Mean Convergence of Interpolation at Zeros of Airy Functions

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Abstract The classical Erdős-Turán theorem established mean convergence of Lagrange interpolants at zeros of orthogonal polynomials. A non-polynomial extension of this was established by Ian Sloan in 1983. Mean convergence of interpolation by entire functions has been investigated by Grozev, Rahman, and Vértesi. In this spirit, we establish an Erdős-Turán theorem for interpolation by entire functions at zeros of the Airy function.

1 Introduction

The classical Erdős-Turán theorem involves a weight w on an compact interval, which we take as $[-1,1]$. We assume that $w \geq 0$ and is positive on a set of positive measure. Let p_n denote the corresponding orthonormal polynomial of degree $n \geq 0$, so that for $m, n \geq 0$,

$$\int_{-1}^{1} p_n p_m w = \delta_{mn}.$$

Let us denote the zeros of p_n in $[-1,1]$ by

$$-1 < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < 1.$$

Given $f : [-1,1] \to \mathbb{R}$, let $L_n[f]$ denote the Lagrange interpolation polynomial to f at $\{x_{jn}\}_{j=1}^{n}$, so that $L_n[f]$ has degree at most $n-1$ and

$$L_n[f](x_{jn}) = f(x_{jn}), \quad 1 \leq j \leq n.$$

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Theorem 1 (Erdős-Turán Theorem). *Let $f : [-1,1] \to \mathbb{R}$ be continuous. For $n \geq 1$, let $L_n[f]$ denote the Lagrange interpolation polynomial to f at the zeros of p_n. Then*

$$\lim_{n\to\infty} \int_{-1}^{1} (f - L_n[f])^2 w = 0.$$

The ramifications of this result continue to be explored to this day. It has been extended in numerous directions: for example, rather than requiring f to be continuous, we can allow it to be Riemann integrable. We may replace w by a positive measure μ, which may have non-compact support. In addition, convergence in L_2 may be replaced, under additional conditions on w, by convergence in L_p. There is a very large literature on all of this. See [10, 11, 12, 13, 22, 23] for references and results.

Ian Sloan and his coauthor Will Smith ingeniously used results on mean convergence of Lagrange interpolation in various L_p norms, to establish the definitive results on convergence of product integration rules [14, 16, 18, 19, 20, 21]. This is a subject of substantial practical importance, for example in numerical solution of integral equations.

One can speculate that it was this interest in product integration that led to Ian Sloan extending the Erdős-Turán theorem to non-polynomial interpolation. Here is an important special case of his general result [17, p. 99]:

Theorem 2 (Sloan's Erdös-Turán Theorem on Sturm-Liouville Systems). *Consider the eigenvalue problem*

$$p(x)\, u''(x) + q(x)\, u'(x) + [r(x) + \lambda]\, u(x) = 0$$

with boundary conditions

$$(\cos\alpha)\, u(a) + (\sin\alpha)\, u'(a) = 0;$$
$$(\cos\beta)\, u(b) + (\sin\beta)\, u'(b) = 0.$$

Assume that p'', q', r are continuous and real valued on $[a,b]$, that $p > 0$ there, while α, β are real. Let $\{u_n\}_{n\geq 0}$ be the eigenfunctions, ordered so that the corresponding eigenvalues $\{\lambda_n\}$ are increasing. Given continuous $f : [a,b] \to \mathbb{R}$, let $\mathscr{L}_n[f]$ denote the linear combination of $\{u_j\}_{j=0}^{n}$ that coincides with f at the $n+1$ zeros of u_{n+1} in the open interval (a,b). Let

$$w(x) = \frac{1}{p(x)} \exp\left(\int_a^x \frac{q(t)}{p(t)} dt \right), \quad x \in [a,b].$$

Then

$$\lim_{n\to\infty} \int_a^b (f(x) - \mathscr{L}_n[f](x))^2\, w(x)\, dx = 0,$$

provided $f(a) = 0$ if $\sin\alpha = 0$ and $f(b) = 0$ if $\sin\beta = 0$. Moreover, there is a constant c independent of n and f such that for all such f,

$$\int_a^b (f(x) - \mathscr{L}_n[f](x))^2 w(x)\,dx$$
$$\leq C \inf_{c_0, c_1, \ldots, c_n} \int_a^b \left(f(x) - \sum_{j=0}^{n} c_j u_j(x) \right)^2 w(x)\,dx.$$

As a specific example, Sloan considers the Bessel equation. His general theorem [17, p. 102], from which the above result is deduced, involves orthonormal functions, associated reproducing kernels, and interpolation points satisfying two boundedness conditions. In 1988, M. R. Akhlaghi [2] extended Sloan's result to convergence in L_p for $p \geq 1$.

Interpolation by trigonometric polynomials is closely related to that by algebraic polynomials, in as much as every even trigonometric polynomial has the form $P(\cos\theta)$ where P is an algebraic polynomial. From trigonometric polynomials, one can pass via scaling limits to entire functions of exponential type, and the latter have a long and gloried history associated with sampling theory. However, to this author's knowledge, the first general result on mean convergence of entire interpolants at equispaced points is due to Rahman and Vértesi [15, Theorem 1, p. 304]. Define the classic sinc kernel

$$\mathbb{S}(t) = \begin{cases} \frac{\sin \pi t}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Given a function $f : \mathbb{R} \to \mathbb{R}$, and $\tau > 0$, define the (formal) Lagrange interpolation series

$$L_\tau[f;x] = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\tau}\right) \mathbb{S}\left(\tau\left(x - \frac{k\pi}{\tau}\right)\right).$$

It is easily seen that this series converges uniformly in compact sets if for some $p > 1$, we have

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\tau}\right) \right|^p < \infty.$$

Theorem 3 (Theorem of Rahman and Vértesi). *Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable over every finite interval and satisfy for some $\beta > \frac{1}{p}$,*

$$|f(x)| \leq C(1 + |x|)^{-\beta}, \quad x \in \mathbb{R}.$$

Then

$$\lim_{\tau \to \infty} \int_{-\infty}^{\infty} |f(x) - L_\tau[f;x]|^p\,dx = 0.$$

Butzer, Higgins and Stens later showed that this result is equivalent to the classical sampling theorem, and as such is an example of an approximate sampling theorem [3]. Of course there are sampling theorems at nonequally spaced points (see for example [7, 25]), and in the setting of de Branges spaces, there are more general expansions involving interpolation series. However, as far as this author is aware, there are no analogues of the Rahman-Vértesi theorem in that more general setting. Ganzburg [5] and Littman [9] have explored other aspects of convergence of Lagrange interpolation by entire functions.

