UNIVERSALITY LIMITS IN THE BULK FOR ARBITRARY MEASURES ON COMPACT SETS

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ABSTRACT. We present a new method for establishing universality limits in the bulk, based on the theory of entire functions of exponential type. Let μ be a measure on a compact subset of the real line. Assume that μ is absolutely continuous in a neighborhood of some point x in the support, and that μ' is bounded above and below near x, which is assumed to be a Lebesgue point of μ'. Then universality holds at x iff it holds "along the diagonal", that is

$$\lim_{n \to \infty} \frac{K_n\left(x + \frac{a}{n}, x + \frac{a}{n}\right)}{K_n(x, x)} = 1,$$

for all real a. The method does not require regularity of the measure μ as did earlier methods. Moreover, the assumption on the diagonal is certainly satisfied in the case of regular measures, so that we obtain another proof of some recent results of Simon and Totik.

1. INTRODUCTION AND RESULTS[1]

Let μ be a finite positive Borel measure with compact support $\text{supp}[\mu]$ and infinitely many points in the support. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

Throughout we use

$$w = \frac{d\mu}{dx}$$

to denote the Radon-Nikodym derivative of μ. We say that μ is *regular* (in the sense of Stahl and Totik [23]) if

$$\lim_{n \to \infty} \gamma_n^{1/n} = \frac{1}{cap\,(\text{supp}\,[\mu])},$$

where *cap* denotes logarithmic capacity. When the support is $[-1, 1]$, the condition reduces to

$$\lim_{n \to \infty} \gamma_n^{1/n} = 2.$$

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Orthogonal polynomials play an important role in random matrix theory [1], [2], [8], [18]. One of the key limits there involves the reproducing kernel

$$(1.1) \qquad\qquad K_n\left(x,y\right) = \sum_{k=0}^{n-1} p_k\left(x\right) p_k\left(y\right).$$

Because of the Christoffel-Darboux formula, it may also be expressed as

$$(1.2) \qquad K_n\left(x,y\right) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n\left(x\right) p_{n-1}\left(y\right) - p_{n-1}\left(x\right) p_n\left(y\right)}{x-y}.$$

Define the normalized kernel

$$(1.3) \qquad\qquad \widetilde{K}_n\left(x,y\right) = w\left(x\right)^{1/2} w\left(y\right)^{1/2} K_n\left(x,y\right).$$

The simplest case of the universality law is the limit

$$(1.4) \qquad \lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\widetilde{K}_n\left(\xi,\xi\right)} = \frac{\sin\pi\left(a-b\right)}{\pi\left(a-b\right)}.$$

Typically this holds uniformly for ξ in a compact subinterval of $(-1,1)$ and a,b in compact subsets of the real line. Of course, when $a=b$, we interpret $\frac{\sin\pi(a-b)}{\pi(a-b)}$ as 1.

In [15], we presented a new approach, based on localization and smoothing. It was assumed that μ is a finite positive measure with support $[-1,1]$, that is regular (in the sense of Stahl and Totik). Moreover, it was assumed that in a neighborhood of some compact set $J \subset (-1,1)$, μ is absolutely continuous, while $w = \mu'$ is positive and continuous at each point of J. The universality limit (1.4) was established uniformly for $\xi \in J$ and a,b in compact subsets of the real line. If J consists of just a single point x, then the hypothesis is that μ is absolutely continuous in some neighborhood $(x-\varepsilon, x+\varepsilon)$ of x, while $w\left(x\right) > 0$ and w is continuous at x. This alone is sufficient for universality at x.

Subsequently, Totik [26], his student Findley [3], and Simon [22] presented far reaching extensions of this result. For example, Totik showed that the same result holds for regular measures on a general compact subset of the real line, instead of $[-1,1]$, and moreover, we may relax the requirement of continuity of w. We only need $\log w$ to be integrable in a neighborhood of the points where universality is desired, together with a Lebesgue point type condition on a certain local Szegő function. In particular, we obtain universality a.e. in any neighborhood where $\log w$ is integrable. Totik's method was based on that in [15], together with "polynomial pullbacks" and potential theory. Simon [22] used the ideas of [15] together with Jost functions, and other tools, to prove universality at points of continuity for any regular measure. We emphasize that all these results require regularity of the measure μ, and use this property in an essential way.

Traditionally, universality results had been proved by substituting higher order asymptotics into the Christoffel-Darboux formula. Levin [11] observed

that mere first order asymptotics are sufficient. The error can then be controlled by a suitable Markov-Bernstein inequality. Together with the method of [15], Levin's fundamental idea lead to universality results for fixed exponential weights on the real line. This circle of methods has also been exploited for orthogonal polynomials on the unit circle, and for universality at the edge of the spectrum [12], [13], [16], [17].

In this paper, we present a new method, based on the theory of entire functions of exponential type, that works for arbitrary, possibly non-regular, measures with compact support. The hypotheses involve the kernel along the diagonal, or its reciprocal, the Christoffel function

$$(1.5) \qquad \lambda_n(x) = 1/K_n(x,x).$$

Our main result is that universality is equivalent to "universality along the diagonal", or alternatively, ratio asymptotics for Christoffel functions.

Theorem 1.1
Let μ be a finite positive Borel measure on the real line with compact support. Let $J \subset \operatorname{supp}[\mu]$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume that w is positive and continuous at each point of J. The following are equivalent:
(I) Uniformly for $\xi \in J$ and a in compact subsets of the real line,

$$(1.6) \qquad \lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{n}, \xi + \frac{a}{n}\right)}{K_n(\xi,\xi)} = 1.$$

(II) Uniformly for $\xi \in J$ and a, b in compact subsets of the complex plane, we have

$$(1.7) \qquad \lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}.$$

Remarks
(a) If we restrict a, b to lie in compact subsets of the real line, then we may reformulate the limit (1.7) as

$$(1.8) \qquad \lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)},$$

since the quantity on the left is well defined.
(b) Of course the hypothesis (1.6) can be reformulated in terms of Christoffel functions: uniformly for $\xi \in J$ and a in compact subsets of the real line,

$$\lim_{n\to\infty} \lambda_n\left(\xi + \frac{a}{n}\right) / \lambda_n(\xi) = 1.$$

If we assume instead of (1.6), that for some subsequence $\mathcal{S}$ of integers,

$$(1.9) \qquad \lim_{n\to\infty, n\in\mathcal{S}} \lambda_n\left(\xi + \frac{a}{n}\right) / \lambda_n(\xi) = 1,$$

again with the relevant uniformity, then the conclusion (1.7) holds as $n \to \infty$ through $\mathcal{S}$.

