UNIVERSALITY LIMITS FOR EXPONENTIAL WEIGHTS

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Abstract. We establish universality in the bulk for fixed exponential weights on the whole real line. Our methods involve first order asymptotics for orthogonal polynomials and localization techniques. In particular we allow exponential weights such as $|x|^{2\beta} g^2(x) \exp(-2Q(x))$, where $\beta > -1/2$, Q is convex and Q'' satisfies some regularity conditions, while g is positive, and has uniformly continuous and slowly growing or decaying logarithm.

1. Results[1]

Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0, \infty)$ is continuous, and all moments

$$\int_{\mathbb{R}} x^j W^2(x)\, dx,\ j = 0, 1, 2, \ldots ,$$

are finite. Then we may define orthonormal polynomials

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + \ldots, \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int_{\mathbb{R}} p_n p_m W^2 = \delta_{mn}.$$

One of the key limits in random matrix theory, the so-called universality limit [4], involves the reproducing kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x)\, p_k(y)$$

and its normalized cousin

$$\widetilde{K}_n(x, y) = W(x)\, W(y)\, K_n(x, y)\, .$$

For the weight $W(x) = \exp(-|x|^{\alpha})$, where $\alpha > 0$, the limit *in the bulk* takes the form

$$\tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / \tilde{K}_n(x,x) \to \frac{\sin \pi (b-a)}{\pi (b-a)},$$

uniformly for $|x| \leq (1-\varepsilon)\, C_\alpha n^{1/\alpha}$, and a, b in compact subsets of the real line, as $n \to \infty$. Here $\varepsilon \in (0, 1)$ is arbitrary, and C_α is a constant depending only on α. There are results at the "soft" edge of the spectrum, namely in a neighborhood of the point $x = \pm C_\alpha n^{1/\alpha}$, where the sin kernel is replaced by the Airy kernel. Moreover, universality is also often established for varying weights. Most of the existing rigorous results have been established for weights of the form $\exp(-Q)$,

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Date: March 11, 2008.
[1]Research supported by NSF grant DMS0400446 and US-Israel BSF grant 2004353

where Q is analytic or piecewise analytic. Some of the important references are [1], [2], [3], [4], [5], [6], [7], [9], [10], [17], [21].

In this paper, we show that the first order asymptotics for orthogonal polynomials established by the authors in [11] imply universality in the bulk for fixed exponential weights on $\mathbb{R}$. We then use a localization technique, developed in [12], [13] for weights on $[-1,1]$, to extend the range of weights that we can treat. Our class of weights is:

Definition 1.1
Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0,\infty)$ satisfies the following conditions:
(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) Q'' exists and is positive in $\mathbb{R}\setminus\{0\}$.
(c)

$$\lim_{|t|\to\infty} Q(t) = \infty.$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)},\ t \neq 0,$$

is quasi-increasing in $(0,\infty)$, in the sense that for some $C > 0$,

$$0 < x < y \Rightarrow T(x) \leq CT(y).$$

We assume an analogous restriction for $y < x < 0$. In addition, we assume that for some $\Lambda > 1$,

$$T(t) \geq \Lambda \text{ in } \mathbb{R}\setminus\{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{Q'(x)}{Q(x)} \text{ a.e. } x \in \mathbb{R}\setminus\{0\}.$$

Then we write $W \in \mathcal{F}(C^2)$.

This class of weights is a special case of the class of weights considered in [11, p. 7]; there more general intervals than the real line were permitted. Examples of weights in this class are $W = \exp(-Q)$, where

$$Q(x) = \begin{cases} Ax^\alpha, & x \in [0,\infty) \\ B|x|^\beta, & x \in (-\infty,0) \end{cases},$$

where $\alpha,\beta > 1$ and $A, B > 0$. More generally, if $\exp_k = \exp(\exp(\ldots\exp()))$ denotes the kth iterated exponential, we may take

$$Q(x) = \begin{cases} \exp_k(Ax^\alpha) - \exp_k(0), & x \in [0,\infty) \\ \exp_\ell\left(B|x|^\beta\right) - \exp_\ell(0), & x \in (-\infty,0) \end{cases}$$

where $k,\ell \geq 1$, $\alpha,\beta > 1$.

A key descriptive role is played by the Mhaskar-Rakhmanov-Saff numbers

$$a_{-n} < 0 < a_n,$$

defined for $n \geq 1$ by the equations

$$n = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx; \tag{1.1}$$

$$0 = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx. \tag{1.2}$$

In the case where Q is even, $a_{-n} = -a_n$. The existence and uniqueness of these numbers is established in the monographs [11], [15], [19], but goes back to earlier work of Mhaskar, Rakhmanov, and Saff. One illustration of their role is the Mhaskar-Saff identity:

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[a_{-n},a_n]},$$

valid for $n \geq 1$ and all polynomials P of degree $\leq n$.

We also define,

$$\beta_n = \frac{1}{2}(a_n + a_{-n}) \text{ and } \delta_n = \frac{1}{2}(a_n + |a_{-n}|), \tag{1.3}$$

which are respectively the center, and half-length of the Mhaskar-Rakhmanov-Saff interval

$$\Delta_n = [a_{-n}, a_n].$$

The linear transformation

$$L_n(x) = \frac{x-\beta_n}{\delta_n}$$

maps Δ_n onto $[-1,1]$. Its inverse

$$L_n^{[-1]}(u) = \beta_n + u\delta_n$$

maps $[-1,1]$ onto Δ_n. For $0 < \varepsilon < 1$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n]. \tag{1.4}$$

The smallest and largest zeros of $p_n(W^2, x)$ are very close to a_{-n} and a_n. Moreover, $\left\{p_n \circ L_n^{[-1]}\right\}_{n\geq 1}$ behaves much like a sequence of orthonormal polynomials on $[-1,1]$. In particular, staying well inside of the Mhaskar-Rakhmanov-Saff interval $\Delta_n = [a_{-n}, a_n]$ gives us the *bulk of the spectrum*, while $a_{\pm n}$ are the *edges*, in the parlance of universality theory.

Our first result is:

Theorem 1.2
Let $W = \exp(-Q) \in \mathcal{F}(C^2)$. Let $0 < \varepsilon < 1$. Then uniformly for a, b in compact subsets of the real line, and $x \in J_n(\varepsilon)$, we have as $n \to \infty$,

$$\tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / \tilde{K}_n(x,x) = \frac{\sin\pi(b-a)}{\pi(b-a)} + o(1). \tag{1.5}$$

In particular, if W is even, this holds uniformly for $|x| \leq (1-\varepsilon)a_n$.

We note that the proof works without change for a larger class of weights, namely the class $\mathcal{F}\left(lip\frac{1}{2}\right)$ in [11, p. 12]. However, the definition of that class is more implicit, so is omitted.