One setting where mean convergence has been explored, is interpolation at zeros of Bessel functions, notably by Grozev and Rahman [6, Theorem 1, p. 48]. Let $\alpha > -1$ and

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\alpha+k+1)}$$

denote the Bessel function of order α. It is often convenient to instead use its entire cousin,

$$G_\alpha(z) = z^{-\alpha} J_\alpha(z).$$

J_α has positive zeros

$$j_{\alpha,1} < j_{\alpha,2} < j_{\alpha,3} < \cdots,$$

and matching negative zeros

$$j_{\alpha,-k} = -j_{\alpha,k}, \quad k \geq 1,$$

so for $f : \mathbb{R} \to \mathbb{C}$, and $\tau > 0$, one can define the formal interpolation series

$$L_{\alpha,\tau}[f;x] = \sum_{k=-\infty, k\neq 0}^{\infty} f\left(\frac{j_{\alpha,k}}{\tau}\right) \ell_{\alpha,k}(\tau z),$$

where for $k \neq 0$,

$$\ell_{\alpha,k}(z) = \frac{G_\alpha(z)}{G'_\alpha(j_{\alpha,k})(z - j_{\alpha,k})}.$$

Theorem 4 (Theorem of Rahman and Grozev). *Let $\alpha \geq -\frac{1}{2}$ and $p > 1$, or let $-1 < \alpha < -\frac{1}{2}$ and $1 < p < \frac{2}{|2\alpha+1|}$. Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable over every finite interval and satisfy for some $\delta > 0$,*

$$|f(x)| \leq C(1+|x|)^{-\alpha-\frac{1}{2}-\frac{1}{p}-\delta}, \quad x \in \mathbb{R}.$$

Then

$$\lim_{\tau\to\infty} \int_{-\infty}^{\infty} \left| |x|^{\alpha+\frac{1}{2}} (f(x) - L_\tau[f;x]) \right|^p dx = 0.$$

Note that $p = 2$ is always included. The proof of this theorem involves a lot of tools: detailed properties of entire functions of exponential type and of Bessel functions, and a converse Marcinkiewicz-Zygmund inequality that is itself of great interest.

In this paper, we explore convergence of interpolation at scaled zeros of Airy functions. Recall that the Airy function Ai is given on the real line by [1, 10.4.32, p. 447]

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt.$$

The Airy function Ai is an entire function of order $\frac{3}{2}$, with only real negative zeros $\{a_j\}$, where

$$0 > a_1 > a_2 > a_3 > \cdots.$$

Ai satisfies the differential equation

$$Ai''(z) - zAi(z) = 0.$$

The Airy kernel $\mathbb{A}i(\cdot,\cdot)$, much used in random matrix theory, is defined [8] by

$$\mathbb{A}i(a,b) = \begin{cases} \frac{Ai(a)Ai'(b)-Ai'(a)Ai(b)}{a-b}, & a \neq b, \\ Ai'(a)^2 - aAi(a)^2, & a = b. \end{cases}$$

Observe that

$$\ell_j(z) = \frac{\mathbb{A}i(z,a_j)}{\mathbb{A}i(a_j,a_j)} = \frac{Ai(z)}{Ai'(a_j)(z-a_j)},$$

is the Airy analogue of a fundamental of Lagrange interpolation, satisfying

$$\ell_j(a_k) = \delta_{jk}.$$

There is an analogue of sampling series and Lagrange interpolation series involving $\{\ell_j\}$:

Definition 1. Let $\mathscr{G}$ be the class of all functions $g : \mathbb{C} \to \mathbb{C}$ with the following properties:

(a) g is an entire function of order at most $\frac{3}{2}$;
(b) There exists $L > 0$ such that for $\delta \in (0,\pi)$, some $C_\delta > 0$, and all $z \in \mathbb{C}$ with $|\arg z| \leq \pi - \delta$,

$$|g(z)| \leq C_\delta (1+|z|)^L \left| \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) \right|;$$

(c)

$$\sum_{j=1}^{\infty} \frac{|g(a_j)|^2}{|a_j|^{1/2}} < \infty.$$

In [8, Corollary 1.3, p. 429], it was shown that each $g \in \mathscr{G}$ admits the expansion

$$g(z) = \sum_{j=1}^{\infty} g(a_j) \frac{\mathbb{A}i(z,a_j)}{\mathbb{A}i(a_j,a_j)}.$$

Moreover, for $f, g \in \mathscr{G}$, there is the quadrature formula [8, Corollary 1.4, p. 429]

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \sum_{j=1}^{\infty} \frac{(fg)(a_j)}{\mathbb{A}i(a_j,a_j)}.$$

In analogy with the entire interpolants of Grozev-Rahman, we define for $f : \mathbb{R} \to \mathbb{R}$, the formal series

$$\mathbb{L}_\tau [f;z] = \sum_{j=1}^{\infty} f\left(\frac{a_j}{\tau}\right) \ell_j(\tau z) = \sum_{j=1}^{\infty} f\left(\frac{a_j}{\tau}\right) \frac{\mathbb{A}i(\tau z, a_j)}{\mathbb{A}i(a_j, a_j)}. \quad (1)$$

Note that it samples f only in $(-\infty, 0)$.

