Mutually regular measures have similar universality limits

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Abstract. We use a localization technique to compare universality limits for two different measures. Assume that μ and ν are mutually regular measures, and are mutually absolutely continuous in some closed neighborhood J of a given point x_0 in their support (whether in the bulk or the edge). Assume that at x_0, the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is positive and continuous. Then under further assumptions on one of the measures, the two measures share a similar universality law at x_0.

§1. Results

Let μ be a finite positive Borel measure with compact support E on the real line. Then we may define orthonormal polynomials

$$p_n(x) = p_n^{\mu}(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int_E p_n p_m \, d\mu = \delta_{mn}.$$

These orthonormal polynomials satisfy a recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} > 0 \text{ and } b_n \in \mathbb{R}, \quad n \geq 0,$$

and we use the convention $p_{-1} = 0$. Throughout $w = \frac{d\mu}{dx}$ denotes the absolutely continuous part of μ with respect to Lebesgue measure. The measure μ is said to be *regular* in the sense of Stahl and Totik [11], if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap(E)},$$

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where $cap\,(E)$ denotes the logarithmic capacity of E. In particular, if $E=[-1,1]$, this requires that

$$\lim_{n\to\infty}\gamma_n^{1/n}=2.$$

One of the key limits in random matrix theory, the so-called universality limit [1], involves the reproducing kernel

$$K_n^{\mu}(x,y)=\sum_{k=0}^{n-1}p_k(x)\,p_k(y)$$

and its normalized cousin

$$\widetilde{K}_n^{\mu}(x,y)=w(x)^{1/2}\,w(y)^{1/2}\,K_n^{\mu}(x,y)\,.$$

In [6], we presented a new approach to this universality limit, proving:

Theorem 1. *Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Let $\mathcal{K}$ be a compact subset of $(-1,1)$ such that μ is absolutely continuous in an open interval containing I. Assume that w is positive and continuous at each point of $\mathcal{K}$. Then*

$$\lim_{n\to\infty}\frac{\widetilde{K}_n^{\mu}\left(x+\frac{a}{\widetilde{K}_n^{\mu}(x,x)},x+\frac{b}{\widetilde{K}_n^{\mu}(x,x)}\right)}{\widetilde{K}_n^{\mu}(x,x)}=\frac{\sin\pi\,(a-b)}{\pi\,(a-b)},$$

uniformly for $x\in I$ and a,b in compact subsets of the real line.

We also established L_p analogues assuming less on w. Subsequently, Vili Totik [13] established a far reaching extension, replacing $[-1,1]$ by general compact sets, but also allowing Lebesgue points instead of points of continuity.

In [7], we showed how localization and smoothing can be applied at the edge 1 of the spectrum. For $\alpha>-1$, let

$$\mathbb{J}_{\alpha}(u,v)=\frac{J_{\alpha}(\sqrt{u})\,\sqrt{v}J'_{\alpha}(\sqrt{v})-J_{\alpha}(\sqrt{v})\,\sqrt{u}J'_{\alpha}(\sqrt{u})}{2\,(u-v)}$$

be the Bessel kernel of order α, where J_{α} is the usual Bessel function of the first kind and order α. Our result for the edge was:

Theorem 2. *Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Assume that for some $\rho>0$, μ is absolutely continuous in $J=[1-\rho,1]$, and in J, its absolutely continuous component has the form $w(x)=h(x)\,(1-x)^{\alpha}\,(1+x)^{\beta}$, where $\alpha,\beta>-1$. Assume that $h(1)>0$*

and h is continuous at 1. Then uniformly for a, b in compact subsets of $(0, \infty)$, we have

$$\lim_{n\to\infty} \frac{1}{2n^2} \widetilde{K}_n^{\mu} \left(1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_{\alpha}(a, b). \tag{1}$$

If $\alpha \geq 0$, we may allow compact subsets of $[0, \infty)$.

The proof of Theorem 1 involved reducing the measure μ to a Legendre weight near $\mathcal{K}$, while the proof of Theorem 2 reduced μ to a Jacobi weight near 1.

In this paper, we show how the same localization principle offers a unified framework for universality limits in the bulk, or at the edge, of the spectrum. We need:

Definition 1. Let μ, ν be measures with compact support. We say they are *mutually regular*, if as $n \to \infty$,

$$\sup_{\deg(P)\leq n} \left(\frac{\int P^2 d\mu}{\int P^2 d\nu}\right)^{1/n} \to 1,$$

and

$$\sup_{\deg(P)\leq n} \left(\frac{\int P^2 d\nu}{\int P^2 d\mu}\right)^{1/n} \to 1.$$

Note that if μ is regular in the sense of Stahl and Totik, they show that it is mutually regular with the Legendre weight $\nu' = 1$ having the same support as μ. Indeed, this is a key tool in the proofs in [6] and [7].