(c) We can reformulate the hypotheses so that we avoid assuming uniformity in a. Instead of (1.6), we can assume that for each fixed a, we have uniformly for $\xi \in J$,

$$\lim_{n \to \infty} \frac{K_n \left(\xi + \frac{a}{\widetilde{K}_n(\xi, \xi)}, \xi + \frac{a}{\widetilde{K}_n(\xi, \xi)} \right)}{K_n (\xi, \xi)} = 1.$$

Moreover, instead of assuming this for all a, it suffices that it holds for a denumerable set of a with a finite limit point.

(d) If J consists of a single point ξ, and we make the weaker hypothesis

$$\liminf_{n \to \infty} \lambda_n \left(\xi + \frac{a}{n} \right) / \lambda_n (\xi) \geq 1,$$

then we can still prove (1.7) when $a = 0$ or $b = 0$. However for general a, b we cannot prove this.

(e) If μ is a regular measure with support E, then it is known that for $\xi \in J$,

$$(1.10) \qquad \lim_{n \to \infty} n \lambda_n \left(\xi + \frac{a}{n} \right) = w (\xi) / \nu_E (\xi),$$

where ν_E is the equilibrium measure of E. This remarkable result was proved by Totik [26], [25], extending a great deal of earlier work. Totik actually proved convergence a.e., but as noted in [26], his proof also establishes the limit uniformly in compact sets where there is continuity, allowing in addition the parameter a. Another proof is given in [22]. So for regular measures on general compact sets, we obtain an alternative, and more direct, proof of many of the aforementioned results of Totik [26] and Simon [22].

(f) It is likely that the ratio limit (1.6) can hold even when

$$\lim_{n \to \infty} \frac{1}{n} K_n (x, x)$$

does not exist.

Instead of assuming continuity on J, we can assume a Lebesgue point type condition. It was Vili Totik who first observed that universality can be proved at Lebesgue points, rather than just points of continuity [26].

Theorem 1.2

Let μ be a finite positive Borel measure with compact support. Let $J \subset \mathrm{supp}[\mu]$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume that w is bounded above and below by positive constants in that open set. Assume, moreover, that uniformly for $\xi \in J$, we have

$$(1.11) \qquad \lim_{s \to 0+} \frac{1}{s} \int_{\xi - s}^{\xi + s} |w (t) - w (\xi)| \, dt = 0.$$

Then the equivalence of (I), (II) in Theorem 1.1 remains valid.

Of course, when $J = \{\xi\}$, (1.11) just asserts that ξ is a Lebesgue point of w. When J has non-empty interior, the uniformity of (1.11) forces w to be continuous in the interior of J. As noted above, for regular measures, Totik and Findley assumed less on w. They assumed that $\log w$ is integrable in a neighborhood of ξ, rather than w being bounded above and below by positive constants there. The methods of this paper may also be applied to varying weights [14].

The paper is organised as follows. In Section 2, we present the ideas of proof. In Section 3, we present further notation and background. In section 4, we survey some results that we shall need on entire functions of exponential type. In Section 5, we use the theory of normal families to establish some preliminary results. In Section 6, we prove Theorem 1.2, and deduce Theorem 1.1.

2. The Ideas of Proof

Suppose that μ is a measure with compact support and that $w = \mu'$ is bounded above and below in some open set containing the compact set J. Then it is well known that for some $C_1, C_2 > 0$,

$$(2.1) \qquad C_1 \leq \frac{1}{n} K_n(x, x) \leq C_2,$$

in any proper open subset of that open set, and hence, via Cauchy-Schwarz inequality, that

$$\frac{1}{n} |K_n(\xi, t)| \leq C$$

for ξ, t in the latter open set. We can use this and Bernstein's inequality

$$|P(z)| \leq \left| z + \sqrt{z^2 - 1} \right|^n \|P\|_{L_\infty[-1,1]}, \; z \in \mathbb{C},$$

valid for polynomials P of degree $\leq n$, to show that for fixed $\xi \in J$, and a, b in any given compact subset of the complex plane,

$$\left\{ \frac{1}{n} K_n \left(\xi + \frac{a}{n}, \xi + \frac{b}{n} \right) \right\}_{n=1}^\infty$$

is uniformly bounded. Hence it is a normal family. In view of (2.1), the same is true of $\{f_n(a, b)\}_{n=1}^\infty$, where

$$(2.2) \qquad f_n(a, b) = \frac{K_n \left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)} \right)}{K_n(\xi, \xi)}.$$

In fact, Bernstein's inequality yields a lot more: there exist $C_1, C_2 > 0$ with the following property. Given $A > 0$, we have for $n \geq n_0(A)$ and $|a|, |b| \leq A$, that

$$(2.3) \qquad |f_n(a, b)| \leq C_1 e^{C_2(|\text{Im}\, a| + |\text{Im}\, b|)}.$$

We emphasize that C_1 and C_2 are independent of n, A, a, b.

Let $f(a, b)$ be the limit of some subsequence $\{f_n(\cdot, \cdot)\}_{n \in \mathcal{S}}$ of $\{f_n(\cdot, \cdot)\}_{n=1}^{\infty}$. It is an entire function in a, b, but (2.3) shows even more: namely that for all complex a, b,

$$(2.4) \qquad |f(a, b)| \leq C_1 e^{C_2(|\operatorname{Im} a| + |\operatorname{Im} b|)}.$$

So f is bounded for $a, b \in \mathbb{R}$, and is an entire function of exponential type in each variable. We can then apply the very rich theory of entire functions of exponential type.