We prove:

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Riemann integrable in each finite interval, with $f(x) = 0$ in $[0,\infty)$. Assume in addition that for some $\beta > \frac{1}{2}$, and $x \in \mathbb{R}$,*

$$|f(x)| \le C(1+|x|)^{-\beta}. \quad (2)$$

Then

$$\lim_{\tau\to\infty} \int_{-\infty}^{\infty} (f(x) - \mathbb{L}_\tau [f;x])^2 \, dx = 0. \quad (3)$$

Observe that the integration is over the whole real line. We expect that there is an analogue of this theorem at least for all $p > 1$. However, this seems to require a converse Marcinkiewicz-Zygmund inequality estimating L_p norms of appropriate classes of entire functions in terms of their values at Airy zeros. This is not available, so we content ourselves with a weaker result for the related operator

$$\mathbb{L}^*_\tau [f;z] = \sum_{j=1}^{\infty} f\left(\frac{a_{2j-1}}{\tau}\right) \left[\ell_{2j-1}(\tau z) + \ell_{2j}(\tau z)\right].$$

This interpolates f at each $\frac{a_{2j-1}}{\tau}$, but not at $\frac{a_{2j}}{\tau}$.

Theorem 6. *(a) For bounded functions $f : \mathbb{R} \to \mathbb{R}$, and $\tau \ge |a_1|$,*

$$\sup_{x\in\mathbb{R}} |\mathbb{L}^*_\tau [f;x]| \le C \sup_j \left| f\left(\frac{a_{2j-1}}{\tau}\right)\right|, \quad (4)$$

where C is independent of $\tau \ge 1$ and f.

(b) Let $\frac{4}{5} < p < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Riemann integrable in each finite interval, with $f(x) = 0$ in $[0,\infty)$.Assume in addition that for some $\beta > \frac{1}{p}$, and $x \in \mathbb{R}$, we have (2)*. Then*

$$\lim_{\tau\to\infty} \int_{-\infty}^{\infty} |f(x) - \mathbb{L}^*_\tau [f;x]|^p \, dx = 0. \quad (5)$$

Note that $\mathbb{L}^*_\tau \left[f; \frac{z}{\tau}\right] \in \mathscr{G}$, so this also establishes density of that class of functions in a suitable space of functions containing those in Theorem 6. The usual approach to Erdős-Turan theorems is via quadrature formulae and density of polynomials, or entire functions of exponential type, in appropriate spaces. The latter density is not available for $\mathscr{G}$. So in Section 2, we establish convergence for characteristic functions of intervals. We prove Theorems 5 and 6 in Section 3. Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, z, t, τ, and possibly other specified quantities. The same symbol does not necessarily denote the same constant in different occurrences, even when used in the same line.

2 Interpolation of Step Functions

We prove:

Theorem 7. *Let $r > 0$, and f denote the characteristic function $\chi_{[-r,0]}$ of the interval $[-r,0]$. Then for $p > \frac{4}{5}$,*

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty}|\mathbb{L}_\tau[f;x]-f(x)|^p\,dx=0. \tag{6}$$

and

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty}|\mathbb{L}_\tau^*[f;x]-f(x)|^p\,dx=0. \tag{7}$$

This section is organized as follows: we first recall some asymptotics associated with Airy functions. Then we prove some estimates on integrals involving the fundamental polynomials ℓ_j. Next we prove the case $p=2$ of Theorem 7. Then we estimate a certain sum and finally prove the general case of Theorem 7.

Firstly, the following asymptotics and estimates are listed on pages 448–449 of [1]: see (10.4.59–61) there.

$$Ai(x)=\frac{1}{2\pi^{1/2}}x^{-1/4}\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)(1+o(1)),\quad x\to\infty; \tag{8}$$

$$Ai(-x)=\pi^{-1/2}x^{-1/4}\left[\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)+O\left(x^{-\frac{3}{2}}\right)\right],\quad x\to\infty. \tag{9}$$

Then as Ai is entire, for $x\in[0,\infty)$,

$$\begin{aligned}|Ai(x)|&\le C(1+x)^{-1/4}\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)\text{ and}\\ |Ai(-x)|&\le C(1+x)^{-1/4};\end{aligned} \tag{10}$$

$$\begin{aligned}Ai'(-x)=&-\pi^{-1/2}x^{1/4}\cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)\left(1+O\left(x^{-\frac{4}{3}}\right)\right)\\ &+O\left(x^{-\frac{2}{3}}\right),\quad x\to\infty.\end{aligned} \tag{11}$$

Next, the zeros $\{a_j\}$ of Ai satisfy [1, p. 450, (10.4.94,96)]

$$\begin{aligned}a_j&=-[3\pi(4j-1)/8]^{2/3}\left(1+O\left(\frac{1}{j^2}\right)\right)\\ &=-\left(\frac{3\pi j}{2}\right)^{2/3}(1+o(1)).\end{aligned} \tag{12}$$

Consequently,

$$\left|a_{j+1}\right|-\left|a_j\right|=\frac{\pi}{\left|a_j\right|^{1/2}}\left(1+o\left(1\right)\right). \tag{13}$$

In addition,

$$\begin{aligned}Ai'\left(a_j\right)&=\left(-1\right)^{j-1}\pi^{-1/2}\left(\frac{3\pi}{8}\left(4j-1\right)\right)^{1/6}\left(1+O\left(j^{-2}\right)\right)\\&=\left(-1\right)^{j-1}\pi^{-1/2}\left|a_j\right|^{1/4}\left(1+o\left(1\right)\right).\end{aligned} \tag{14}$$

A calculation shows that

$$\left|Ai'\left(a_j\right)\right|-\left|Ai'\left(a_{j-1}\right)\right|=C_0 j^{-5/6}(1+O\left(j^{-1}\right)),\quad C_0=\frac{1}{6}\left(\frac{3}{2\pi^2}\right)^{1/6}. \tag{15}$$