Recall that the nth Christoffel function for μ is

$$\lambda_n^{\mu}(x) = 1/K_n^{\mu}(x, x) = \min_{\deg(P)\leq n-1} \left(\int P^2 d\mu\right) / P^2(x).$$

When dealing with a positive measure ν, we shall denote its reproducing kernel by K_n^{ν} and its normalized reproducing kernel by $\tilde{K}_n^{\nu}$. We shall also use the superscript ν to indicate other quantities associated with the measure μ. The result of this paper is:

Theorem 3. *Let μ and ν be measures with compact support that are mutually regular. Let J be a compact subset of the support* $\operatorname{supp}[\mu]$ *of μ. Assume that I is an open set containing J, such that in $I \cap \operatorname{supp}[\mu]$, μ and ν are mutually absolutely continuous. Assume moreover, that at each point of J, the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is positive and continuous. Let $g : J \to (0, \infty)$ be a function defined on J. Assume that for some positive numbers d and c,*

$$\lim_{n\to\infty} n^d \lambda_n^{\nu}\left(x + an^{-c}\right) = g(x), \tag{2}$$

uniformly for $x \in J$ and a in compact subsets of the real line. Then uniformly for a, b in compact subsets of the real line, and $x \in J$, with $x + an^{-c}, x + bn^{-c}$ restricted to supp$[\mu]$,

$$\lim_{n\to\infty} \frac{\left| \frac{d\mu}{d\nu}(x) K_n^{\mu}(x + an^{-c}, x + bn^{-c}) - K_n^{\nu}(x + an^{-c}, x + bn^{-c}) \right|}{K_n^{\mu}(x, x)} = 0. \tag{3}$$

Thus μ and ν share similar universality limits on J. Of course, J could consist of a single point at the edge of the spectrum, namely at points where the support of the measures meets its complement. An example would be the endpoint 1 of the interval $[-1, 1]$, as in Theorem 2, where ν can be a Jacobi weight and $c = 2$. In this case, d depends on the particular Jacobi weight. Or, J could also be a single point in the interior of the support, such as a point in $(-1, 1)$, the situation in Theorem 1, where ν can be taken as the Legendre weight and $c = 1$. We emphasize that our hypothesis on continuity of $\frac{d\mu}{d\nu}$ in J, involves approach to J from all points of the support of μ.

We may replace the sequence $\{n^{-c}\}$ by a more general sequence $\{\varepsilon_n\}$. Moreover, we may replace the hypothesis (2) by a more general one. In its formulation, we need more notation. For $x \in \mathbb{R}$ and $\delta > 0$, we set

$$I(x, \delta) = [x - \delta, x + \delta].$$

The distance from a point x to a set J is denoted $dist(x, J)$. For such a set J, we set

$$I(J, \delta) = \{x : dist(x, J) \le \delta\}.$$

$[x]$ denotes the greatest integer $\le x$.

Theorem 4. *Let μ and ν be measures with compact support that are mutually regular. Let J be a compact subset of the support* supp$[\mu]$ *of μ. Assume that I is an open set containing J, such that in $I \cap$* supp$[\mu]$, *μ and ν are mutually absolutely continuous. Assume moreover, that at each point of J, the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is positive and continuous. Assume that $\{\varepsilon_n\}$ is a sequence of positive numbers with limit 0, such that*

$$\lim_{n\to\infty} \lambda_n^{\nu}(x + a\varepsilon_n) / \lambda_n^{\nu}(x) = 1, \tag{4}$$

uniformly for $x \in J$ and a in compact subsets of the real line, with $x + a\varepsilon_n$ restricted to supp$[\nu]$. *Assume, moreover, that for each $A > 0$,*

$$\lim_{\eta\to 0+} \left[\limsup_{n\to\infty} \frac{\lambda_{n-[\eta n]}^{\nu}(x + a\varepsilon_n)}{\lambda_n^{\nu}(x + a\varepsilon_n)} \right] = 1, \tag{5}$$

uniformly for $x \in J$, and $|a| \leq A$. Then uniformly for a, b in compact subsets of the real line, and $x \in J$, with $x + a\varepsilon_n$ and $x + b\varepsilon_n$ restricted to supp$[\mu]$,

$$\lim_{n\to\infty} \frac{\left| \frac{d\mu}{d\nu}(x)\, K_n^\mu(x + a\varepsilon_n, x + b\varepsilon_n) - K_n^\nu(x + a\varepsilon_n, x + b\varepsilon_n) \right|}{K_n^\mu(x, x)} = 0. \quad (6)$$

This paper is organised as follows. In the next section, we establish asymptotics for Christoffel functions. In section 3, we prove Theorem 4 and then deduce Theorem 3.