The proof of Theorem 1.2 involves a careful substitution of the first order asymptotics for p_n, derived in [11], into the Christoffel-Darboux formula, for the case where $a \neq b$ in (1.5). An extra argument is then used to deal with the case where $b - a \to 0$.

Using a localization technique, we shall extend this to other classes of weights. Typically, we shall deal with weights

$$W^h = hW$$

as well as $W^*, W^\#$ (These will be defined later). Their reproducing kernels will be denoted respectively by $K_n^h(x,t)$, $K_n^*(x,t)$ and $K_n^\#(x,t)$, and in normalized form respectively by $\tilde{K}_n^h(x,t)$, $\tilde{K}_n^*(x,t)$ and $\tilde{K}_n^\#(x,t)$. The superscripts $h, *$ and $\#$ will also be used to indicate other quantities associated with these weights.

Recall that a generalized Jacobi weight w has the form

$$w(x) = \prod_{j=1}^{N} |x - \alpha_j|^{\beta_j}, \tag{1.6}$$

where all $\{\alpha_j\}$ are distinct, and all $\beta_j > -1$.

Theorem 1.3
Let $W = \exp(-Q) \in \mathcal{F}(C^2)$. Let $h : \mathbb{R} \to [0,\infty)$ be a function that is square integrable over every finite interval. Assume that there is a generalized Jacobi weight w, a compact interval J, and $C > 0$ such that

$$h^2 \geq Cw \text{ in } J, \tag{1.7}$$

while

$$\lim_{r\to\infty} \frac{\log \|\log h\|_{L_\infty([0,r]\setminus J)}}{\log Q(r)} = 0, \tag{1.8}$$

with an analogous limit as $r \to -\infty$. Assume that $\mathcal{K}$ is a closed subset of $\mathbb{R}$ in which $\log h$ is uniformly continuous. Let $0 < \varepsilon < 1$. Then uniformly for a, b in compact subsets of the real line, and $x \in J_n(\varepsilon) \cap \mathcal{K}$,

$$\tilde{K}_n^h\left(x + \frac{a}{\tilde{K}_n^h(x,x)}, x + \frac{b}{\tilde{K}_n^h(x,x)}\right) / \tilde{K}_n^h(x,x) = \frac{\sin \pi (b-a)}{\pi (b-a)} + o(1). \tag{1.9}$$

The uniform continuity of $\log h$ in $\mathcal{K}$ is assumed in the following "global" sense: given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in \mathcal{K}$, and $|t - x| < \delta$ (with possibly t lying outside $\mathcal{K}$), we have

$$|\log h(t) - \log h(x)| < \varepsilon.$$

Of course, this forces h to be positive in the set $\mathcal{K}$ in which universality is desired.

Note that we can take

$$h = w^{1/2} g,$$

where w is a generalized Jacobi weight, and g is a positive continuous function, with $\log g$ uniformly continuous in the real line, and

$$\lim_{|x|\to\infty} \frac{\log|\log g(x)|}{\log|x|} = 0.$$

Such a choice satisfies (1.8) since $Q(x)$ grows faster than $|x|$ at ∞. This rate of growth/ decay of g is similar to that for entire functions of order 0. In this case, the set $\mathcal{K}$ could be taken as the real line with small intervals removed around the zeros and infinities of w. At the other extreme, our theorem does yield universality at a single point if we assume that $\log h$ is continuous only at a single point.

One may replace the condition (1.7) by a more general but implicit one. We can assume that for $n \geq 1$, and all polynomials P of degree $\leq n$, we have

$$\int_J P^2 \leq N_n \int_J (Ph)^2 ,$$

where for each $\delta > 0$,

$$\log N_n = O\left(n^\delta\right), \ n \to \infty.$$

In addition, one could replace W^h over J by a measure satisfying some similar inequality. One may also weaken the growth restriction (1.8) on h if we assume h is differentiable, and satisfies some other conditions.

The proof of Theorem 1.3 involves reduction to the situation of Theorem 1.2 by a localization technique. When we want universality at a given x_0, we fix $\tau > 0$, and replace W^h outside $[x_0 - \tau, x_0 + \tau]$ by the weight $h(x_0) W(x)$. Subsequently, we use the fact that if τ is small enough, then W^h is almost $h(x_0) W$ inside $[x_0 - \tau, x_0 + \tau]$ because of the continuity of h at x_0. The details are substantially more complicated than in the finite interval case, since we wish to prove universality uniformly for $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$, and $J_n(\varepsilon)$ grows with n. In [12], we could instead just use a compactness argument to prove uniformity.

This paper is organised as follows. In the next section, we present some technical estimates. In Section 3, we prove Theorem 1.2. We recommend that at a first reading, the reader skip Section 2, and focus on Section 3. In section 4, we establish asymptotics of Christoffel functions. In Section 5 we localize, and in section 6, we prove Theorem 1.3. In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. Given sequences $\{c_n\}, \{d_n\}$ of real numbers, we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that for $n \geq 1$,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions. $[x]$ denotes the greatest integer $\leq x$.

2. Technical Estimates

Throughout, we assume $W \in \mathcal{F}\left(C^2\right)$. The class $\mathcal{F}\left(C^2\right)$ is contained in the classes $\mathcal{F}\left(Lip\frac{1}{2}\right), \mathcal{F}\left(lip\frac{1}{2}\right), \mathcal{F}$ in [11], see p. 13 there. So we can apply estimates for all these classes from there. We define for $n \geq 1$ the square root factor

$$\rho_n(x) = \sqrt{(x - a_{-n})(a_n - x)}, \ x \in \Delta_n. \tag{2.1}$$

Our first lemma deals with estimates involving $a_{\pm n}$:

Lemma 2.1
(a) Let $\Lambda > 1$ be as in Definition 1.1. Then

$$\delta_n, \quad |a_{\pm n}| = O\left(n^{1/\Lambda}\right). \tag{2.2}$$

(b) For $\frac{1}{2} \leq \frac{m}{n} \leq 2$,

$$\left|\frac{a_m}{a_n} - 1\right| \sim \frac{1}{T(a_n)}\left|\frac{m}{n} - 1\right|. \tag{2.3}$$

Moreover,

$$\left|\frac{\delta_m}{\delta_n} - 1\right| = O\left(\left|\frac{m}{n} - 1\right|\right). \tag{2.4}$$