Define the Scorer function [1, p. 448, (10.4.42)]

$$Gi\left(x\right)=\frac{1}{\pi}\int_0^\infty \sin\left(\frac{t^3}{3}+xt\right)dt. \tag{16}$$

We shall use an identity for the Hilbert transform of the Airy function [24, p. 71, eqn. (4.4)]:

$$\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{Ai\left(t\right)}{t-x}dt=-Gi\left(x\right). \tag{17}$$

Here PV denotes Cauchy principal value integral. We also use [1, p. 450, eqn. (10.4.87)]

$$Gi\left(-x\right)=\pi^{-1/2}x^{-1/4}\left[\cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)+o\left(1\right)\right],\quad x\to\infty. \tag{18}$$

Finally the Airy kernel $\mathbb{A}i\left(a,b\right)$ satisfies [8, p. 432]

$$\int_{-\infty}^{\infty}\mathbb{A}i\left(a_j,s\right)\mathbb{A}i\left(s,a_k\right)ds=\delta_{jk}\mathbb{A}i\left(a_j,a_j\right)=\delta_{jk}Ai'\left(a_j\right)^2.$$

Thus

$$\int_{-\infty}^{\infty}\ell_j\left(s\right)\ell_k\left(s\right)ds=\delta_{jk}\frac{1}{\mathbb{A}i\left(a_j,a_j\right)}=\delta_{jk}\frac{1}{Ai'\left(a_j\right)^2}. \tag{19}$$

Lemma 1. *(a) As $j\to\infty$,*

$$\int_{-\infty}^{\infty}\ell_j\left(t\right)dt=\frac{\pi}{\left|a_j\right|^{1/2}}(1+o\left(1\right)). \tag{20}$$

(b) Uniformly in j,r with $r>\left|a_j\right|$,

$$\int_{-r}^{0}\ell_j\left(t\right)dt=\frac{\pi}{\left|a_j\right|^{1/2}}(1+o\left(1\right)+O\left(\frac{1}{\left|a_j\right|^{1/4}r^{3/4}\left(r-\left|a_j\right|\right)}\right). \tag{21}$$

Proof. (a) Now (17) yields

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\ell_j(t)\,dt = -\frac{Gi(a_j)}{Ai'(a_j)} \tag{22}$$

Here using (12),

$$\cos\left(\frac{2}{3}\left|a_j\right|^{\frac{3}{2}}+\frac{\pi}{4}\right) = (-1)^j + O\left(\frac{1}{j}\right),$$

so from (18),

$$Gi(a_j) = \pi^{-1/2}\left|a_j\right|^{-1/4}(-1)^j(1+o(1)).$$

Substituting this and (14) into (22) gives

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\ell_j(t)\,dt = \frac{1}{\left|a_j\right|^{1/2}}(1+o(1)).$$

(b) Using the bound (10),

$$\begin{aligned}\int_0^{\infty}\left|\ell_j(t)\right|dt &\leq \frac{C}{\left|Ai'(a_j)\right|}\int_0^{\infty}\frac{t^{-1/4}\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)}{t-a_j}dt \\ &\leq \frac{C}{\left|a_j Ai'(a_j)\right|} \leq C\left|a_j\right|^{-5/4},\end{aligned} \tag{23}$$

recall (14). Next,

$$\begin{aligned}\left|\int_{-\infty}^{-r}\ell_j(t)\,dt\right| &= \frac{1}{\left|Ai'(a_j)\right|}\left|\int_r^{\infty}\frac{Ai(-x)}{x+a_j}dx\right| \\ &= \frac{1}{\left|Ai'(a_j)\right|}\left|\pi^{-1/2}\int_r^{\infty}\frac{x^{-1/4}\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)}{x+a_j}dx\right. \\ &\quad \left. + O\left(\int_r^{\infty}\frac{x^{-7/4}}{\left|x+a_j\right|}dx\right)\right|,\end{aligned} \tag{24}$$

by (9). Here,

$$
\begin{aligned}
I &= \int_r^\infty \frac{x^{-1/4}\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)}{x+a_j}dx \\
&= \int_r^\infty \frac{-x^{-\frac{3}{4}}\frac{d}{dx}\left[\cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)\right]}{x+a_j}dx \\
&= \frac{\cos\left(\frac{2}{3}r^{3/2}+\frac{\pi}{4}\right)}{r^{3/4}\left(r-\left|a_j\right|\right)} + \int_r^\infty \cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)\frac{d}{dx}\left[\frac{1}{x^{3/4}(x-\left|a_j\right|)}\right]dx \\
&= O\left(\frac{1}{r^{3/4}\left(r-\left|a_j\right|\right)}\right) + O\left(\int_r^\infty \left|\frac{d}{dx}\left[\frac{1}{x^{3/4}(x-\left|a_j\right|)}\right]\right|dx\right) \\
&= O\left(\frac{1}{r^{3/4}\left(r-\left|a_j\right|\right)}\right), \qquad (25)
\end{aligned}
$$

as $\frac{1}{x^{3/4}(x-\left|a_j\right|)}$ is decreasing in $[r,\infty)$. Next,

$$\int_r^\infty \frac{x^{-7/4}}{\left|x+a_j\right|}dx \le \frac{1}{r-\left|a_j\right|}\int_r^\infty x^{-7/4}dx \le \frac{C}{r^{3/4}\left(r-\left|a_j\right|\right)}.$$

Thus, using also (25) in (24),

$$\left|\int_{-\infty}^{-r} \ell_j(t)\,dt\right| \le \frac{C}{\left|a_j\right|^{1/4} r^{3/4}\left(r-\left|a_j\right|\right)}.$$

Together with (20) and (23), this gives the result (21). □

Lemma 2. *Let* $L \ge 1$, *and*

$$S_L(x) = \sum_{j=1}^{L}\ell_j(x) = \sum_{j=1}^{L}\frac{\mathbb{A}i(x,a_j)}{\mathbb{A}i(a_j,a_j)}. \qquad (26)$$