Our goal is to show that

$$(2.5) \qquad f(a, b) = \frac{\sin \pi(a - b)}{\pi(a - b)}.$$

So we study the properties of f. Our main tool is to take elementary properties of the reproducing kernel K_n, and then after scaling and taking limits, to deduce that an analogous property is true for f. Let us fix a. Since for each real ξ, $K_n(\xi, t)$ has only real zeros, the same is true of $f(a, \cdot)$. Moreover, $f(a, \cdot)$ has countably many such zeros. Using elementary properties of the reproducing kernel K_n, we can show that for all $a \in \mathbb{C}$,

$$(2.6) \qquad \int_{-\infty}^{\infty} |f(a, s)|^2 \, ds \leq f(a, \bar{a}).$$

If σ is the exponential type of $f(a, \cdot)$, we can show that σ is independent of a, using interlacing properties of zeros of K_n. Using the fact that $\frac{\sin \pi s}{\pi s}$ is a reproducing kernel for the entire functions of exponential type that are also in $L_2(\mathbb{R})$, we can establish the useful inequality

$$0 \leq \int_{\mathbb{R}} \left(\frac{f(a, s)}{f(a, a)} - \frac{\sin \sigma(s - a)}{\sigma(s - a)} \right)^2 ds$$

$$(2.7) \qquad \leq \quad \frac{1}{f(a, a)} - \frac{\pi}{\sigma}.$$

From this we deduce

$$\sigma \geq \pi \sup_{x \in \mathbb{R}} f(x, x) \geq \pi.$$

For the converse inequality, we use Markov-Stieltjes inequalities, and a formula relating exponential type of entire functions and their zero distribution, to obtain

$$\sigma \leq \pi \sup_{x \in \mathbb{R}} f(x, x).$$

Thus,

$$\sigma = \pi \sup_{x \in \mathbb{R}} f(x, x),$$

and (2.7) becomes

$$\int_{-\infty}^{\infty} \left(\frac{f(a, s)}{f(a, a)} - \frac{\sin \sigma(a - s)}{\sigma(a - s)} \right)^2 ds$$

$$(2.8) \qquad \leq \quad \frac{1}{f(a, a)} - \frac{1}{\sup_{x \in \mathbb{R}} f(x, x)}.$$

Assuming (1.6), we immediately obtain $f(x, x) = 1$ for all x, and then $\sigma = \pi$. Substituting this back into (2.8), completes the proof of Theorem 1.2.

3. NOTATION AND BACKGROUND

In this section, we record our notation, though some of it has already been introduced earlier. In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, y, s, t. The same symbol does not necessarily denote the same constant in different occurences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. We use $\sim$ in the following sense: given real sequences $\{c_n\}, \{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 with

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

Throughout, μ denotes a finite positive Borel measure with compact support. Its Radon-Nikodym derivative is $w = \mu'$. The corresponding ortho-normal polynomials are denoted by $\{p_n\}_{n=0}^{\infty}$, so that

$$\int p_n p_m d\mu = \delta_{mn}.$$

We denote the zeros of p_n by

(3.1) $$x_{nn} < x_{n-1,n} < ... < x_{2n} < x_{1n}.$$

The reproducing kernel $K_n(x, t)$, is defined by (1.2), while the normalized reproducing kernel is defined by (1.3). The nth Christoffel function is

(3.2) $$\lambda_n(x) = 1/K_n(x, x) = \inf_{\deg(P) \leq n-1} \frac{\int P^2 d\mu}{P^2(x)}.$$

The Gauss quadrature formula asserts that whenever P is a polynomial of degree $\leq 2n - 1$,

(3.3) $$\sum_{j=1}^{n} \lambda_n(x_{jn}) P(x_{jn}) = \int P \, d\mu.$$

In addition to this, we shall need another Gauss type of quadrature formula [4, p. 19 ff.]. Given a real number ξ, there are n or $n-1$ points $t_{jn} = t_{jn}(\xi)$, one of which is ξ, such that

(3.4) $$\sum_{j} \lambda_n(t_{jn}) P(t_{jn}) = \int P \, d\mu,$$

whenever P is a polynomial of degree $\leq 2n - 2$. The $\{t_{jn}\}$ are zeros of

(3.5) $$\psi_n(\xi, t) = p_n(\xi) p_{n-1}(t) - p_{n-1}(\xi) p_n(t),$$

regarded as a function of t.

In order to prove that universality holds uniformly for ξ in J, we shall consider a sequence $\{\xi_n\}$ of points in J, rather than a fixed ξ. At the nth stage, we shall consider the quadrature that includes ξ_n, so that

$$t_{jn} = t_{jn}(\xi_n) \text{ for all } j.$$

Because we wish to focus on ξ_n, we shall set $t_{0n} = \xi_n$, and order the $\{t_{jn}\}$ around ξ_n, treated as the origin:

(3.6) $$\ldots < t_{-2,n} < t_{-1,n} < t_{0n} = \xi_n < t_{1n} < \ldots .$$

Of course the sequence $\{t_{jn}\}$ consists of either $n-1$ or n points, so terminates, and it is possible that all t_{jn} lie to the left or right of ξ_n. However in the limiting situations we treat, where ξ_n lies in the interior of the support, this will not occur. It is known [4, p. 19] that when $(p_n p_{n-1})(\xi_n) \neq 0$, then one zero of $\psi_n(\xi_n, t)$ lies in $(x_{jn}, x_{j-1,n})$ for each j, and the remaining zero lies outside (x_{nn}, x_{1n}).

Throughout J will be the compact set in Theorem 1.2. We let J_1 and J_2 denote compact sets, each consisting of finitely many closed intervals, such that

$$J_2^o \supset J_1 \text{ and } J_1^o \supset J,$$

while μ is absolutely continuous in J_2, with

(3.7) $$C_1 \leq w \leq C_2 \text{ in } J_2.$$

Recall, we assumed that w is bounded above and below in an open set containing J, so such a J_1 and J_2 exist. Furthermore, since J_1 and J_2 consist of finitely many intervals, we may assume they are each just one interval. For, proving Theorem in 1.2 uniformly for that part of J contained in such intervals, then yields the uniformity in the general case. So in the sequel, we assume J_1 and J_2 are compact intervals.

For the given sequence $\{\xi_n\}$ in J, we shall define for $n \geq 1$,

(3.8) $$f_n(a,b) = \frac{K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)}.$$

The zeros of

$$f_n(0,t) = \frac{K_n\left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)}$$

will be denote by $\{\rho_{jn}\}_{j\neq 0}$. Since $\{t_{jn}\} = \{t_{jn}(\xi_n)\}$ are the zeros of $\psi_n(\xi_n, t) = \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}(\xi_n - t)K_n(\xi_n, t)$, we have

$$\rho_{jn} = \tilde{K}_n(\xi_n, \xi_n)(t_{jn} - \xi_n).$$

We also set,

$$\rho_{0n} = 0,$$

corresponding to $t_{0n} = \xi_n$.