In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, θ. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α.

§2. Christoffel functions

The methods used to prove the following result are well known, coming primarily from a seminal paper of Máté, Nevai and Totik [8].

Theorem 5. *Assume the hypotheses of Theorem 4. Let $A > 0$. Then uniformly for $|a| < A$, and $x \in J$ with $x + a\varepsilon_n \in \operatorname{supp}[\mu]$, we have*

$$\lim_{n\to\infty} \lambda_n^\mu(x + a\varepsilon_n) / \lambda_n^\nu(x) = \frac{d\mu}{d\nu}(x). \quad (7)$$

Proof: We first prove that uniformly for $x \in J$ and $|a| \leq A$, with $x + a\varepsilon_n$ restricted to supp$[\mu]$,

$$\limsup_{n\to\infty} \frac{\lambda_n^\mu(x + a\varepsilon_n)}{\lambda_n^\nu(x)\, \frac{d\mu}{d\nu}(x)} \leq 1. \quad (8)$$

Let $\varepsilon > 0$ and choose $\delta > 0$ such that μ and ν are mutually absolutely continuous in $I(J, \delta) \cap \operatorname{supp}[\mu]$, and such that

$$\begin{aligned}(1 + \varepsilon)^{-1} &\leq \frac{d\mu}{d\nu}(x) / \frac{d\mu}{d\nu}(y) \\ &\leq 1 + \varepsilon, \quad x, y \in I(J, \delta) \cap \operatorname{supp}[\mu] \text{ with } |x - y| \leq \delta. \quad (9)\end{aligned}$$

This is possible because of compactness of J and continuity and positivity of $\frac{d\mu}{d\nu}$ at every point of J. Let us fix $x_0 \in J$ and recall that $I(x_0, \delta) = [x_0 - \delta, x_0 + \delta]$. Define a measure μ^* with

$$\mu^* = \mu \text{ in } \operatorname{supp}[\mu] \backslash I(x_0, \delta)$$

and in $I(x_0,\delta)$, let μ^* be absolutely continuous with respect to ν, with Radon-Nikodym derivative w.r.t. ν satisfying

$$\frac{d\mu^*}{d\nu} = \frac{d\mu}{d\nu}(x_0)(1+\varepsilon) \text{ in } I(x_0,\delta). \tag{10}$$

Because of (9), $\mu \le \mu^*$, so that if $\lambda_n^{\mu^*}$ is the nth Christoffel function for μ^*, we have for all x,

$$\lambda_n^{\mu}(x) \le \lambda_n^{\mu^*}(x). \tag{11}$$

We now find an upper bound for $\lambda_n^{\mu^*}(x)$ for $x \in I(x_0,\delta/2)$. Let d be the diameter of $\mathrm{supp}[\mu] \cup \mathrm{supp}[\nu]$. There exists $r \in (0,1)$ depending only on δ such that

$$0 \le 1 - \left(\frac{t-x}{d}\right)^2 \le r \text{ for } x \in I(x_0,\delta/2) \cap \mathrm{supp}[\mu]$$
$$\text{and } t \in (\mathrm{supp}[\mu] \cup \mathrm{supp}[\nu]) \setminus I(x_0,\delta). \tag{12}$$

Let $\eta \in \left(0,\frac{1}{2}\right)$ and choose $\sigma > 1$ so close to 1 that

$$\sigma^{1-\eta} < r^{-\eta/4}. \tag{13}$$

Let $m = m(n) = n - 2[\eta n/2]$. Fix $x \in I(x_0,\delta/2) \cap \mathrm{supp}[\mu]$ and choose a polynomial P_m of degree $\le m-1$ such that

$$\lambda_m^{\nu}(x) = \int P_m^2 d\nu \text{ and } P_m^2(x) = 1.$$

Thus P_m is the minimizing polynomial in the Christoffel function for the measure ν at x. Let

$$S_n(t) = P_m(t)\left(1 - \left(\frac{t-x}{d}\right)^2\right)^{[\eta n/2]},$$

a polynomial of degree $\le m-1+2[\eta n/2] \le n-1$ with $S_n(x) = 1$. Then using (10) and (12),