Then

$$\lim_{L\to\infty}\frac{1}{|a_{L+1}|}\int_{-\infty}^{\infty}\left(S_L(x)-\chi_{[a_{L+1},0]}(x)\right)^2dx = 0. \qquad (27)$$

Proof. Using (19), and then (14),

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left(S_L(x)-\chi_{[a_{L+1},0]}(x)\right)^2 dx\\
&=\sum_{j=1}^{L}\frac{1}{Ai'(a_j)^2}-2\sum_{j=1}^{L}\int_{a_{L+1}}^{0}\ell_j(x)\,dx+|a_{L+1}|\\
&=\sum_{j=1}^{L}\frac{\pi}{|a_j|^{1/2}}(1+o(1))\\
&\quad-2\sum_{j=1}^{L}\left\{\frac{\pi}{|a_j|^{1/2}}\left(1+o(1)+O\left(\frac{1}{|a_{L+1}|^{3/4}|a_j|^{1/4}\left(|a_{L+1}|-|a_j|\right)}\right)\right)\right\}+|a_{L+1}|\\
&=|a_{L+1}|-\sum_{j=1}^{L}\frac{\pi}{|a_j|^{1/2}}(1+o(1))+O\left(|a_{L+1}|^{-3/4}\sum_{j=1}^{L}\frac{1}{|a_j|^{1/4}\left(|a_{L+1}|-|a_j|\right)}\right),
\end{aligned}
\tag{28}
$$

by (14) and (21). Here using (13),

$$
\sum_{j=1}^{L}\frac{\pi}{|a_j|^{1/2}}=\sum_{j=1}^{L}\left(|a_{j+1}|-|a_j|\right)(1+o(1))=|a_{L+1}|(1+o(1)). \tag{29}
$$

Also, from (12),

$$
\begin{aligned}
|a_{L+1}|^{-3/4}&\sum_{j=1}^{L}\frac{1}{|a_j|^{1/4}\left(|a_{L+1}|-|a_j|\right)}\\
&\le C|a_{L+1}|^{-7/4}\sum_{j=1}^{L}\frac{1}{j^{1/6}\left(1-\frac{j}{L+1}\right)}\\
&\le C|a_{L+1}|^{-7/4}\int_0^L\frac{1}{x^{1/6}\left(1-\frac{x}{L+1}\right)}dx\\
&\le C|a_{L+1}|^{-7/4}L^{5/6}\log L\\
&\le C|a_{L+1}|^{-1/2}\log L.
\end{aligned}
\tag{30}
$$

Substituting this and (29) into (28), gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left(S_L(x)-\chi_{[a_{L+1},0]}(x)\right)^2 dx\\
&\quad=o(|a_{L+1}|)+O\left(C|a_{L+1}|^{-1/2}\log L\right)=o(|a_{L+1}|).
\end{aligned}
$$

□

Proof of Theorem 7 *for* $p=2$. Given $\tau\ge|a_1|/r$, choose $L=L(\tau)$ by the inequality

$$
|a_L|\le\tau r<|a_{L+1}|. \tag{31}
$$

Then

$$\mathbb{L}_\tau [f;x] = \sum_{a_j/\tau \in [-r,0]} \ell_j (\tau x) = \sum_{j=1}^{L} \ell_j (\tau x) . \tag{32}$$

By Lemma 2,

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\mathbb{L}_\tau [f;x] - \chi_{[a_{L+1},0]} (\tau x)\right)^2 dx \\ &\quad = \frac{1}{\tau} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{L} \ell_j (t) - \chi_{[a_{L+1},0]} (t) \right)^2 dt \\ &\quad = \frac{1}{\tau} o(|a_{L+1}|) = o(1), \end{aligned}$$

as $\tau \to \infty$. Also, as $\tau \to \infty$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\chi_{[r,0]} (x) - \chi_{[a_{L+1},0]} (\tau x)\right)^2 dx \\ &\quad = \int_{-\infty}^{\infty} \chi_{\left[\frac{a_{L+1}}{\tau},r\right]} (x)^2 dx \\ &\quad = \frac{|a_{L+1}|}{\tau} - r \leq \frac{|a_{L+1}| - |a_L|}{\tau} = O\left(L^{-1/3}\tau^{-1}\right) = o(1). \end{aligned}$$

Then (6) follows. Since $\mathbb{L}_\tau [f;x] = \mathbb{L}^*_\tau [f;x]$ if L above is even, (7) also follows. The case of odd L is easily handled by estimating separately the single extra term. □

Next we bound a generalization of $S_L (x)$:

Lemma 3. *Let $A > 0$, $0 \leq \beta < \frac{5}{4}$ and $\{c_j\}_{j=1}^{\infty}$ be real numbers such that for $j \geq 1$,*

$$c_{2j} = c_{2j-1}, \tag{33}$$

and

$$\left|c_{2j-1}\right| \leq A\left(1 + \left|a_{2j}\right|\right)^{-\beta}. \tag{34}$$

Let

$$\hat{S}(x) = \sum_{j=1}^{\infty} c_j \ell_j (x).$$

Then the series converges and for all real x,

$$\left|\hat{S}(x)\right| \leq CA(1 + |x|)^{-\beta}. \tag{35}$$

Here C is independent of x, A, and $\{c_j\}_{j=1}^{\infty}$.