In particular, $a_{2n} \sim a_n$ *and* $\delta_{2n} \sim \delta_n$ for $n \geq 1$.
(c) For $n \geq 1$ *and* $x \in \Delta_n$,

$$|Q'(x)| \leq C\frac{n}{\rho_n(x)}. \tag{2.5}$$

(d) For $n \geq 1$ *and* $x \in \Delta_n$,

$$Q(x) \leq Cn. \tag{2.6}$$

(e) Let $\varepsilon \in (0,1)$. *For* $n \geq 1$ *and* $x \in J_n(\varepsilon)$,

$$\rho_n(x) \sim \delta_n. \tag{2.7}$$

(f) Let $\varepsilon \in (0,1)$. *There exists* $s \in (0,1)$ *such that for large enough* n,

$$J_n(\varepsilon) \subset \Delta_{sn}. \tag{2.8}$$

(g) There exists C_0 *such that if* $\eta \in (0, C_0)$, *then for all* $\varepsilon \in (0,1)$ *and* $n \geq 1$,

$$J_n(\varepsilon) \supset J_{n-[\eta n]}(2\varepsilon).$$

(h) For $n \geq 1$ *and polynomials* P *of degree* $\leq n$,

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[a_{-n}, a_n]}. \tag{2.9}$$

Moreover, given $p > 0$ *and* $r > 1$, *there exist* C_1, C_2 *such that for* $n \geq 1$ *and polynomials* P *of degree* $\leq n$,

$$\|PW\|_{L_p(\mathbb{R}\setminus\Delta_{rn})} \leq C_1 \exp\left(-n^{C_2}\right) \|PW\|_{L_p(\mathbb{R})}. \tag{2.10}$$

Proof
(a) See (3.30) in [11, Lemma 3.5, p. 72].
(b) See (3.51) in [11, Lemma 3.11, p. 81] for the first relation (2.3). Straightforward manipulations then yield (2.4).
(c) See [11, Lemma 3.8(a), p. 77].
(d) See (3.18) in [11, Lemma 3.4, p. 69], and also use the fact that $T \geq \Lambda$ there.
(e) This follows as in $J_n(\varepsilon)$,

$$\begin{aligned} 2\delta_n &\geq a_n - x \geq \varepsilon\delta_n; \\ 2\delta_n &\geq x - a_{-n} \geq \varepsilon\delta_n. \end{aligned}$$

(f) The right endpoint of $J_n(\varepsilon)$ is $a_n - \varepsilon\delta_n$ while that of Δ_{sn} is a_{sn}, so we want

$$a_n - \varepsilon\delta_n < a_{sn}$$

$$\iff a_n - a_{sn} < \varepsilon\delta_n.$$

This follows from (b) for some s close enough to 1. The left endpoints can be similarly compared.

(g) Comparing the right endpoints of $J_n(\varepsilon)$ and $J_{n-[\eta n]}(2\varepsilon)$, we see that their difference is

$$\begin{aligned} & (a_n - \varepsilon\delta_n) - \left(a_{n-[\eta n]} - 2\varepsilon\delta_{n-[\eta n]}\right) \\ \geq \; & \varepsilon\delta_{n-[\eta n]}\left(2 - \frac{\delta_n}{\delta_{n-[\eta n]}}\right), \end{aligned}$$

as a_n increases with n. By (b) of this lemma,

$$1 - \frac{\delta_n}{\delta_{n-[\eta n]}} = O(\eta),$$

uniformly for $n \geq 1$, so there exists C_0 such that for $\eta \in (0, C_0)$, and $n \geq 1$,

$$(a_n - \varepsilon\delta_n) - \left(a_{n-[\eta n]} - 2\varepsilon\delta_{n-[\eta n]}\right) \geq 0.$$

Comparison of the left endpoints is similar.
(h) This is classical, see for example [11, (4.7), p. 97]. ∎

Next, we define the equilibrium density

$$\sigma_n(x) = \frac{\rho_n(x)}{\pi^2}\int_{a_{-n}}^{a_n} \frac{Q'(s) - Q'(x)}{s - x}\frac{ds}{\rho_n(s)},\ x \in \Delta_n. \tag{2.11}$$

It satisfies the equation for the equilibrium potential [11, p. 16]:

$$\int_{a_{-n}}^{a_n} \log\frac{1}{|x-s|}\sigma_n(s)\,ds + Q(x) = C,\ x \in \Delta_n;$$

$$\int_{a_{-n}}^{a_n} \sigma_n = n,$$

and admits the alternative representation [11, p. 46]

$$\sigma_n(x) = \frac{1}{\pi}\int_{|b_x|}^{n} \frac{ds}{\rho_s(x)},\ x \in \Delta_n, \tag{2.12}$$

where b is the inverse function of the map $t \to a_t$, $t \in \mathbb{R}$, that is $b(a_t) = t$, $t \in \mathbb{R}$. Sometimes, we also use the density transformed to $[-1, 1]$,

$$\sigma_n^*(x) = \frac{\delta_n}{n}\sigma_n\left(L_n^{[-1]}(x)\right),\ x \in [-1, 1], \tag{2.13}$$

which has total mass 1. Recall that the nth Christoffel function for W^2 is

$$\lambda_n\left(W^2, x\right) = 1/K_n\left(W^2, x, x\right) = \min_{\deg(P)\leq n-1}\left(\int_{\mathbb{R}} P^2W^2\right)/P^2(x).$$

Our next lemma deals with σ_n and λ_n :

Lemma 2.2
Let $0 < \varepsilon, s < 1, A > 0$.
(a) Uniformly for $x \in \Delta_{sn}$,

$$\tilde{K}_n(x, x) = \lambda_n^{-1}\left(W^2, x\right)W^2(x) = \sigma_n(x)(1 + o(1)). \tag{2.14}$$

In particular, this holds uniformly for $x \in J_n(\varepsilon)$.
(b) Uniformly for $x \in J_n(\varepsilon)$,

$$\sigma_n(x) \sim \frac{n}{\delta_n}. \tag{2.15}$$

(c) Uniformly for $|a| \leq A$ and $x \in J_n(\varepsilon)$, we have

$$\left[1 \pm L_n\left(x + \frac{a}{\tilde{K}_n(x,x)}\right)\right] = \left[1 \pm L_n(x)\right](1 + o(1)) \sim 1. \tag{2.16}$$

(d) Uniformly for $|a| \leq A, n \geq 1$, and $x \in J_n(\varepsilon)$, we have

$$W\left(x + \frac{a\delta_n}{n}\right)/W(x) = \exp(O(|a|)) \sim 1. \tag{2.17}$$

(e) Uniformly for $|a| \leq A$, and $x \in J_n(\varepsilon)$, we have

$$\sigma_n\left(x + \frac{a}{\tilde{K}_n(x,x)}\right)/\sigma_n(x) = 1 + o(1). \tag{2.18}$$

A similar statement holds if we replace $\frac{a}{\tilde{K}_n(x,x)}$ by $a\frac{\delta_n}{n}$.
(f) Uniformly for $|a| \leq A$, and $x \in J_n(\varepsilon)$, we have

$$\tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{a}{\tilde{K}_n(x,x)}\right)/\tilde{K}_n(x,x) = 1 + o(1). \tag{2.19}$$

(g) For $\frac{n}{2} \leq m \leq n$ and $x \in \Delta_{sm}$,

$$1 \leq \frac{\sigma_n(x)}{\sigma_m(x)} \leq 1 + C\left(1 - \frac{m}{n}\right). \tag{2.20}$$