$$\begin{aligned}
\lambda_n^{\mu^*}(x) &\le \int S_n^2 d\mu^* \\
&\le \frac{d\mu}{d\nu}(x_0)(1+\varepsilon)\int_{I(x_0,\delta)} P_m^2 d\nu + r^{2[\eta n/2]} \int_{\mathrm{supp}[\mu]\setminus I(x_0,\delta)} P_m^2 d\mu \\
&\le \frac{d\mu}{d\nu}(x_0)(1+\varepsilon)\lambda_m^{\nu}(x) + r^{2[\eta n/2]} \int_{\mathrm{supp}[\mu]\setminus I(x_0,\delta)} P_m^2 d\mu.
\end{aligned}$$

Now we use the key idea of regularity, probably first used in this context by Máté, Nevai and Totik [8, Lemma 9, p. 450]. By the mutual regularity defined in Definition 1, for $m \ge m_0(\sigma)$, we have

$$\int_{\mathrm{supp}[\mu]\setminus I(x_0,\delta)} P_m^2 d\mu \le \sigma^m \int P_m^2 d\nu = \sigma^m \lambda_m^{\nu}(x).$$

Then from (13), uniformly for $x \in I(x_0, \delta/2)$,

$$\begin{aligned}\lambda_n^{\mu^*}(x) &\leq \frac{d\mu}{d\nu}(x_0)(1+\varepsilon)\lambda_m^{\nu}(x)\left\{1 + C\left[\sigma^{1-\eta}r^{\eta}\right]^n\right\} \\ &\leq \frac{d\mu}{d\nu}(x_0)(1+\varepsilon)\lambda_m^{\nu}(x)\{1+o(1)\},\end{aligned}$$

so as $\lambda_n^{\mu} \leq \lambda_n^{\mu^*}$, for all $x \in I(x_0, \delta/2) \cap \operatorname{supp}[\mu]$,

$$\lambda_n^{\mu}(x)/\lambda_n^{\nu}(x) \leq \frac{d\mu}{d\nu}(x_0)(1+\varepsilon)\{1+o(1)\}\lambda_m^{\nu}(x)/\lambda_n^{\nu}(x). \tag{14}$$

The $o(1)$ term is independent of x_0. Using (9) again, we obtain for $n \geq n_0(x_0, \delta)$, and for all $x \in I(x_0, \delta/2) \cap \operatorname{supp}[\mu]$, that

$$\lambda_n^{\mu}(x)/\lambda_n^{\nu}(x) \leq \frac{d\mu}{d\nu}(x)(1+\varepsilon)^2\lambda_m^{\nu}(x)/\lambda_n^{\nu}(x).$$

By covering J with finitely many such intervals $I(x_0, \delta/2)$, we obtain for some maximal threshold n_1, that for $n \geq n_1 = n_1(\varepsilon, \delta, J)$, that this last inequality holds for all $x \in I(J, \delta/2) \cap \operatorname{supp}[\mu]$. Now let $A > 0$ and $|a| \leq A$. There exists $n_2 = n_2(A, J, \delta)$ such that for $n \geq n_2$ and all $|a| \leq A$ and all $x \in J$, we have $x + a\varepsilon_n \in I(J, \delta/2)$. Recall too that $m = n - 2[\eta n/2]$. By our hypothesis (5), we can choose $\eta > 0$ small enough and n_3 such that for $|a| \leq A, x \in J$, and $n \geq n_3$,

$$\lambda_m^{\nu}(x + a\varepsilon_n)/\lambda_n^{\nu}(x + a\varepsilon_n) \leq 1 + \varepsilon.$$

We deduce that

$$\limsup_{n\to\infty}\left(\sup_{\substack{a\in[-A,A],x\in J\\ x+a\varepsilon_n\in\operatorname{supp}[\mu]}} \frac{\lambda_n^{\mu}(x+a\varepsilon_n)}{\lambda_n^{\nu}(x+a\varepsilon_n)\frac{d\mu}{d\nu}(x+a\varepsilon_n)}\right) \leq (1+\varepsilon)^3.$$

As the left-hand side is independent of the parameter ε, and $\frac{d\mu}{d\nu}$ is continuous on J, we deduce that

$$\limsup_{n\to\infty}\left(\sup_{\substack{a\in[-A,A],x\in J\\ x+a\varepsilon_n\in\operatorname{supp}[\mu]}} \frac{\lambda_n^{\mu}(x+a\varepsilon_n)}{\lambda_n^{\nu}(x+a\varepsilon_n)\frac{d\mu}{d\nu}(x)}\right) \leq 1. \tag{15}$$