Proof. We may assume that $A = 1$. We assume first that $x \in (-\infty, 0)$, as this is the most difficult case. Set $a_0 = 0$. Choose an even integer $j_0 \geq 2$ such that $x \in [a_{j_0}, a_{j_0-2})$. Let us first deal with central terms: assume that $j \geq 1$ and $|j - j_0| \leq 3$. Then

$$\left|\ell_j (x)\right| = \frac{1}{\left|Ai' (a_j)\right|} \left| \frac{Ai (x) - A_i (a_j)}{x - a_j} \right| = \left| \frac{Ai' (t)}{Ai' (a_j)} \right|,$$

for some t between x and a_j, so $(|t|+1)/|a_j| \sim 1$. Using (11), (14), and the continuity of Ai', we see that

$$\left|\ell_j(x)\right| \leq C\frac{(1+|t|)^{1/4}}{\left|a_j\right|^{1/4}} \leq C.$$

Thus as $|x|+1 \sim \left|a_{j_0}\right|$, and (34) holds,

$$\sum_{j:|j-j_0|\leq 3} \left|c_j\ell_j(x)\right| \leq C(1+|x|)^{-\beta}. \tag{36}$$

Again, we emphasize that C is independent of L and x and $\{c_j\}$. We turn to the estimation of

$$S^*(x) = \left(\sum_{j=1}^{j_0-4} + \sum_{j=j_0+3}^{\infty}\right) c_j\ell_j(x).$$

Recall from (14) that $Ai'(a_j)$ has sign $(-1)^{j-1}$. Then

$$\begin{aligned}
&S^*(x)\\
&= Ai(x)\left(\sum_{k=1}^{(j_0-4)/2} + \sum_{k=j_0/2+2}^{\infty}\right) c_{2k-1}\left[\frac{1}{|Ai'(a_{2k-1})|(x-a_{2k-1})} - \frac{1}{|Ai'(a_{2k})|(x-a_{2k})}\right]\\
&= Ai(x)(\Sigma_1+\Sigma_2), \qquad (37)
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1 &= \left(\sum_{k=1}^{(j_0-4)/2} + \sum_{k=j_0/2+2}^{\infty}\right) \frac{c_{2k-1}}{|Ai'(a_{2k-1})|}\left(\frac{1}{x-a_{2k-1}} - \frac{1}{x-a_{2k}}\right)\\
&= \left(\sum_{k=1}^{(j_0-4)/2} + \sum_{k=j_0/2+2}^{\infty}\right) \frac{c_{2k-1}}{|Ai'(a_{2k-1})|}\frac{a_{2k-1}-a_{2k}}{(x-a_{2k-1})(x-a_{2k})} \qquad (38)
\end{aligned}$$

and

$$\Sigma_2 = \left(\sum_{k=1}^{(j_0-4)/2} + \sum_{k=j_0/2+2}^{\infty}\right) c_{2k-1}\left(\frac{1}{|Ai'(a_{2k-1})|} - \frac{1}{|Ai'(a_{2k})|}\right)\frac{1}{x-a_{2k}}.$$

Then if $\mathscr{I}$ denotes the set of integers k with either $1 \leq k \leq (j_0-4)/2$ or $k \geq j_0/2+2$, we see that $\frac{|x-a_{2k}|}{|x-a_{2k-1}|}$, and $\frac{|a_{2k-1}-a_{2k}|}{|a_{2k-2}-a_{2k}|}$ are bounded above and below by positive constants independent of k,x, so using (14) and (34), so

$$
\begin{aligned}
\Sigma_1 &\leq C \sum_{k\in\mathscr{I}} \frac{1}{|a_{2k}|^{1/4+\beta}} \frac{|a_{2k-2}-a_{2k}|}{(x-a_{2k})^2} \\
&\leq C \int_{[|a_1|,\infty)\setminus\left[|a_{j_0-3}|,|a_{j_0+1}|\right]} \frac{1}{t^{1/4+\beta}} \frac{1}{(|x|-t)^2} dt \\
&= \frac{C}{\left|a_{j_0}\right|^{5/4+\beta}} \int_{\left[\frac{|a_1|}{|a_{j_0}|},\infty\right)\setminus\left[\frac{|a_{j_0-3}|}{|a_{j_0}|},\frac{|a_{j_0+1}|}{|a_{j_0}|}\right]} \frac{1}{s^{1/4+\beta}} \frac{1}{\left(\frac{|x|}{|a_{j_0}|}-s\right)^2} ds \\
&\leq \frac{C}{\left|a_{j_0}\right|^{5/4+\beta}} \left(\left\{ \begin{array}{ll} \left|a_{j_0}\right|^{\beta-3/4}, & \beta > 3/4 \\ \log\left|a_{j_0},\right| & \beta = 3/4 \\ 1, & \beta < 3/4 \end{array} \right\} + \frac{\left|a_{j_0}\right|}{\left||x|-\left|a_{j_0-3}\right|\right|} + \frac{\left|a_{j_0}\right|}{\left||x|-\left|a_{j_0+1}\right|\right|} \right) \\
&\leq C \left(\frac{1}{\left|a_{j_0}\right|^{-1/4+\beta}} + \frac{\left|a_{j_0}\right|^{1/2}}{\left|a_{j_0}\right|^{1/4+\beta}} \right) \\
&\leq C(1+|x|)^{1/4-\beta},
\end{aligned} \tag{39}
$$

recall that $\beta < \frac{5}{4}$, and that $\left||x|-\left|a_{j_0-3}\right|\right| \geq \left|\left|a_{j_0}\right|-\left|a_{j_0-2}\right|\right| \geq C\left|a_{j_0}\right|^{-1/2}$, by (13). Next, using (13–15) and (34),