For an appropriate subsequence $\mathcal{S}$ of integers, we shall let

(3.9) $$f(a,b) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a,b).$$

The zeros of $f(0,\cdot)$ will be denoted by $\{\rho_j\}_{j\neq 0}$, and we set $\rho_0 = 0$. Our ordering of zeros is

$$... \leq \rho_{-2} \leq \rho_{-1} < \rho_0 = 0 < \rho_1 \leq \rho_2 \leq ... .$$

We shall denote the (exponential) type of $f(a,\cdot)$ by σ_a. We shall show that σ_a is independent of a, and then just use σ to denote the type. Initially, this type will be associated with the specific subsequence $\mathcal{S}$.

4. Entire functions of Exponential Type

In this section, we review some theory that we shall use about entire functions of exponential type. Most of this can be found in the elegant series of lectures of B. Ja. Levin [10]. Recall that if g is entire of order 1, then its *exponential type* σ is

(4.1) $$\sigma = \limsup_{r\to\infty} \frac{\max_{|z|=r} \log|g(z)|}{r}.$$

We say that an entire function g belongs to the *Cartwright class* and write $g \in \mathcal{C}$ if it is of exponential type and

(4.2) $$\int_{-\infty}^{\infty} \frac{\log^+|g(t)|}{1+t^2} dt < \infty.$$

Here $\log^+ s = \max\{0, \log s\}$.

We let $n(g,r)$ denote the number of zeros of g in the ball center 0, radius r, counting multiplicity. An important result is that for $g \in \mathcal{C}$, that is real valued on the real axis,

(4.3) $$\lim_{r\to\infty} \frac{n(g,r)}{2r} = \frac{\sigma}{\pi}.$$

For this, see [10, Theorem 1, p. 127]. A simpler proof, in the case of g having all real zeros, is given in [7, p. 66].

When f is entire of exponential type σ and bounded along the real axis, we have [10, p. 38, Theorem 3]

(4.4) $$|f(z)| \leq e^{\sigma|\operatorname{Im} z|} \|f\|_{L_\infty(\mathbb{R})}, \ z \in \mathbb{C}.$$

When g is entire of exponential type σ and $g \in L_2(\mathbb{R})$, we write, as did Levin, $g \in L_\sigma^2$. Here, we have instead of the last inequality, [10, p. 149]

(4.5) $$|g(z)| \leq \left(\frac{2}{\pi}\right)^{1/2} e^{\sigma(|\operatorname{Im} z|+1)} \|g\|_{L_2(\mathbb{R})}, \ z \in \mathbb{C}.$$

An important identity is the reproducing kernel identity [24, p. 95], [6, (6.75), p. 58]

$$(4.6) \qquad g(x) = \int_{-\infty}^{\infty} g(t) \frac{\sin \sigma (x - t)}{\pi (x - t)} dt, \ x \in \mathbb{R},$$

when $g \in L_\sigma^2$. One possible proof of this involves the Paley-Wiener Theorem, and the observation that $\frac{\sin \sigma(x-t)}{\pi(x-t)}$ is the Fourier transform of the characteristic function of $[-\sigma, \sigma]$. We shall also use [5, p. 414, no. 3.741.3] (or, choose $g(t) = \frac{\sin \pi(x-t)}{\pi(x-t)}$ in the previous identity)

$$(4.7) \qquad \int_{-\infty}^{\infty} \left(\frac{\sin \pi s}{\pi s} \right)^2 ds = 1.$$

5. Normality

We begin with a straightforward consequence of Bernstein's growth inequality for polynomials in the complex plane. Throughout this section, J, J_1 and J_2 are as in Section 3, so that J_1 and J_2 are closed intervals, while $\{\xi_n\}$ is a sequence in J. We shall assume the hypotheses of Section 3, but shall not assume (1.6).

Lemma 5.1
Let $[c, d]$ be a real interval and $\mathcal{K}$ be a compact subset of (c, d). Let $A, \eta > 0$ and

$$\Gamma = \sup_{x \in \mathcal{K}} \frac{1}{\sqrt{(d - x)(x - c)}}.$$

There exists $n_0 = n_0(A, \mathcal{K}, \eta, \Gamma)$ such that for $n \geq n_0$, polynomials P of degree $\leq n$, $x \in \mathcal{K}$ and $|a| \leq A$,

$$(5.1) \qquad \left| P\left(x + i\frac{a}{n} \right) \right| \leq e^{(1+\eta)\Gamma|a|} \|P\|_{L_\infty[c,d]}.$$

Proof
We prove this for $[c, d] = [-1, 1]$. The general case follows by a linear transformation. Let $x \in \mathcal{K}$ and $z = x + i\frac{a}{n}$. By Bernstein's inequality,

$$(5.2) \qquad |P(z)| \leq \left| z + \sqrt{z^2 - 1} \right|^n \|P\|_{L_\infty[-1,1]}.$$

As $\left| x + \sqrt{x^2 - 1} \right| = 1$, we obtain

$$\log \left| z + \sqrt{z^2 - 1} \right|$$

$$= \operatorname{Re} \left(\int_0^{a/n} \frac{d}{du} \log \left(u + \sqrt{u^2 - 1} \right)_{|u=x+is} i \, ds \right)$$

$$= \operatorname{Re} \left(\int_0^{a/n} \frac{1}{\sqrt{u^2 - 1}}_{|u=x+is} i \, ds \right)$$

$$= \frac{1}{\sqrt{1 - x^2}} \operatorname{Re} \left(\int_0^{a/n} \frac{ds}{\sqrt{1 + \frac{s^2 - 2ixs}{1 - x^2}}} \right)$$

$$= \frac{1}{\sqrt{1 - x^2}} \left(\frac{a}{n} + O \left(\frac{a}{n} \right)^2 / \left(1 - x^2 \right) \right).$$

On substituting this into (5.2), we obtain (5.1) in the special case $[c, d] = [-1, 1]$. ∎

Lemma 5.2
For $n \geq 1$, let

$$(5.3) \qquad f_n (u, v) = \frac{K_n \left(\xi_n + \frac{u}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{v}{\tilde{K}_n(\xi_n, \xi_n)} \right)}{K_n (\xi_n, \xi_n)}.$$