Proof
(a) This is Theorem 1.25 in [11, Theorem 1.25, p. 26].
(b) From Theorem 5.2(b) in [11, Theorem 5.2, p. 110], for any fixed $s \in (0,1)$,

$$\sigma_n(x) \sim \frac{n}{\rho_n(x)} \text{ in } \Delta_{sn} \tag{2.21}$$

uniformly in n, x. Then (2.15) follows from Lemma 2.1(e) and (f).
(c) For $x \in J_n(\varepsilon)$,

$$1 \pm L_n(x) \geq \varepsilon,$$

while

$$\begin{aligned} &\left(1 \pm L_n\left(x + \frac{a}{\tilde{K}_n(x,x)}\right)\right) - (1 \pm L_n(x)) \\ = \;& \pm\frac{a}{\delta_n \tilde{K}_n(x,x)} = O\left(\frac{1}{n}\right), \end{aligned}$$

uniformly for $|a| \leq A$ and $x \in J_n(\varepsilon)$. Then (2.16) follows.
(d) For some ξ between x and $x + \frac{a\delta_n}{n}$,

$$\begin{aligned} &\left|Q\left(x + \frac{a\delta_n}{n}\right) - Q(x)\right| \\ = \;& \left|Q'(\xi)\, a\frac{\delta_n}{n}\right| \leq C|a|, \end{aligned}$$

by Lemma 2.1(c) and (e).
(e) To prove (2.18), we use the smoothness estimate for σ_n^* from [11, Theorem

6.3(a), p.148] with $\psi(t)=t^{1/2}$ there: for $n\geq 1$ and $u,v\in(-1,1)$,

$$\begin{aligned}&\left|\sigma_n^*(u)\sqrt{1-u^2}-\sigma_n^*(v)\sqrt{1-v^2}\right|\\ \leq\ & C\left(\frac{|u-v|}{1-u^2}\right)^{1/4}.\end{aligned}$$

Setting $u=L_n(x)$ and $v=L_n\left(x+\frac{a}{\tilde{K}_n(x,x)}\right)=L_n(x)+\frac{a}{\delta_n\tilde{K}_n(x,x)}$, and recalling the definition (2.13) of σ_n^*, and that $\rho_n(x)=\delta_n\sqrt{1-L_n^2(x)}$, we obtain

$$\begin{aligned}&\frac{1}{n}\left|(\sigma_n\rho_n)(x)-(\sigma_n\rho_n)\left(x+\frac{a}{\tilde{K}_n(x,x)}\right)\right|\\ \leq\ & C\left(\frac{|a|}{\delta_n\tilde{K}_n(x,x)}\left(\frac{\delta_n}{\rho_n(x)}\right)^2\right)^{1/4}\leq Cn^{-1/4},\end{aligned}$$

so

$$\left|1-\frac{(\sigma_n\rho_n)\left(x+\frac{a}{\tilde{K}_n(x,x)}\right)}{(\sigma_n\rho_n)(x)}\right|\leq Cn^{-1/4},$$

by (2.21) of this lemma. Finally, as $\rho_n(x)\geq\varepsilon\delta_n$, it is easily seen that

$$\begin{aligned}&\rho_n\left(x+\frac{a}{\tilde{K}_n(x,x)}\right)-\rho_n(x)\\ =\ & O\left(\frac{\delta_n}{\rho_n(x)}\frac{a}{\tilde{K}_n(x,x)}\right)=O\left(\frac{\delta_n}{n}\right)=O\left(\frac{\rho_n(x)}{n}\right),\end{aligned}$$

so

$$\rho_n\left(x+\frac{a}{\tilde{K}_n(x,x)}\right)/\rho_n(x)=1+o(1).$$

(f) This follows from (e) and (a).
(g) From (2.12), for $x\in\Delta_{sm}$,

$$\begin{aligned}0\ &<\ \sigma_n(x)-\sigma_m(x)\\ &=\ \frac{1}{\pi}\int_m^n\frac{ds}{\rho_s(x)}\\ &\leq\ \frac{1}{\pi}\frac{n-m}{\rho_m(x)}\leq C\sigma_m(x)\left(\frac{n}{m}-1\right),\end{aligned}$$

by (2.21). ■

Next, we record some asymptotics for orthonormal polynomials:

Lemma 2.3
(a)

$$\frac{\gamma_{n-1}}{\gamma_n}=\frac{\delta_n}{2}(1+o(1)). \tag{2.22}$$

(b) Let $0<\varepsilon<1$. *Uniformly for* $x\in J_n(\varepsilon)$,

$$\delta_n^{1/2}(p_nW)(x)=\left(1-L_n(x)^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\theta_n(x)+o(1), \tag{2.23}$$

where

$$\theta_n(x) = \frac{1}{2}\arccos L_n(x) + \pi\int_x^{a_n}\sigma_n - \frac{\pi}{4}. \tag{2.24}$$

Moreover,

$$\delta_n^{1/2}(p_{n-1}W)(x) = \left(1 - L_n(x)^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\psi_n(x) + o(1), \tag{2.25}$$

where

$$\psi_n(x) = \theta_n(x) - \arccos L_n(x). \tag{2.26}$$

Proof
(a) This is (1.124) of Theorem 1.23 in [11, p. 26] Note that there $A_n = \frac{\gamma_{n-1}}{\gamma_n}$.
(b) In [11, Theorem 15.3, p. 403], it is shown that there exists $\eta > 0$ such that for a range of m that includes $m = n-1, n$, and uniformly for for $|u| \leq 1 - n^{-\eta}$,

$$\begin{aligned} &\delta_n^{1/2}(p_mW)\left(L_n^{[-1]}(u)\right)\left(1-u^2\right)^{1/4} \\ = &\sqrt{\frac{2}{\pi}}\cos\left(\left(m-n+\frac{1}{2}\right)\arccos u + n\pi\int_u^1\sigma_n^* - \frac{\pi}{4}\right) + O\left(n^{-\eta}\right). \end{aligned}$$

Setting $u = L_n(x)$, and noting the relationship (2.13) between σ_n and σ_n^*, we obtain the result. We also use that for $x \in J_n(\varepsilon) \Leftrightarrow u = L_n^{[-1]}(x) \in [-1+\varepsilon, 1-\varepsilon]$, we have

$$\sqrt{1-u^2} = \sqrt{1 - L_n(x)^2} \geq \sqrt{\varepsilon}.$$

■

Our final lemma concerns derivatives of orthogonal polynomials.