$$
\begin{aligned}
|\Sigma_2| &\leq C \sum_{k\in\mathscr{I}} \frac{k^{-5/6}}{|a_{2k}|^{\beta}\,|Ai'(a_{2k-1})|\,|Ai'(a_{2k})|} \frac{1}{|x-a_{2k}|} \\
&\leq C \sum_{k\in\mathscr{I}} \frac{(|a_{2k}|-|a_{2k-2}|)}{|a_{2k}|^{5/4+\beta}} \frac{1}{|x-a_{2k}|} \\
&\leq C \int_{[|a_1|,\infty)\setminus\left[|a_{j_0-3}|,|a_{j_0+1}|\right]} \frac{1}{t^{5/4+\beta}\,||x|-t|} dt \\
&\leq \frac{C}{\left|a_{j_0}\right|^{5/4+\beta}} \int_{\left[\frac{|a_1|}{|a_{j_0}|},\infty\right)\setminus\left[\frac{|a_{j_0-3}|}{|a_{j_0}|},\frac{|a_{j_0+1}|}{|a_{j_0}|}\right]} \frac{1}{s^{5/4+\beta}} \frac{1}{\left|\frac{|x|}{|a_{j_0}|}-s\right|} ds \\
&\leq \frac{C}{\left|a_{j_0}\right|^{5/4+\beta}} \left(\left|a_{j_0}\right|^{1/4+\beta} + \left|\log\left|\frac{|x|}{|a_{j_0}|}-\frac{|a_{j_0-3}|}{|a_{j_0}|}\right|\right| + \left|\log\left|\frac{|x|}{|a_{j_0}|}-\frac{|a_{j_0+1}|}{|a_{j_0}|}\right|\right| \right) \\
&\leq C\left(\left|a_{j_0}\right|^{-1} + \left|a_{j_0}\right|^{-5/4-\beta}\log j_0\right) \leq C(1+|x|)^{1/4-\beta},
\end{aligned}
$$

recall $\beta < \frac{5}{4}$. Substituting this and (39) into (37) gives

$$
|S^*(x)| \leq C\,|Ai(x)|\,(1+|x|)^{1/4-\beta} \leq C(1+|x|)^{-\beta},
$$

in view of (10). This and (36) gives (35). Finally, the case where $x \geq 0$ is easier. □

We deduce:

Proof of Theorem 7 for the general case. Recall that $f = \chi_{[-r,0]}$. Assume first $p > 2$. The previous lemma (with all $c_j = 1$ and $\beta = 0$) shows that

$$\left|\mathbb{L}_{\tau}[f;x]-f(x)\right| \leq \sup_{x\in\mathbb{R}}\left|\mathbb{L}_{\tau}[f;x]\right|+1 \leq C,$$

where C is independent of τ. We can then apply the case $p=2$ of Theorem 7:

$$\begin{aligned}&\limsup_{\tau\to\infty}\int_{-\infty}^{\infty}\left|\mathbb{L}_{\tau}[f;x]-f(x)\right|^{p}dx\\&\quad\leq C^{p-2}\limsup_{\tau\to\infty}\int_{-\infty}^{\infty}\left|\mathbb{L}_{\tau}[f;x]-f(x)\right|^{2}dx=0.\end{aligned}$$

Next, if $\frac{4}{5}<p<2$, and $s\geq 2r$, Hölder's inequality gives

$$\begin{aligned}&\limsup_{\tau\to\infty}\int_{-s}^{s}\left|\mathbb{L}_{\tau}[f;x]-f(x)\right|^{p}dx\\&\quad\leq \limsup_{\tau\to\infty}\left(\int_{-s}^{s}\left|\mathbb{L}_{\tau}[f;x]-f(x)\right|^{2}dx\right)^{\frac{p}{2}}(2s)^{1-\frac{p}{2}}=0.\end{aligned} \tag{40}$$

Next, for $|x|\geq s>2r$, and all τ, we have

$$\begin{aligned}\left|\mathbb{L}_{\tau}[f;x]-f(x)\right| &\leq \sum_{a_j\in[-\tau r,0]}\frac{|Ai(\tau x)|}{|Ai'(a_j)|\,|\tau x-a_j|}\\&\leq C\frac{|Ai(\tau x)|}{\tau|x|}\sum_{a_j\in[-\tau r,0]}\frac{1}{|a_j|^{1/4}}\\&\leq C(\tau|x|)^{-5/4}\sum_{j\leq C(\tau r)^{3/2}}\frac{1}{j^{1/6}}\\&\leq C(\tau|x|)^{-5/4}(\tau r)^{5/4}=Cr^{5/4}|x|^{-5/4}.\end{aligned}$$

Thus

$$\limsup_{\tau\to\infty}\int_{|x|\geq s}\left|\mathbb{L}_{\tau}[f;x]-f(x)\right|^{p}dx\leq C\int_{|x|\geq s}|x|^{-5p/4}dx\leq Cs^{1-5p/4}\to 0$$

as $s\to\infty$, recall $p>5/4$. Together with (40), this gives (6). Of course (7) also follows as $\mathbb{L}_{\tau}^{*}[f;x]$ differs from $\mathbb{L}_{\tau}[f;x]$ in at most one term, which can easily be estimated. □

3 Proof of Theorems 5 and 6

Proof of Theorem 5. Suppose first that f is bounded and Riemann integrable, and supported in $(-r,0]$, some $r>0$. Let $\varepsilon>0$. Then we can find a (piecewise constant) step function g also compactly supported in $(-r,0]$ such that both

$$g \geq f \text{ in } \mathbb{R} \text{ and } \int_{-\infty}^{\infty} (f-g)^2 < \varepsilon^2.$$

This follows directly from the theory of Riemann sums and the boundedness of f. Theorem 7 implies that for any such step function g,

$$\lim_{\tau\to\infty} \left(\int_{-\infty}^{\infty} (g(x) - \mathbb{L}_\tau[g;x])^2 \, dx \right)^{1/2} = 0.$$