(a) $\{f_n (u, v)\}_{n=1}^{\infty}$ is uniformly bounded for u, v in compact subsets of the plane.
(b) Let $f (u, v)$ denote the locally uniform limit of some subsequence $\{f_n (u, v)\}_{n \in \mathcal{S}}$ of $\{f_n (u, v)\}_{n=1}^{\infty}$. Then for each fixed real number u, $f (u, \cdot)$ is an entire function of exponential type. Moreover, for some C_1 and C_2 independent of u, v, and the subsequence $\mathcal{S}$,

$$(5.4) \qquad |f (u, v)| \leq C_1 e^{C_2 (|\operatorname{Im} u| + |\operatorname{Im} v|)}.$$

(c) For each fixed real number u, $f (u, \cdot)$ has only real zeros.
Proof
(a) We note first that since μ is absolutely continuous in the larger interval J_2, and w is bounded above and below there (recall (3.7)), we have the well known bound [19, Theorem 20, p. 116]

$$(5.5) \qquad K_n (x, x)^{-1} = \lambda_n (x) \sim \frac{1}{n} \text{ in } J_1.$$

Indeed, in one direction, this follows by comparing $\lambda_n(x)$ to the Christoffel functions of weights that are constant in J_2. By Cauchy-Schwarz, we have

$$\frac{1}{n}\left|K_n\left(\xi,t\right)\right|$$

$$\leq \quad \left(\frac{1}{n}K_n\left(\xi,\xi\right)\right)^{1/2}\left(\frac{1}{n}K_n\left(t,t\right)\right)^{1/2} \leq C$$

for $\xi,t \in J_1$. By Lemma 5.1, applied separately in each variable, we then have for $\xi,t \in J_1, |a|,|b| \leq A$ and $n \geq n_0(A)$,

$$(5.6) \qquad \frac{1}{n}\left|K_n\left(\xi+i\frac{a}{n},t+i\frac{b}{n}\right)\right| \leq Ce^{C_2(|a|+|b|)}.$$

(Strictly speaking, we have to take a slightly smaller interval than J_1, but can relabel). C_2 is independent of A,ξ,t,a,b. It depends only on the intervals J_1 and J_2. As C is also independent of A,ξ,a, the stated uniform boundedness follows. Of course if u,v lie in a bounded subset of the plane, and $\xi \in J_1$, then for n large enough, we may write $\xi + \frac{u}{n} = \xi + \frac{\text{Re}(u)}{n} + i\frac{\text{Im}(u)}{n}$, where $\xi + \frac{\text{Re}(u)}{n}$ is contained in a slightly large interval than J_1. By relabelling, we may assume it lies in J_1. Then we may recast (5.6) in the form

$$(5.7) \qquad \frac{1}{n}\left|K_n\left(\xi+\frac{u}{n},\xi+\frac{v}{n}\right)\right| \leq Ce^{C_2(|\text{Im}\,u|+|\text{Im}\,v|)}.$$

Since

$$\tilde{K}_n\left(\xi_n,\xi_n\right) \sim n,$$

we see also that for $|u|,|v| \leq A$ and $n \geq n_0(A)$

$$|f_n(u,v)| \leq Ce^{C_2(|\text{Im}\,u|+|\text{Im}\,v|)},$$

where C_1,C_2 are independent of n,u,v,A.

(b) Now $\{f_n(u,v)\}_{n=1}^{\infty}$ is a normal family of two variables u,v. If $f(u,v)$ is the locally uniform limit through the subsequence $\mathcal{S}$ of integers, we see that $f(u,v)$ is an entire function in u,v satisfying for all complex u,v,

$$(5.8) \qquad |f(u,v)| \leq Ce^{C_2(|\text{Im}\,u|+|\text{Im}\,v|)}.$$

In particular, $f(u,v)$ is bounded for $u,v \in \mathbb{R}$, and is an entire function of exponential type in each variable.

(c) It is shown in [4, p. 18], that for each real ξ, $K_n(\xi,t)$ has only real simple zeros. Hence for real u, $f_n(u,v)$ has only real simple zeros as a function of v. Hurwitz's theorem shows that $f(u,v)$ has only real zeros. ∎

Remark

The only places in this paper where we use the upper bound $w \leq C$ are in deducing (5.5) above, and in Lemma 5.3(a) below. We can instead assume that there exists $C > 0$, with the following property: given $A > 0$, there exists $n_0 = n_0(A)$ such that for $n \geq n_0$, $\xi \in J$, and $|a| \leq A$,

$$K_n\left(\xi+\frac{a}{n},\xi+\frac{a}{n}\right) \geq Cn.$$

Lemma 5.3

(a) Uniformly for $u \in \mathbb{R}$,

(5.9)
$$f(u, u) \sim 1.$$

(b) Assume

(5.10)
$$\lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_{\xi_n - \varepsilon}^{\xi_n + \varepsilon} |w(t) - w(\xi_n)| \, dt = 0.$$

Then for all $u \in \mathbb{C}$,

(5.11)
$$\int_{-\infty}^{\infty} |f(u, s)|^2 \, ds \le f(u, \bar{u}).$$

(c) For each $a \in \mathbb{R}$, $f(a, \cdot)$ has infinitely many real zeros.

Proof

(a) Since for some $\eta > 0$, for $|t| \le \eta$, and for large enough n,

$$C_1 \le \frac{K_n(\xi_n + t, \xi_n + t)}{K_n(\xi_n, \xi_n)} \le C_2,$$

(recall (5.5)), while

$$f(u, u) = \lim_{n \to \infty, n \in \mathcal{S}} \frac{K_n\left(\xi_n + \frac{u}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{u}{\tilde{K}_n(\xi_n, \xi_n)}\right)}{K_n(\xi_n, \xi_n)},$$

we obtain (5.9).