Lemma 2.4
Let $\varepsilon \in \left(0, \frac{1}{3}\right)$. There exists $C > 0$ such that for $n \geq 1$,

$$\|p_n'' W\|_{L_\infty(J_n(\varepsilon))} \leq C\frac{n^2}{\delta_n^{5/2}}. \tag{2.27}$$

Proof
By Theorem 1.17 of [11, p. 22], for each $s \in (0,1)$, there exists $C = C(s)$ such that for $n \geq 1$,

$$\|p_n W\|_{L_\infty(J_n(s))} \leq C\delta_n^{-1/2}. \tag{2.28}$$

Moreover, from Theorem 1.18 there, there exist $C_1, C_2 > 0$ such that for $n \geq 1$,

$$\|p_n W\|_{L_\infty(\mathbb{R})} \leq C_1 n^{C_2}. \tag{2.29}$$

(The factors $T(a_{\pm n})$ there are $o\left(n^2\right)$). We multiply p_n by a fast decreasing polynomial S_m of appropriate degree, and then apply a Markov-Bernstein inequality. More specifically, by Theorem 7.5 in [11, p. 172], given $\xi_{\pm m} \in \left(0, \frac{1}{3}\right)$ with $m^2\xi_{\pm m} \to \infty$ as $m \to \infty$, there exist polynomials S_m such that

$$|S_m(x) - 1| \leq e^{-C_0 m\sqrt{\min\{\xi_{-m}, \xi_m\}}},\ x \in \left[-1 + \frac{3}{2}\xi_{-m}, 1 - \frac{3}{2}\xi_m\right];$$

$$0 < S_m(x) \leq C,\ x \in [-1, 1];$$

$$0 < S_m(x) \leq e^{-C_0 m\sqrt{\xi_m}},\ x \in [1 - \frac{1}{2}\xi_m, 1),$$

with a similar relation in $[-1,-1+\frac{1}{2}\xi_{-m}]$. We choose for some large enough K (chosen so that $C_0K\sqrt{\varepsilon}\gg C_2$, where C_2 is as in (2.29)),

$$m=m\left(n\right)=\left[K\log n\right];$$

and choose $\xi_{\pm m}$ so that

$$\begin{aligned}1-\frac{3}{2}\xi_m &= L_{m+n}\circ L_n^{[-1]}\left(1-\varepsilon\right);\\ -1+\frac{3}{2}\xi_{-m} &= L_{m+n}\circ L_n^{[-1]}\left(-1+\varepsilon\right),\end{aligned}$$

and set

$$R_m\left(x\right)=S_m\left(L_{m+n}\left(x\right)\right).$$

Note that

$$\begin{aligned}L_{m+n}\circ L_n^{[-1]}\left(1-\varepsilon\right) &= L_{m+n}\left(a_n-\varepsilon\delta_n\right)\\ &= 1+\frac{a_n-a_{m+n}-\varepsilon\delta_n}{\delta_{m+n}}\\ &= 1-\varepsilon+O\left(\frac{m}{n}\right)=1-\varepsilon+o\left(1\right),\end{aligned}$$

by Lemma 2.1(b). So $\frac{3}{2}\xi_m=\varepsilon+o\left(1\right)$. Similarly, $\frac{3}{2}\xi_{-m}=\varepsilon+o\left(1\right)$. Then the conditions on $\xi_{\pm m}$ are met, and we have for some fixed $0<\varepsilon'<\varepsilon$,

$$\left|R_m\left(x\right)-1\right|\le n^{-C_0K\sqrt{\varepsilon}/2},\ x\in J_n\left(\varepsilon\right); \tag{2.30}$$

$$0<R_m\left(x\right)\le C,\ x\in\Delta_{m+n}; \tag{2.31}$$

$$R_m\left(x\right)\le Cn^{-C_0K\sqrt{\varepsilon}/2},\ x\in\Delta_{m+n}\backslash J_n\left(\varepsilon'\right). \tag{2.32}$$

From (2.28), (2.29), (2.31) and (2.32), and the Mhaskar-Saff identity, we see that

$$\left\|p_nR_mW\right\|_{L_\infty(\mathbb{R})}=\left\|p_nR_mW\right\|_{L_\infty(\Delta_{m+n})}\le C\delta_n^{-1/2}.$$

Now we apply the Markov-Bernstein inequality in [11, Theorem 1.15, p. 21],

$$\left\|\left(p_nR_mW\right)'\varphi_n\right\|_{L_\infty(\mathbb{R})}\le C\left\|p_nR_mW\right\|_{L_\infty(\Delta_{m+n})}\le C\delta_n^{-1/2},$$

where φ_n is a function defined in [11, (1.92), p. 19]. It is shown in [11, p. 112] that given $s\in(0,1)$, we have for $n\ge1$ and $x\in\Delta_{sn}$,

$$\varphi_n\left(x\right)\sim\sigma_n^{-1}\left(x\right).$$

Then (2.8) and (2.15) imply that for $n\ge1$ and $x\in J_n\left(\varepsilon\right)$, we have

$$\varphi_n\left(x\right)\sim\frac{\delta_n}{n}.$$

Thus for $x\in J_n\left(\varepsilon\right)$,

$$\left|p_n'R_mW\right|\left(x\right)\le\left|p_nR_m'W\right|\left(x\right)+\left|Q'\left(x\right)\right|\left|p_nR_mW\right|\left(x\right)+C\frac{n}{\delta_n^{3/2}}.$$

Here as R_m has degree $O\left(\log n\right)$, and is bounded (uniformly in n) in Δ_{m+n}, Markov's inequality gives

$$\left|R_m'\left(x\right)\right|=O\left(\log n\right)^2,$$

while by Lemma 2.1(c), and (e),

$$\left|Q'\left(x\right)\right|\le C\frac{n}{\delta_n}.$$

Since also $R_m \sim 1$ in $J_n(\varepsilon)$, it follows that (for each fixed $\varepsilon \in (0,1)$)

$$\|p_n' W\|_{L_\infty(J_n(\varepsilon))} \leq C \frac{n}{\delta_n^{3/2}}.$$

Moreover, the global bound (2.29) on $p_n W$, and the Markov inequality in [11, Cor. 1.16, p. 21] gives for some $C_1, C_2 > 0$,

$$\|p_n' W\|_{L_\infty(\mathbb{R})} \leq C_1 n^{C_2}.$$

These last two bounds are analogous to (2.28) and (2.29). Applying the same argument as above once more, then gives (2.27). ■

3. Proof of Theorem 1.2

We shall use the Christoffel-Darboux formula

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(t) - p_{n-1}(x) p_n(t)}{x-t}$$

and its confluent form

$$K_n(x,x) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n'(x) p_{n-1}(x) - p_{n-1}'(x) p_n(x) \right).$$