Then using also the orthonormality relation (19),

$$\begin{aligned}
&\limsup_{\tau\to\infty} \left(\int_{-\infty}^{\infty} (f(x) - \mathbb{L}_\tau[f;x])^2 \, dx \right)^{1/2} \\
&\quad \leq \left(\int_{-\infty}^{\infty} (f(x) - g(x))^2 \, dx \right)^{1/2} + \limsup_{\tau\to\infty} \left(\int_{-\infty}^{\infty} (g(x) - \mathbb{L}_\tau[g;x])^2 \, dx \right)^{1/2} \\
&\qquad + \limsup_{\tau\to\infty} \left(\int_{-\infty}^{\infty} \mathbb{L}_\tau[g-f;x]^2 \, dx \right)^{1/2} \\
&\quad \leq \varepsilon + 0 + \limsup_{\tau\to\infty} \left(\frac{1}{\tau} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{\infty} (f-g)\left(\frac{a_j}{\tau}\right) \ell_j(x) \right)^2 dx \right)^{1/2} \\
&\quad = \varepsilon + \limsup_{\tau\to\infty} \left(\frac{1}{\tau} \sum_{a_j \in (-\tau r, 0)} \frac{(f-g)^2 \left(\frac{a_j}{\tau}\right)}{Ai'(a_j)^2} \right)^{1/2} \\
&\quad = \varepsilon + C \limsup_{\tau\to\infty} \left(\sum_{\frac{a_j}{\tau} \in (-r,0)} \left(\frac{|a_j|}{\tau} - \frac{|a_{j-1}|}{\tau} \right) (f-g)^2 \left(\frac{a_j}{\tau}\right) \right)^{1/2} \\
&\quad = \varepsilon + C \left(\int_{-r}^{0} |f-g|^2(x) \, dx \right)^{1/2} \leq C\varepsilon.
\end{aligned}$$

Here C is independent of ε, g and f, and we have used (13), (14), that $\max_j \left(\frac{|a_j|}{\tau} - \frac{|a_{j-1}|}{\tau} \right) \to 0$ as $\tau \to \infty$, and the theory of Riemann sums. So we have the result for such compactly supported f. Now assume that f is supported in $(-\infty, -r)$ and for some $\beta > \frac{1}{2}$, (2) holds. Then using (19) again,

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty}(f(x)-\mathbb{L}_\tau[f;x])^2\,dx\right)^{1/2}\\
&\leq\left(\int_r^{\infty}f^2(x)\,dx\right)^{1/2}+\left(\frac{1}{\tau}\sum_{a_j\in(-\infty,-\tau r)}\frac{f^2\left(\frac{a_j}{\tau}\right)}{Ai'(a_j)^2}\right)^{1/2}\\
&\leq C\left(\int_r^{\infty}(1+|x|)^{-2\beta}\,dx\right)^{1/2}+C\left(\frac{1}{\tau}\sum_{a_j\in(-\infty,-\tau r)}\frac{\left(\frac{|a_j|}{\tau}\right)^{-2\beta}}{|a_j|^{1/2}}\right)\\
&\leq Cr^{\frac{1}{2}-\beta}+C\left(\frac{1}{\tau}\sum_{a_j\in(-\infty,-\tau r)}\left(\frac{|a_j|}{\tau}\right)^{-2\beta}\left(|a_j|-|a_{j-1}|\right)\right)^{1/2}\\
&\leq Cr^{\frac{1}{2}-\beta}+C\left(\tau^{-1+2\beta}\int_{\tau r}^{\infty}t^{-2\beta}dt\right)^{1/2}\leq Cr^{1/2-\beta},
\end{aligned}$$

where C is independent of r and τ. So

$$\limsup_{\tau\to\infty}\left(\int_{-\infty}^{\infty}(f(x)-\mathbb{L}_\tau[f;x])^2\,dx\right)^{1/2}\leq Cr^{1/2-\beta}.$$

This can be made arbitrarily small for large enough r. Together with the case above, this easily implies the result. □

Proof of Theorem 6. (a) From Lemma 3 with $\beta=0$,

$$\sup_{x\in\mathbb{R}}\left|\sum_{j=1}^{\infty}f\left(\frac{a_{2j-1}}{\tau}\right)\left(\ell_{2j-1}(\tau x))+\ell_{2j}(\tau x)\right)\right|\leq C\sup_j\left|f\left(\frac{a_{2j-1}}{\tau}\right)\right|.$$

Here C is independent of f,τ.

(b) For $p=2$, the exact same proof as of Theorem 5 gives

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty}(\mathbb{L}_\tau^*[f;x]-f(x))^2\,dx=0.$$

If $p>2$, we can use the boundedness of the operators, to obtain

$$\begin{aligned}
&\limsup_{\tau\to\infty}\int_{-\infty}^{\infty}|\mathbb{L}_\tau^*[f;x]-f(x)|^p\,dx\\
&\quad\leq\limsup_{\tau\to\infty}\left(\sup_{x\in\mathbb{R}}|\mathbb{L}_\tau^*[f;x]-f(x)|\right)^{p-2}\int_{-\infty}^{\infty}|\mathbb{L}_\tau^*[f;x]-f(x)|^2\,dx\\
&\quad\leq C\|f\|_{L_\infty(\mathbb{R})}^{p-2}\limsup_{\tau\to\infty}\int_{-\infty}^{\infty}|\mathbb{L}_\tau^*[f;x]-f(x)|^2\,dx=0.
\end{aligned}$$

Now let $\frac{4}{5}<p<2$. Let $s>0$. Hölder's inequality gives

$$\limsup_{\tau\to\infty}\int_{-s}^{s}\left|\mathbb{L}_{\tau}^{*}\left[f;x\right]-f\left(x\right)\right|^{p}dx$$
$$\leq \limsup_{\tau\to\infty}\left(\int_{-s}^{s}\left|\mathbb{L}_{\tau}^{*}\left[f;x\right]-f\left(x\right)\right|^{2}dx\right)^{\frac{p}{2}}(2s)^{1-\frac{p}{2}}=0, \quad (41)$$

by the case $p=2$. Next, our bound (2) on f and Lemma 3 show that for all x,

$$\left|\mathbb{L}_{\tau}^{*}\left[f;x\right]-f\left(x\right)\right|\leq C\left(1+|x|\right)^{-\beta}.$$

Then

$$\int_{|x|\geq s}\left|\mathbb{L}_{\tau}^{*}\left[f;x\right]-f\left(x\right)\right|^{p}dx$$
$$\leq C\int_{s}^{\infty}\left(1+|x|\right)^{-p\beta}dx\leq Cs^{1-p\beta}\to 0 \text{ as } s\to\infty,$$

as $\beta>\frac{1}{p}$. □

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