(b) Let

(5.12)
$$s = \xi_n + \frac{u}{\tilde{K}_n(\xi_n, \xi_n)}.$$

We use the identity

$$K_n(s, \bar{s}) = \int |K_n(s, t)|^2 \, d\mu(t),$$

valid for all complex s. Let $r > 0$. We drop most of the integral:

(5.13)

$$
\begin{aligned}
1 &\ge \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} \frac{|K_n(s, t)|^2}{K_n(s, \bar{s})} w(t) \, dt \\
&= w(\xi_n) \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} \frac{|K_n(s, t)|^2}{K_n(s, \bar{s})} \, dt + \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} \frac{|K_n(s, t)|^2}{K_n(s, \bar{s})} (w(t) - w(\xi_n)) \, dt \\
&= : I_1 + I_2.
\end{aligned}
$$

Here by Cauchy-Schwarz and the upper bound (5.6),

$$\frac{|K_n(s, t)|^2}{K_n(s, \bar{s})} \le K_n(t, t) \le Cn.$$

Then by the lower bound implicit in (5.5),

$$
\begin{aligned}
|I_2| &\leq Cn \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}} |w(t) - w(\xi_n)| \, dt \\
&\leq \frac{C\tilde{K}_n(\xi_n,\xi_n)}{r} \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}} |w(t) - w(\xi_n)| \, dt \\
&\rightarrow 0, \ n \rightarrow \infty,
\end{aligned}
$$

by (5.10). Next, the substitution $t = \xi_n + \frac{y}{\tilde{K}_n(\xi_n,\xi_n)}$ and (5.12) give

$$
\begin{aligned}
&I_1 \\
&= \int_{-r}^{r} \left| \frac{K_n\left(\xi_n + \frac{u}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} \right|^2 \frac{K_n(\xi_n,\xi_n)}{K_n(s,\bar{s})} dy.
\end{aligned}
$$

As $n \rightarrow \infty$ through $\mathcal{S}$, the last right-hand side has lim inf at least

$$
\int_{-r}^{r} \frac{|f(u,y)|^2}{f(u,\bar{u})} dy,
$$

Substituting into (5.13) gives

$$
1 \geq \int_{-r}^{r} \frac{|f(u,y)|^2}{f(u,\bar{u})} dy.
$$

Finally, let $r \rightarrow \infty$.

(c) We note first that $f(a,\cdot)$ is non-constant, and not a polynomial. Indeed, we have just shown that it belongs to $L_2(\mathbb{R})$ and satisfies $f(a,a) \neq 0$. It also lies in the Cartwright class and is real on the real axis. We can then write [10, p. 130]

$$
f(a, a+z) = f(a,a) \lim_{R \rightarrow \infty} \prod_{b:|b|<R \text{ and } f(a,a+b)=0} \left(1 - \frac{z}{b}\right).
$$

∎

6. Proof of Theorems 1.1 and 1.2

It follows from Lemma 5.2 that for each real a, $f(a,\cdot)$ is entire of exponential type σ_a, say. We first show that σ_a is independent of a. We note that σ_a *could possibly* depend on $\{\xi_n\}$ and the subsequence $\mathcal{S}$.

Lemma 6.1
For $a \in \mathbb{R}$, let $n(f(a,\cdot),r)$ denote the the number of zeros of $f(a,\cdot)$ in the ball center 0, radius r, counting multiplicity. Then for any real a, we have as $r \rightarrow \infty$,

$$
(6.1) \qquad n(f(a,\cdot),r) - n(f(0,\cdot),r) = O(1).
$$

Consequently,

(6.2) $$\sigma_a = \sigma_0 = \sigma, \ say.$$

Moreover, for all $a \in \mathbb{R}$, $f(a, \cdot) \in L_\sigma^2$.

Proof

We use a basic property of

$$\psi_n(\xi, t) = \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} K_n(\xi, t)(\xi - t) = p_n(t) p_{n-1}(\xi) - p_{n-1}(t) p_n(\xi).$$

For real ξ, with $p_{n-1}(\xi) p_n(\xi) \neq 0$, $\psi_n(\xi, t)$ has, as a function of t, simple zeros in each of the intervals

$$(x_{nn}, x_{n-1,n}), (x_{n-1,n}, x_{n-2,n}), ..., (x_{2n}, x_{1n}).$$

There is a single remaining zero, and this lies outside $[x_{nn}, x_{1n}]$. When $p_{n-1}(\xi) p_n(\xi) = 0$, $\psi_n(\xi, t)$ is a multiple of p_n or p_{n-1}. As the zeros of the latter polynomials interlace, we see that in this case, there is a simple zero in each of the intervals

$$[x_{nn}, x_{n-1,n}), [x_{n-1,n}, x_{n-2,n}), ..., [x_{2n}, x_{1n}).$$

For all this, see [4, proof of Theorem 3.1, p. 19]. It follows that whatever is ξ, the number j of zeros of $K_n(\xi, t)$ in $[x_{mn}, x_{kn}]$ satisfies

$$|j - (m - k)| \leq 1.$$

Consider now

$$f_n(a, t) = K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)}\right) / K_n(\xi_n, \xi_n)$$

and

$$f_n(0, t) = K_n\left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)}\right) / K_n(\xi_n, \xi_n)$$

as functions of t. In any fixed interval $[-r, r]$, it follows that the difference between the number of zeros of these two functions is at most 2. Letting $n \to \infty$ through $\mathcal{S}$, we see that (6.1) holds. Then (6.2) follows from (4.3). Finally, $f \in L_2(\mathbb{R})$, by (5.11), so also $f(a, \cdot) \in L_\sigma^2$. $\blacksquare$

Lemma 6.2

(a) Assume (5.10). Then we have for all $a \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \left(\frac{f(a, s)}{f(a, a)} - \frac{\sin \sigma(a - s)}{\sigma(a - s)}\right)^2 ds$$

(6.3) $$\leq \frac{1}{f(a, a)} - \frac{\pi}{\sigma}.$$

(b)

(6.4) $$\sigma \geq \pi \sup_{a \in \mathbb{R}} f(a, a) \geq \pi.$$

Proof

(a) The left-hand side in (6.3) equals

$$\frac{1}{f(a,a)^2} \int_{-\infty}^{\infty} f(a,s)^2\, ds - \frac{2}{f(a,a)} \int_{-\infty}^{\infty} f(a,s)\, \frac{\sin\sigma(a-s)}{\sigma(a-s)}\, ds + \int_{-\infty}^{\infty} \left(\frac{\sin\sigma(a-s)}{\sigma(a-s)}\right)^2 ds$$

$$\leq \frac{1}{f(a,a)} - 2\frac{\pi}{\sigma} + \frac{\pi}{\sigma},$$

by (5.11) of Lemma 5.3(b), and the identities (4.6) and (4.7). Recall that $f(a,\cdot) \in L_\sigma^2$, so (4.6) is applicable.