We shall make the change of variable

$$x \to x + \frac{a}{\tilde{K}_n(x,x)}.$$

This is permissible, in view of Lemma 2.2(f) and the fact that we shall prove uniformity in b. Thus it suffices to establish the limit

$$\lim_{n\to\infty} \tilde{K}_n\left(x + \frac{b}{\tilde{K}_n(x,x)}, x \right) / \tilde{K}_n(x,x) = \frac{\sin \pi b}{\pi b}, \tag{3.1}$$

uniformly for b in compact subsets of the real line and $x \in J_n(\varepsilon)$. Let us set, for a given x,

$$x_{n,b} = x + \frac{b}{\tilde{K}_n(x,x)} = x + O\left(\frac{\delta_n}{n} \right),$$

recall Lemma 2.2(a), (b). From Lemma 2.2(c),

$$1 - L_n^2(x_{n,b}) = \left(1 - L_n^2(x) \right)(1 + o(1)). \tag{3.2}$$

Moreover, uniformly in b and x, Lemma 2.2 (a), (e) give

$$\begin{aligned} \int_x^{x_{n,b}} \sigma_n &= (x_{n,b} - x)\, \sigma_n(x)(1 + o(1)) \\ &= b + o(1), \end{aligned}$$

so recalling the notation (2.24),

$$\begin{aligned} & \theta_n(x) - \theta_n(x_{n,b}) \\ = & \frac{1}{2} \left[\arccos L_n(x) - \arccos L_n(x_{n,b}) \right] + \pi \int_x^{x_{n,b}} \sigma_n \\ = & \pi b + o(1), \end{aligned}$$

by Lemma 2.2(c). Also, by (2.26), we then have

$$\psi_n(x) - \psi_n(x_{n,b}) = \pi b + o(1).$$

From Lemma 2.3, and the above considerations, the asymptotics for p_n and p_{n-1} at $x_{n,b}$ take the form

$$\delta_n^{1/2}\left(p_n W\right)\left(x_{n,b}\right) = \left(1 - L_n\left(x\right)^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\left(\theta_n\left(x\right) - \pi b\right) + o\left(1\right); \tag{3.3}$$

$$\delta_n^{1/2}\left(p_{n-1} W\right)\left(x_{n,b}\right) = \left(1 - L_n\left(x\right)^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\left(\psi_n\left(x\right) - \pi b\right) + o\left(1\right). \tag{3.4}$$

For $b = 0$, the relation (3.1) is immediate, as the right-hand side is 1. Now assume $b \neq 0$. The Christoffel-Darboux formula gives

$$\begin{aligned} & \tilde{K}_n\left(x_{n,b}, x\right) / \tilde{K}_n\left(x, x\right) \\ = \; & \frac{1}{b}\frac{\gamma_{n-1}}{\gamma_n}\left(p_n\left(x_{n,b}\right) p_{n-1}\left(x\right) - p_{n-1}\left(x_{n,b}\right) p_n\left(x\right)\right) W\left(x_{n,b}\right) W\left(x\right). \end{aligned}$$

Inserting here the expressions (3.2), (3.3), (3.4), (2.22), (2.23) and (2.25), we obtain uniformly in $x \in J_n\left(\varepsilon\right)$ and b in a compact subset of $\mathbb{R}\setminus\{0\}$,

$$\begin{aligned} & \tilde{K}_n\left(x_{n,b}, x\right) / \tilde{K}_n\left(x, x\right) \\ = \; & \left(1 + o\left(1\right)\right)\frac{1}{\pi b}\left(1 - L_n^2\left(x\right)\right)^{-1/2} \times \\ & \times\left\{\cos\left(\theta_n\left(x\right) - \pi b\right)\cos\left(\psi_n\left(x\right)\right) - \cos\left(\psi_n\left(x\right) - \pi b\right)\cos\theta_n\left(x\right) + o\left(1\right)\right\}. \end{aligned}$$

After some simple trigonometry and using (2.26), the cosine terms are reduced to

$$\begin{aligned} & \sin\left(\pi b\right)\sin\left(\theta_n\left(x\right) - \psi_n\left(x\right)\right) \\ = \; & \sin\left(\pi b\right)\sin\left(\arccos L_n\left(x\right)\right) \\ = \; & \sin\left(\pi b\right)\sqrt{1 - L_n^2\left(x\right)}, \end{aligned}$$

and we finally obtain

$$\tilde{K}_n\left(x_{n,b}, x\right) / \tilde{K}_n\left(x, x\right) = \left(1 + o\left(1\right)\right)\left(\frac{\sin\pi b + o\left(1\right)}{\pi b}\right).$$

This gives the result, but the uniformity in b follows only for b in compact subsets of $\mathbb{R}\setminus\{0\}$. To complete the proof, it suffices to show that given a sequence $\{b_n\}$ of non-zero numbers with limit 0, and a sequence $\{x_n\}$ with $x_n \in J_n\left(\varepsilon\right)$, we have

$$\lim_{n\to\infty}\tilde{K}_n\left(x_n + \frac{b_n}{\tilde{K}_n\left(x_n\right)}, x_n\right) / \tilde{K}_n\left(x_n\right) = 1, \tag{3.5}$$

where we now use the abbreviation

$$\tilde{K}_n\left(x_n\right) = \tilde{K}_n\left(x_n, x_n\right).$$

Note that $\tilde{K}_n\left(x_n\right) \sim n/\delta_n$. We again use the Christoffel-Darboux formula, and expand $p_n\left(x_n + b/\tilde{K}_n\left(x_n\right)\right)$ and $p_{n-1}\left(x_n + b/\tilde{K}_n\left(x_n\right)\right)$ about x_n to the second order. We also use the identity

$$\tilde{K}_n\left(x_n\right) = \frac{\gamma_{n-1}}{\gamma_n}\left(p_n' p_{n-1} - p_{n-1}' p_n\right)\left(x_n\right) W^2\left(x_n\right),$$

and the following consequence of Lemma 2.2(d):

$$W\left(x_n + \frac{b_n}{\tilde{K}_n\left(x_n\right)}\right) / W\left(x_n\right) = \exp\left(O\left(\left|b_n\right|\right)\right) = 1 + o\left(1\right).$$

We obtain

$$\begin{aligned}
&\tilde{K}_n\left(x_n+\frac{b_n}{\tilde{K}_n(x_n)},x_n\right)/\tilde{K}_n(x_n)\\
&= \frac{1+o(1)}{b_n}\frac{\gamma_{n-1}}{\gamma_n}\left\{\begin{array}{c} 0+\frac{b_n}{\tilde{K}_n(x_n)}\left(p_n'p_{n-1}-p_{n-1}'p_n\right)(x_n)W^2(x_n) \\ +O\left(\left(\frac{b_n}{\tilde{K}_n(x_n)}\right)^2 \max_{j=n-1,n}\|p_jW\|_{L_\infty(J_n(\varepsilon))}\max_{j=n-1,n}\left\|p_j''W\right\|_{L_\infty(J_n(\varepsilon))}\right)\end{array}\right\}\\
&= 1+O\left(b_n\frac{\delta_n^3}{n^2}\max_{j=n-1,n}\|p_jW\|_{L_\infty(J_n(\varepsilon))}\max_{j=n-1,n}\left\|p_j''W\right\|_{L_\infty(J_n(\varepsilon))}\right)+o(1)\\
&= 1+O(b_n)+o(1),
\end{aligned}$$

by Lemma 2.4, and (2.28), completing the proof. ■

4. Christoffel functions

In this section, we show that for a suitable range of x,

$$\lambda_n\left(\left(W^h\right)^2,x\right)/\lambda_n\left(W^2,x\right)=h^2(x)(1+o(1)).$$