Finally, our hypothesis (4) gives (8). In a similar way, we can establish the converse bound

$$\limsup_{n\to\infty}\frac{\lambda_n^{\nu}(x)\frac{d\mu}{d\nu}(x)}{\lambda_n^{\mu}(x+a\varepsilon_n)} \leq 1, \tag{16}$$

uniformly for $x \in J$, $|a| \leq A$, and $x + a\varepsilon_n$ restricted to supp$[\mu]$. Indeed with m, x and η as above, let us choose a polynomial P of degree $\leq m-1$ such that

$$\lambda_m^{\mu}(x) = \int P_m^2(t)\, d\mu(t) \text{ and } P_m^2(x) = 1.$$

Then with S_n as above, and proceeding as above,

$$\lambda_n^{\nu}(x) \leq \int S_n^2 d\nu$$

$$\leq \left[\frac{d\mu}{d\nu}(x_0)^{-1}(1+\varepsilon)\right] \int_{I(x_0,\delta)} P_m^2 d\mu + r^{2[\eta n/2]} \int_{\text{supp}[\mu]\setminus I(x_0,\delta)} P_m^2 d\nu$$
$$\leq \left[\frac{d\mu}{d\nu}(x_0)^{-1}(1+\varepsilon)\right] \lambda_m^{\mu}(x) \left\{1 + C\left[\sigma^{1-\eta} r^{\eta}\right]^n\right\},$$

and so as above,

$$\sup_{x \in I(x_0,\delta/2) \cap \text{supp}[\mu]} \lambda_m^{\nu}(x) / \lambda_m^{\mu}(x)$$
$$\leq \left[\frac{d\mu}{d\nu}(x_0)^{-1}(1+\varepsilon)(1+o(1))\right] \sup_{x \in I(x_0,\delta/2)} \lambda_m^{\nu}(x)/\lambda_n^{\nu}(x)$$
$$\leq \left[\frac{d\mu}{d\nu}(x_0)^{-1}(1+\varepsilon)^3\right].$$

As n runs through all the positive integers, so does $m = n - 2[\eta/2]$. (Indeed, the difference between successive such m is at most 1.) Then (16) follows using monotonicity of λ_n in n, much as above. Together (16) and (8) give (7). □

§3. Localization

Theorem 6. *Assume that μ satisfies the hypotheses of Theorem 4. Assume moreover, that μ^* is a measure with compact support that satisfies the same hypotheses in Theorem 4 as does μ. Assume that*

$$\frac{d\mu}{d\mu^*} = 1 \textit{ in } J.$$

Let $A > 0$. Then as $n \to \infty$,

$$\sup_{a,b \in [-A,A], x \in J} \left| \left(K_n^{\mu} - K_n^{\mu^*}\right)(x + a\varepsilon_n, x + b\varepsilon_n) \right| / K_n^{\mu}(x,x) = o(1). \quad (17)$$

Proof: We initially assume that globally

$$\mu \leq \mu^*. \tag{18}$$

Now

$$\begin{aligned}
&\int \left(K_n^{\mu}(x,t) - K_n^{\mu^*}(x,t)\right)^2 d\mu(t) \\
&= \int K_n^{\mu 2}(x,t)\, d\mu(t) - 2\int K_n^{\mu}(x,t)\, K_n^{\mu^*}(x,t)\, d\mu(t) \\
&\quad + \int K_n^{\mu^* 2}(x,t)\, d\mu(t) \\
&= K_n^{\mu}(x,x) - 2K_n^{\mu^*}(x,x) + \int K_n^{\mu^* 2}(x,t)\, d\mu(t),
\end{aligned}$$

by the reproducing kernel property. As $\mu \leq \mu^*$, we also have

$$\int K_n^{\mu^* 2}(x,t)\, d\mu(t) \leq \int K_n^{\mu^* 2}(x,t)\, d\mu^*(t) = K_n^{\mu^*}(x,x).$$

So

$$\int \left(K_n^{\mu}(x,t) - K_n^{\mu^*}(x,t)\right)^2 d\mu(t) \leq K_n^{\mu}(x,x) - K_n^{\mu^*}(x,x). \tag{19}$$