(c) Since the left-hand side of (6.3) is nonnegative, we obtain for all real a,

$$\sigma \geq \pi f(a,a).$$

As $f(0,0) = 1$, we then obtain (6.4). ∎

Recall from Section 3, the Gauss type quadrature formula, with nodes $\{t_{jn}\}$ including the point $\xi = \xi_n$:

$$\sum_j \lambda_n(t_{jn}) P(t_{jn}) = \int P(t)\, d\mu(t),$$

for all polynomials P of degree $\leq 2n - 2$. Recall that we order the nodes as

$$\ldots < t_{-2,n} < t_{-1,n} < t_{0,n} = \xi_n < t_{1,n} < t_{2,n} < \ldots < .$$

and write

(6.5)
$$t_{jn} = \xi_n + \frac{\rho_{jn}}{\tilde{K}_n(\xi_n, \xi_n)}.$$

Lemma 6.3

(a) For each fixed j, as $n \to \infty$ through $\mathcal{S}$,

(6.6)
$$\rho_{jn} \to \rho_j,$$

where $\rho_0 = 0$ and

$$\ldots \leq \rho_{-2} \leq \rho_{-1} < 0 < \rho_1 \leq \rho_2 \leq \ldots .$$

(b) There exists C_1 such that for all integers j,

(6.7)
$$\rho_j - \rho_{j-1} \leq C_1.$$

(c) The function $f(0,z)$ has (possibly multiple) zeros at $\rho_j, j \neq 0$, and no other zeros.

Proof

(a), (c) We know that $f_n(0,z) = K_n\left(\xi_n, \xi_n + \frac{z}{\tilde{K}_n(\xi_n,\xi_n)}\right) / K(\xi_n, \xi_n)$ has simple zeros at ρ_{jn}, and no other zeros. Moreover as $n \to \infty$ through our subsequence, this sequence converges to $f(0,z)$, uniformly for z in compact sets, and $f(0,z)$ is not identically 0. The result then follows by Hurwitz'

theorem.

(b) Fix j. We use the Markov-Stieltjes inequalities [4, (5.10), p. 33]:

$$\int_{t_{j-1,n}}^{t_{j,n}} w \leq \int_{t_{j-1,n}}^{t_{jn}} d\mu \leq \lambda_n\left(t_{j-1,n}\right) + \lambda_n\left(t_{jn}\right).$$

Using our upper bounds for the Christoffel function, valid in an open set containing J, and the fact that each point of J is a limit point of zeros [4, Theorem 2.4, p. 67], we see that the last right-hand side is $O\left(\frac{1}{n}\right)$. Moreover, by hypothesis, w is bounded below in an open set containing J. It follows that for $n \geq n_0\left(j\right)$,

$$t_{jn} - t_{j-1,n} \leq \frac{C}{n}.$$

It is crucial here that C does nor depend on j, as it arises from the upper bound for the Christoffel functions and the lower bound for w. Then from (6.5) and (5.5), for $n \geq n_0\left(j\right),$

$$\rho_{j,n} - \rho_{j-1,n} \leq C,$$

where C is independent of j. Letting $n \to \infty$ through $\mathcal{S}$ gives the result. ∎

We note that methods and results of Nevai [19, Theorem 20, p. 164] and Last and Simon [9, Sections 8 and 9] show more, namely that

$$\frac{C_1}{n} \leq t_{jn} - t_{j-1,n} \leq \frac{C_2}{n}$$

and hence

$$C_3 \leq \rho_j - \rho_{j-1} \leq C_4.$$

Next, we can deduce:

Lemma 6.4

Assume (5.10) and let

$$\Lambda = \sup_{x \in \mathbb{R}} f\left(x, x\right).$$

For each real a, $f\left(a, \cdot\right)$ is entire of exponential type $\sigma = \pi\Lambda$.

Proof

In view of Lemma 6.1, it suffices to show that $f\left(0, \cdot\right)$ is entire of exponential type $\sigma = \Lambda\pi$. To do this, we consider the zero distribution of $f\left(0, \cdot\right)$. We again need the Markov-Stieltjes inequalities [4, p. 33] associated with these zeros: for each $1 \leq \ell \leq n$,

$$\sum_{j=1}^{\ell-1} \lambda_n\left(t_{jn}\right) \leq \int_{-\infty}^{t_{\ell n}} d\mu \leq \sum_{j=1}^{\ell} \lambda_n\left(t_{jn}\right).$$

We consider this also for k :

$$\sum_{j=1}^{k-1} \lambda_n\left(t_{jn}\right) \leq \int_{-\infty}^{t_{kn}} d\mu \leq \sum_{j=1}^{k} \lambda_n\left(t_{jn}\right).$$

Assume $\ell < k$. Subtracting the relevant parts of the two-sets of inequalities yields

$$\sum_{j=\ell+1}^{k-1} \lambda_n\left(t_{jn}\right) \leq \int_{t_{\ell n}}^{t_{kn}} d\mu\left(t\right).$$

Now assume that $t_{\ell n}, t_{kn} \in J_2$. Then by absolute continuity of μ in J_2, and the substitution $t = \xi_n + \frac{s}{\tilde{K}_n(\xi_n, \xi_n)}$, we obtain

$$\sum_{j=\ell+1}^{k-1} \frac{K_n\left(\xi_n, \xi_n\right)}{K_n\left(t_{jn}, t_{jn}\right)} \leq \int_{\rho_{\ell n}}^{\rho_{kn}} \frac{w\left(\xi_n + \frac{s}{\tilde{K}_n(\xi_n, \xi_n)}\right)}{w\left(\xi_n\right)} ds.$$

Now let $n \to \infty$ through $\mathcal{S}$. In view of (6.6), and our hypothesis (5.10), the last right-hand side converges to

$$\int_{\rho_\ell}^{\rho_k} ds = \rho_k - \rho_\ell.$$

Next, for each fixed j, as $n \to \infty$ through $\mathcal{S}$,

$$\frac{K_n\left(t_{jn}, t_{jn}\right)}{K_n\left(\xi_n, \xi_n\right)} = \frac{K_n\left(\xi_n + \frac{\rho_{jn}}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{\rho_{jn}}{\tilde{K}_n(\xi_n, \xi_n)}\right)}{K_n\left(\xi_n, \xi_n\right)} = f_n\left(\rho_{jn}, \rho_{jn}\right)$$

$$\to f\left(\rho_j, \rho_j\right).$$

Thus for each fixed k, ℓ,

$$\sum_{j=\ell+1}^{k-1} \frac{1}{f\left(\rho_j, \rho_j\right)} \leq \rho_k - \rho_\ell.$$

