of Theorem 3(a). We proceed with the proof of (1.10). For $b \in (0,1)$, let

$$q := \exp\left(-\frac{\pi K'(b)}{K(b)}\right) = \exp\left(-\pi^2 F(b)\right).$$

Then there is the identity [1, p.924, 8.197, no. 3]

$$4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)^4 = b.$$

We see then that as $b \to 0+$,

$$\begin{aligned} b &= 4\sqrt{q}\big(1+o(1)\big) \\ &= 4\exp\left(-\frac{\pi^2}{2}F(b)\right)\big(1+o(1)\big) \end{aligned}$$

and (1.10) follows.

Finally, for (1.11), we note that since $b \to 1-$, we may introduce an extra factor of $2x$ in the numerator and denominator of (4.3). Then a substitution $t = x^2$ and standard integrals give the result. Indeed,

$$\begin{aligned} F(b) &= \big(1+o(1)\big)\int_b^1 \frac{2x}{\sqrt{(x^2-b^2)(1-x^2)}} \Big/ \int_b^1 \log\left|\frac{(1+x)^2}{1-x^2}\right| \frac{2x\,dx}{\sqrt{(x^2-b^2)(1-x^2)}} \\ &= \big(1+o(1)\big)\int_{b^2}^1 \frac{dt}{\sqrt{(t-b^2)(1-t)}} \Big/ \left[\int_{b^2}^1 \log\frac{4}{1-t}\frac{dt}{\sqrt{(t-b^2)(1-t)}} + o(1)\right] \\ &= \big(1+o(1)\big)/\left|\log\left(1-b^2\right)\right|. \end{aligned}$$

Here we have used standard integrals in potential theory [4, pp.45-46]

$$\begin{aligned} \int_{b^2}^1 \frac{dt}{\pi\sqrt{(t-b^2)(1-t)}} &= 1; \\ \int_{b^2}^1 \log|1-t|\frac{dt}{\pi\sqrt{(t-b^2)(1-t)}} &= \log\left(\frac{1-b^2}{4}\right). \end{aligned}$$

Then (1.11) follows. □

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