Next for any polynomial P of degree $\leq n-1$, we have the Christoffel function estimate

$$|P(y)| \leq K_n^{\mu}(y,y)^{1/2} \left(\int P^2 d\mu\right)^{1/2}. \tag{20}$$

Applying this to $P(t) = K_n^{\mu}(x,t) - K_n^{\mu^*}(x,t)$ and using (19) gives, for all $x, y \in \mathbb{R}$,

$$\left|K_n^{\mu}(x,y) - K_n^{\mu^*}(x,y)\right| \leq K_n^{\mu}(y,y)^{1/2} \left[K_n^{\mu}(x,x) - K_n^{\mu^*}(x,x)\right]^{1/2}$$

so

$$\begin{aligned}
&\left|K_n^{\mu}(x,y) - K_n^{\mu^*}(x,y)\right| / K_n^{\mu}(x,x) \\
&\leq \left(\frac{K_n^{\mu}(y,y)}{K_n^{\mu}(x,x)}\right)^{1/2} \left[1 - \frac{K_n^{\mu^*}(x,x)}{K_n^{\mu}(x,x)}\right]^{1/2}.
\end{aligned} \tag{21}$$

Now we set $x = x_0 + a\varepsilon_n$ and $y = x_0 + b\varepsilon_n$, where $a, b \in [-A, A]$ and $x_0 \in J$. By Theorem 5, uniformly for such x, $\frac{K_n^{\mu^*}(x,x)}{K_n^{\mu}(x,x)} = 1 + o(1)$, for $\frac{d\mu}{d\nu}(x) = \frac{d\mu^*}{d\nu}(x)$. Moreover, Theorem 5 shows that

$$K_n^{\mu}(x_0 + a\varepsilon_n, x_0 + a\varepsilon_n) / K_n^{\mu}(x_0, x_0) = 1 + o(1).$$

So

$$\sup_{a,b\in[-A,A],x_0\in J}\left|\left(K_n^{\mu}-K_n^{\mu^*}\right)(x_0+a\varepsilon_n,x_0+b\varepsilon_n)\right|/K_n^{\mu}(x_0,x_0)=o(1).$$

Now we drop the extra hypothesis (18). Define a measure ν by $\nu=\mu=\mu^*$ in J; and elsewhere, let $\nu=\mu+\mu^*$. Then $d\mu\le d\nu$ and $d\mu^*\le d\nu$, while for any polynomial P, we have

$$\int P^2d\mu\le\int P^2d\nu\le\int P^2d\mu+\int P^2d\mu^*,$$

so the mutual regularity of μ and μ^* imply the mutual regularity of any two of μ, μ^*, ν. The case above shows that the reproducing kernels for μ and μ^* have the same asymptotics as that for ν, in the sense of (17), and hence the same asymptotics as each other. □

§4. Proof of the Theorems

In this section, we approximate μ of Theorem 4 by a scaled copy $\nu^{\#}$ of ν and then prove Theorem 4.

Theorem 7. *Let μ and ν be as in Theorem 4. Let $A>0,\varepsilon\in\left(0,\frac{1}{2}\right)$ and choose $\delta>0$ such that (9) holds. Let $x_0\in J$. Then there exists C and n_0 such that for $n\ge n_0$, $a,b\in[-A,A]$, $x\in I(x_0,\delta/2)\cap J$ with $x+a\varepsilon_n$ restricted to* supp$[\mu]$,

$$\frac{\left|\frac{d\mu}{d\nu}(x)K_n^{\mu}(x+a\varepsilon_n,x+b\varepsilon_n)-K_n^{\nu}(x+a\varepsilon_n,x+b\varepsilon_n)\right|}{K_n^{\nu}(x,x)}\le C\varepsilon^{1/2}, \quad (22)$$

where C is independent of ε, δ, n, x, and x_0.

Proof: Fix $x_0\in J$ and let $\nu^{\#}$ be the scaled Legendre weight

$$\nu^{\#}=\frac{d\mu}{d\nu}(x_0)\,\nu.$$

Note that

$$K_n^{\nu^{\#}}(x,y)=\left(\frac{d\mu}{d\nu}(x_0)\right)^{-1}K_n^{\nu}(x,y). \quad (23)$$