As $f\left(\rho_j, \rho_j\right) \leq \Lambda$ for all j, we obtain

$$k - \ell - 1 \leq \Lambda\left(\rho_k - \rho_\ell\right).$$

This also implies $\rho_{\ell+2} - \rho_\ell \geq C > 0$, so $f\left(0, \cdot\right)$ has at most double zeros. Moreover, because $\left\{\rho_{jn}\right\}$ are simple zeros of $f_n\left(0, \cdot\right)$, ρ_k can only be a double zero of $f\left(0, \cdot\right)$ if it is repeated in the sequence $\left\{\rho_j\right\}$. Then, in the interval $[\rho_\ell, \rho_k]$, the total number of zeros of $f\left(0, \cdot\right)$, namely $k - \ell + 1$ or $k - \ell + 2$ or $k - \ell + 3$, if 0 does not belong to $[k, \ell]$, and $k - \ell$ or $k - \ell + 1$ or $k - \ell + 2$ if it does, is at most $\Lambda\left(\rho_\ell - \rho_k\right) + 4$. Let us denote by $n\left(r\right)$ the number of zeros of $f\left(0, \cdot\right)$ in $[-r, r]$ (or equivalently in the ball centre 0, radius r). In view of the fact that $C_1 \leq \rho_{j+2} - \rho_j \leq C_2$ and there are infinitely many $\left\{\rho_j\right\}$, we can choose ρ_k a bounded distance from r, and ρ_ℓ a bounded distance from $-r$. We obtain that $n\left(r\right)$ is at most the number of zeros in $[\rho_\ell, \rho_k]$ plus $O\left(1\right)$, and hence at most $\Lambda\left(\rho_\ell - \rho_k\right) + O\left(1\right)$. So

$$n\left(r\right) \leq 2\Lambda r + O\left(1\right).$$

Then, recalling (4.3),

$$\frac{\sigma}{\pi} = \lim_{r \to \infty} \frac{n\left(r\right)}{2r} \leq \Lambda.$$

Together with our lower bound $\sigma \geq \pi\Lambda$ from Lemma 6.2, we obtain the result. $\blacksquare$

Remark

Together with with Lemma 6.2, this gives

(6.8)
$$\int_{-\infty}^{\infty} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \sigma (a-s)}{\sigma (a-s)} \right)^2 ds$$
$$\leq \frac{1}{f(a,a)} - \frac{1}{\sup_{x\in\mathbb{R}} f(x,x)}.$$

In particular, if a attains the sup, so that $f(a,a) = \sup_{x\in\mathbb{R}} f(x,x)$, then for all $s \in \mathbb{R}$,

(6.9)
$$\frac{f(a,s)}{f(a,a)} = \frac{\sin \sigma (a-s)}{\sigma (a-s)}.$$

If the sup is not attained at any finite point, then instead we obtain a sequence $\{a_k\}$ with $\lim_{k\to\infty} |a_k| = \infty$, and

(6.10)
$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \left(\frac{f(a_k,s)}{f(a_k,a_k)} - \frac{\sin \sigma (a_k-s)}{\sigma (a_k-s)} \right)^2 ds = 0.$$

In the converse direction, we note that

$$\int_{-\infty}^{\infty} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \sigma (a-s)}{\sigma (a-s)} \right)^2 ds$$
$$= \frac{1}{f(a,a)^2} \int_{-\infty}^{\infty} f(a,s)^2 \, ds - \frac{1}{\sup_{x\in\mathbb{R}} f(x,x)}.$$

Then the desired identity (6.9) is true iff

(6.11)
$$\int_{-\infty}^{\infty} f(a,s)^2 \, ds = f(a,a)^2 / \sup_{x\in\mathbb{R}} f(x,x).$$

Note that (1.6) is not assumed for any of this! In fact up till now we have not used the hypothesis (1.6).

Proof of Theorem 1.2

Obviously (II) implies (I), so we prove only (I) implies (II). Since $\tilde{K}_n(\xi_n, \xi_n) \sim n$, our hypothesis (1.6) implies also that

$$\lim_{n\to\infty} f_n(a,a) = \lim_{n\to\infty} \frac{K_n \left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)} \right)}{K_n(\xi_n, \xi_n)} = 1,$$

and hence

$$f(a,a) = 1 \text{ for all real } a.$$

(It is only here that we need the uniformity of (1.6) in ξ and a). So

$$\Lambda = \sup_{x\in\mathbb{R}} f(x,x) = 1.$$

By Lemma 6.4, for each fixed a, $f(a, \cdot)$ is entire of exponential type $\sigma = \pi$. By (6.3), we then obtain, for each real a,

$$\int_{-\infty}^{\infty} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \pi (s-a)}{\pi (s-a)} \right)^2 ds = 0,$$

that is

$$f(a,s) = \frac{\sin \pi (s-a)}{\pi (s-a)}.$$

By convergence continuation,

$$\lim_{n \to \infty, n \in \mathcal{S}} \frac{K_n \left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n, \xi_n)} \right)}{K_n (\xi_n, \xi_n)} = \frac{\sin \pi (a-b)}{\pi (a-b)},$$

uniformly for a, b in compact subsets of the plane. (Recall that the left-hand side is uniformly bounded for such a, b). As the limit function is independent of the subsequence $\mathcal{S}$, we obtain

$$\lim_{n \to \infty} \frac{K_n \left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n, \xi_n)} \right)}{K_n (\xi_n, \xi_n)} = \frac{\sin \pi (a-b)}{\pi (a-b)},$$

again with the appropriate uniformity in a, b. Finally as $\{\xi_n\}$ can be any sequence in J, we obtain the conclusion (1.7) of Theorem 1.2, uniformly for $\xi \in J$. ∎

Proof of Theorem 1.1

This follows immediately from Theorem 1.2. Indeed, the uniformity of (1.11) for $\xi \in J$ follows easily from the assumed continuity of w (regarded as a function defined on all of supp$[\mu]$) at each point of compact J. ∎

Acknowledgement

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