In addition, we also need a "localized" form of this result, involving weights that are equal to $W^h=Wh$ in a neighborhood of a given x_0. We shall need some additional notation for this purpose. We choose x_0 and $\tau>0$, and set

$$I(x_0,\tau)=[x_0-\tau,x_0+\tau]. \tag{4.1}$$

We let

$$W^*(x)=W(x)\left\{\begin{array}{ll} h(x), & x\in I(x_0,\tau) \\ h(x_0), & x\in\mathbb{R}\backslash I(x_0,\tau)\end{array}\right. ; \tag{4.2}$$

$$W^{\#}(x)=W(x)\left\{\begin{array}{ll} h(x), & x\in I(x_0,\tau) \\ \max\{h(x),h(x_0)\}, & x\in\mathbb{R}\backslash I(x_0,\tau)\end{array}\right. . \tag{4.3}$$

We shall use the fact that

$$W^h\le W^{\#} \text{ and } W^*\le W^{\#} \text{ in } \mathbb{R}, \tag{4.4}$$

while

$$W^h=W^*=W^{\#} \text{ in } I(x_0,\tau). \tag{4.5}$$

Of course, W^* and $W^{\#}$ depend on x_0, but the estimates and asymptotics will be uniform for a range of x_0. We shall assume throughout that $W\in\mathcal{F}\left(C^2\right)$ and that h satisfies the hypotheses of Theorem 1.3.

Theorem 4.1
Let $0<\varepsilon<1, A>0$. *Then for*

$$W_1=W^h \text{ or } W^* \text{ or } W^{\#},$$

we have

$$\sup_{x_0\in\mathcal{K}\cap J_n(\varepsilon),|a|\le A}\left|\frac{\lambda_n\left(W_1^2,x_0+a\frac{\delta_n}{n}\right)}{\lambda_n\left(W^2,x_0+a\frac{\delta_n}{n}\right)h^2(x_0)}-1\right|=o(1). \tag{4.6}$$

As a first step, we prove the following. We remind the reader that W^* and $W^{\#}$ both depend on τ.

Lemma 4.2
Let $\tau > \delta > 0$, $\eta, \varepsilon \in (0,1)$. There exists n_0 such that for $n \geq n_0$, $x_0 \in \mathcal{K} \cap J_n(\varepsilon)$, $x_1 \in I(x_0, \delta/2)$, we have

$$\lambda_n\left(W_1^2, x_1\right)/\lambda_n\left(W^2, x_1\right) \leq \|h\|^2_{L_\infty(I(x_0,\delta))}(1+\eta) + e^{-n^C} \tag{4.7}$$

and

$$\lambda_n\left(W^2, x_1\right)/\lambda_n\left(W_1^2, x_1\right) \leq \left\|h^{-1}\right\|^2_{L_\infty(I(x_0,\delta))}(1+\eta) + e^{-n^C}. \tag{4.8}$$

The threshhold n_0 is independent of $x_0 \in \mathcal{K} \cap J_n(\varepsilon)$, but depends on $\tau, \delta, \eta, \varepsilon$.
Proof of (4.7)
Let $\eta \in \left(0, \frac{1}{2}\right)$, $n \geq 1$, and $m = n - [\eta n]$. Choose a polynomial R of degree $\leq m-1$ such that

$$\lambda_m\left(W^2, x_1\right) = \int_{\mathbb{R}} (RW)^2 \text{ and } R(x_1) = 1.$$

We shall need the fast decreasing polynomials of Ivanov and Totik [8, p. 2, Theorem 1]. Choosing there

$$\varphi(x) = \min\left\{(n|x|)^2, n|x|\right\},\ x \in [-1,1],$$

there exists $C_1 \geq 1$ and polynomials S_n^* of degree $\leq C_1 n \log n$ such that

$$S_n^*(0) = 1 \text{ and } |S_n^*(t)| \leq e^{-\min\left\{(n|t|)^2, n|t|\right\}},\ t \in [-1,1].$$

In particular $|S_n^*| \leq 1$ in $[-1,1]$ and

$$|S_n^*(t)| \leq e^{-n|t|},\ \frac{1}{n} \leq |t| \leq 1.$$

Let

$$S_n(t) = S^*_{[\eta n/(2C_1 \log n)]}\left(\frac{t - x_1}{2\delta_{2n}}\right),$$

a polynomial of degree $\leq \eta n$, for n exceeding some threshold that depends only on η. Note that for $t \in \Delta_{2n} \backslash I(x_0, \delta)$, we have $|t - x_1| \geq \delta/2$, so

$$|S_n(t)| \leq e^{-C_2 \frac{\eta n}{\log n} \frac{\delta}{2\delta_{2n}}} \leq e^{-n^{C_3}},\ t \in \Delta_{2n} \backslash I(x_0, \delta), \tag{4.9}$$

recall (2.2). Note also that

$$|S_n(t)| \leq 1,\ t \in \Delta_{2n}. \tag{4.10}$$

Let us set

$$P = RS_n,$$

a polynomial of degree $\leq n-1$ with $P(x_1) = 1$. Then

$$\begin{aligned} & \lambda_n\left(W_1^2, x_1\right) \\ \leq\ & \int_{-\infty}^{\infty} (PW_1)^2 \\ =\ & \left[\int_{I(x_0,\delta)} + \int_{J \backslash I(x_0,\delta)} + \int_{\Delta_{2n} \backslash (J \cup I(x_0,\delta))} + \int_{\mathbb{R} \backslash \Delta_{2n}}\right] (PW_1)^2 \\ =\ & : I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.11}$$

Here as $W_1 \leq W \|h\|_{L_\infty(I(x_0,\delta))}$ in $I(x_0,\delta)$, (recall (4.5)), while (4.10) holds, so

$$\begin{aligned} I_1 &\leq \|h\|^2_{L_\infty(I(x_0,\delta))} \int_{I(x_0,\delta)} (RW)^2 \\ &\leq \|h\|^2_{L_\infty(I(x_0,\delta))} \lambda_m\left(W^2, x_1\right). \end{aligned} \tag{4.12}$$

Next, using (4.9), (4.10), and the fact that $W \leq 1$, we see that

$$I_2 \leq e^{-n^{C_3}} \|R\|^2_{L_\infty(J)} \int_J \max\{h, h(x_0)\}^2 .$$

Here using Christoffel function bounds for the Legendre weight [16, p. 106, 108], we see that

$$\begin{aligned} \|R\|^2_{L_\infty(J)} &\leq Cn^C \int_J R^2 \\ &\leq C_1 n^C \int_J R^2 W^2. \end{aligned}$$