Let

$$\mu^*=\mu \text{ in } I(x_0,\delta)$$

and

$$\mu^*=\frac{d\mu}{d\nu}(x_0)\,\nu \text{ outside } I(x_0,\delta).$$

Observe that μ^* and ν are mutually absolutely continuous, since $\frac{d\mu^*}{d\nu}$ is positive and constant outside $I(x_0,\delta)$, and positive and continuous in the interior of $I(x_0,\delta)$. Because of our localization result Theorem 6, we may replace μ by μ^*, without affecting the asymptotics for $K_n^\mu(x+a\varepsilon_n, x+b\varepsilon_n)$ in the interval $I\left(x_0,\frac{\delta}{2}\right)$. So in the sequel, we assume that $\mu=\frac{d\mu}{d\nu}(x_0)\,\nu=\nu^\#$ outside $I(x_0,\delta)$, while not changing μ in $I(x_0,\delta)$. Observe that (9) implies that

$$(1+\varepsilon)^{-1}\nu^\# \le \mu \le (1+\varepsilon)\,\nu^\#, \text{ everywhere.} \tag{24}$$

Then, much as in the previous section,

$$\begin{aligned}
&\int \left(K_n^\mu(x,t)-K_n^{\nu^\#}(x,t)\right)^2 d\nu^\#(t)\\
&= \int K_n^{\mu 2}(x,t)\,d\nu^\#(t) - 2\int K_n^\mu(x,t)\,K_n^{\nu^\#}(x,t)\,d\nu^\#(t)\\
&\quad + \int K_n^{\nu^\# 2}(x,t)\,d\nu^\#(t)\\
&= \int K_n^{\mu 2}(x,t)\,d\mu(t) + \int_{I(x_0,\delta)} K_n^{\mu 2}(x,t)\,d\left(\nu^\#-\mu\right)(t)\,dt\\
&\quad - 2K_n^\mu(x,x) + K_n^{\nu^\#}(x,x)\\
&= K_n^{\nu^\#}(x,x) - K_n^\mu(x,x) + \int_{I(x_0,\delta)} K_n^2(x,t)\,d\left(\nu^\#-\mu\right)(t)\,dt,
\end{aligned}$$

recall that $\mu=\nu^\#$ outside $I(x_0,\delta)$. By (24),

$$\begin{aligned}
\int_{I(x_0,\delta)} K_n^{\mu 2}(x,t)\,d\left(\nu^\#-\mu\right)(t)\,dt &\le \varepsilon\int_{I(x_0,\delta)} K_n^{\mu 2}(x,t)\,d\mu(t)\\
&\le \varepsilon K_n^\mu(x,x).
\end{aligned}$$

So

$$\int \left(K_n^\mu(x,t)-K_n^{\nu^\#}(x,t)\right)^2 d\nu^\#(t) \le K_n^{\nu^\#}(x,x) - (1-\varepsilon)\,K_n^\mu(x,x). \tag{25}$$

Applying an obvious analogue of (20) to $P(t)=K_n(x,t)-K_n^\#(x,t)$ and using (25) gives for all x, y,

$$\begin{aligned}
&\left|K_n^\mu(x,y)-K_n^{\nu^\#}(x,y)\right|\\
&\le K_n^{\nu^\#}(y,y)^{1/2}\left[K_n^{\nu^\#}(x,x)-(1-\varepsilon)\,K_n^\mu(x,x)\right]^{1/2}
\end{aligned}$$

so

$$\left|K_n^{\mu}(x,y) - K_n^{\nu^{\#}}(x,y)\right| / K_n^{\nu^{\#}}(x,x)$$
$$\leq \left(\frac{K_n^{\nu^{\#}}(y,y)}{K_n^{\nu^{\#}}(x,x)}\right)^{1/2}\left[1-(1-\varepsilon)\frac{K_n^{\mu}(x,x)}{K_n^{\nu^{\#}}(x,x)}\right]^{1/2}.$$

In view of (24), we also have

$$\frac{K_n^{\mu}(x,x)}{K_n^{\nu^{\#}}(x,x)} = \frac{\lambda_n^{\nu^{\#}}(x)}{\lambda_n^{\mu}(x)} \geq \frac{1}{1+\varepsilon},$$

so for all x, y,

$$\left|K_n^{\mu}(x,y) - K_n^{\nu^{\#}}(x,y)\right| / K_n^{\nu^{\#}}(x,x)$$
$$\leq \left(\frac{K_n^{\nu^{\#}}(y,y)}{K_n^{\nu^{\#}}(x,x)}\right)^{1/2}\left[1-\frac{1-\varepsilon}{1+\varepsilon}\right]^{1/2}$$
$$\leq \sqrt{2\varepsilon}\left(\frac{K_n^{\nu^{\#}}(y,y)}{K_n^{\nu^{\#}}(x,x)}\right)^{1/2}$$
$$= \sqrt{2\varepsilon}\left(\frac{K_n^{\nu}(y,y)}{K_n^{\nu}(x,x)}\right)^{1/2} = \sqrt{2\varepsilon}\left(\frac{\lambda_n^{\nu}(x)}{\lambda_n^{\nu}(y)}\right)^{1/2}.$$

Here we have used (23). Now we set $x = x_1 + a\varepsilon_n$ and $y = x_1 + b\varepsilon_n$, where $x_1 \in I\left(x_0, \frac{\delta}{2}\right)$ and $a, b \in [-A, A]$. By our hypothesis (4), uniformly for $a, b \in [-A, A]$, and $x_1 \in J$,

$$\frac{\lambda_n^{\nu}(x)}{\lambda_n^{\nu}(y)} \sim 1,$$

and also the constants implicit in $\sim$ are independent of ε, δ and x_1 (this is crucial!). Thus for some C and n_0 depending only on A and J, we have for $n \geq n_0$, $a, b \in [-A, A]$, and $x_1 \in I\left(x_0, \frac{\delta}{2}\right) \cap J$,

$$\sup_{a,b\in[-A,A],x_1\in I\left(x_0,\frac{\delta}{2}\right)\cap J} \frac{\left|\left(K_n^{\mu} - K_n^{\nu^{\#}}\right)(x_1+a\varepsilon_n, x_1+b\varepsilon_n)\right|}{K_n^{\nu^{\#}}(x_1+a\varepsilon_n, x_1+a\varepsilon_n)} \leq C\sqrt{\varepsilon}.$$

Then also, from (23), for the same range of parameters,

$$\frac{\left|\frac{d\mu}{d\nu}(x_0)\, K_n^{\mu}(x_1+a\varepsilon_n, x_1+b\varepsilon_n) - K_n^{\nu}(x_1+a\varepsilon_n, x_1+b\varepsilon_n)\right|}{K_n^{\nu}(x_1+a\varepsilon_n, x_1+a\varepsilon_n)}$$
$$\leq C\sqrt{\varepsilon}.$$

Because of our hypothesis (4), we may replace $K_n^{\nu}(x_1 + a\varepsilon_n, x_1 + a\varepsilon_n)$ in the last denominator by $K_n^{\nu}(x_1, x_1)$. Moreover, by (9), continuity of $\frac{d\mu}{d\nu}$ in J, and this last relation,

$$\left|\frac{d\mu}{d\nu}(x_1) - \frac{d\mu}{d\nu}(x_0)\right| \left|K_n^{\mu}(x_1 + a\varepsilon_n, x_1 + b\varepsilon_n)\right| / K_n^{\nu}(x_1, x_1) \leq C\varepsilon.$$

Combining the last two inequalities gives the result. □

Proof of Theorem 4: Let $A, \varepsilon_1 > 0$. Choose $\varepsilon > 0$ so small that the right-hand side $C\varepsilon^{1/2}$ of (22) is less than ε_1. Choose $\delta > 0$ such that (9) holds. Now cover J by, say M intervals $I\left(x_j, \frac{\delta}{2}\right)$, $1 \leq j \leq M$, each of length δ. For each j, there exists a threshold $n_0 = n_0(j)$ for which (22) holds for $n \geq n_0(j)$ with $I\left(x_0, \frac{\delta}{2}\right)$ replaced by $I\left(x_j, \frac{\delta}{2}\right)$. Let n_1 denote the largest of these. Then we obtain, for $n \geq n_1$, $a, b \in [-A, A]$, and $x_0 \in J$

$$\frac{\left|\frac{d\mu}{d\nu}(x_1) K_n^{\mu}(x_1 + a\varepsilon_n, x_1 + b\varepsilon_n) - K_n^{\nu}(x_1 + a\varepsilon_n, x_1 + b\varepsilon_n)\right|}{K_n^{\nu}(x_1, x_1)} \leq \varepsilon_1.$$

It follows that uniformly for $a, b \in [-A, A]$ and $x_1 \in J$,

$$\lim_{n\to\infty} \frac{\left|\frac{d\mu}{d\nu}(x_1) K_n^{\mu}(x_1 + a\varepsilon_n, x_1 + b\varepsilon_n) - K_n^{\nu}(x_1 + a\varepsilon_n, x_1 + b\varepsilon_n)\right|}{K_n^{\nu}(x_1, x_1)} = 0. \quad \square \tag{26}$$

Proof of Theorem 3: Note first, that as the uniform limit of continuous functions, the function g is continuous. We choose $\varepsilon_n = n^{-c}$ in Theorem 4. The limit (4) follows from (2) and the continuity of g. The limit (5) follows easily from (2). □

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