Thus

$$I_2 \leq C_3 n^C e^{-n^{C_3}} \lambda_m\left(W^2, x_1\right)\left(1 + \|h\|^2_{L_\infty(I(x_0,\delta))}\right). \tag{4.13}$$

Next, $h^{\pm 1}(x_0)$ are bounded for $x_0 \in \mathcal{K} \cap J_n(\varepsilon) \cap J$, while (2.6) and (1.8) imply that uniformly for $x_0 \in \mathcal{K} \cap J_n(\varepsilon) \setminus J$, $\log\left|\log h^{\pm 1}(x_0)\right| = o(\log n)$. Hence, for all $r > 0$,

$$\log \left\|\max\left\{h^{\pm 1}, h^{\pm 1}(x_0)\right\}\right\|_{L_\infty(\Delta_{2n} \setminus J)} = O(n^r). \tag{4.14}$$

Then by (4.4) and (4.9),

$$\begin{aligned} I_3 &\leq e^{-n^{C_3}} \left(e^{n^{C_3/2}} + \|h\|^2_{L_\infty(I(x_0,\delta))}\right) \int_{\Delta_{2n} \setminus J} (RW)^2 \\ &\leq C_1 e^{-n^{C_3/2}} \left(1 + \|h\|^2_{L_\infty(I(x_0,\delta))}\right) \lambda_m\left(W^2, x_1\right), \end{aligned}$$

for n large enough, with the threshhold on n depending only on h. Finally, we note that given $r > 0$, we have for $n \geq n_0$ and all $j \geq 1$,

$$\|W_1/W\|_{L_\infty\left(\Delta_{2^{j+1}n} \setminus \Delta_{2^j n}\right)} \leq \|\max\{h, h(x_0)\}\|_{L_\infty\left(\Delta_{2^{j+1}n} \setminus \Delta_{2^j n}\right)} \leq \exp\left(\left(2^{j+1}n\right)^r\right),$$

so

$$\begin{aligned} I_4 &= \sum_{j=0}^{\infty} \int_{\Delta_{2^{j+1}n} \setminus \Delta_{2^j n}} (PW_1)^2 \\ &\leq \sum_{j=0}^{\infty} \exp\left(2\left(2^{j+1}n\right)^r\right) \int_{\Delta_{2^{j+1}n} \setminus \Delta_{2^j n}} (PW)^2 \\ &\leq \sum_{j=0}^{\infty} \exp\left(2\left(2^{j+1}n\right)^r - \left(2^j n\right)^{C_2}\right) \int_{\mathbb{R}} (PW)^2, \end{aligned}$$

by (2.10) of Lemma 2.1(h), applied to P, regarded as a polynomial of degree $\leq 2^j n$. As we may assume $r < C_2$, we obtain for $n \geq n_0 \neq n_0(x_0)$,

$$I_4 \leq e^{-n^C} \int_{\Delta_{2n}} (PW)^2 \leq e^{-n^C} \int_{\Delta_{2n}} (RW)^2 \leq e^{-n^C} \lambda_m\left(W^2, x_1\right).$$

Adding the estimates for I_1, I_2, I_3, I_4 gives for $n \geq n_0$,

$$\lambda_n\left(W_1^2, x_1\right)/\lambda_m\left(W^2, x_1\right) \leq \|h\|_{L_\infty(I(x_0,\delta))}^2\left(1+e^{-n^{C_5}}\right)+e^{-n^{C_5}}.$$

Here n_0 is independent of $x_1 \in I\left(x_0, \tau\right), x_0 \in \mathcal{K} \cap J_n(\varepsilon)$. Finally, given $0<s<1$, Lemma 2.2(a), (g) give for $x_1 \in \Delta_{sm}$,

(4.15)

$$\lambda_m\left(W^2, x_1\right)/\lambda_n\left(W^2, x_1\right)=\frac{\sigma_n\left(x_1\right)}{\sigma_m\left(x_1\right)}(1+o(1)) \leq 1+C\left(\frac{n}{m}-1\right) \leq 1+C\eta.$$

Combining this estimate and the previous one, and choosing $\eta>0$ small enough, gives the result for $x_1 \in \Delta_{sm} \cap \mathcal{K} \cap J_n(\varepsilon)$. In view of (2.8), we may choose s so close to 1 that $\mathcal{K} \cap J_n(\varepsilon) \subset \Delta_{sm}$.

Proof of (4.8)

Although this is similar to (4.7), there are some significant differences, so we provide some details. Let $\eta \in\left(0, \frac{1}{2}\right)$, $n \geq 1$, and $m=n-[\eta n]$. Choose a polynomial R of degree $\leq m-1$ such that

$$\lambda_m\left(W_1^2, x_1\right)=\int_{\mathbb{R}}\left(R W_1\right)^2 \text{ and } R\left(x_1\right)=1.$$

Let S_n and $P=R S_n$, as above. Then

$$\begin{aligned}
& \lambda_n\left(W^2, x_1\right) \\
\leq & \int_{-\infty}^{\infty}(P W)^2 \\
= & {\left[\int_{I(x_0,\delta)}+\int_{J \backslash I(x_0,\delta)}+\int_{\Delta_{2n} \backslash(J \cup I(x_0,\delta))}+\int_{\mathbb{R} \backslash \Delta_{2n}}\right](P W)^2 } \\
= & : I_1+I_2+I_3+I_4 .
\end{aligned} \tag{4.16}$$

Here $W \leq W_1\left\|h^{-1}\right\|_{L_\infty(I(x_0,\delta))}$ in $I\left(x_0, \delta\right)$, while (4.10) holds, so

$$I_1 \leq\left\|h^{-1}\right\|_{L_\infty(I(x_0,\delta))}^2 \lambda_m\left(W_1^2, x_1\right). \tag{4.17}$$

Next, using (4.9) and (4.10), we see that

$$I_2 \leq C_1 e^{-n^{C_4}} \int_{J \backslash I(x_0,\delta)}(R W)^2 .$$

If $W=W^*$, we continue this as

$$\begin{aligned}
I_2 & \leq C_1 e^{-n^{C_4}} h\left(x_0\right)^{-2} \int_{J \backslash I(x_0,\delta)}\left(R W_1\right)^2 \\
& \leq C_1 e^{-n^{C_4}} h\left(x_0\right)^{-2} \lambda_m\left(W_1^2, x_1\right) .
\end{aligned}$$

If $W_1=W^h$ or $W^{\#}$, we instead use (1.7), namely that h^2 majorizes a generalized Jacobi weight over J, together with the fact that for some $C>0$,

$$\int_J R^2 \leq n^C \int_J R^2 w,$$

see [16, p. 120]. Since

$$\max \left\{h, h\left(x_0\right)\right\}^2 \geq h^2 \geq C w,$$

we see that

$$I_2 \leq C_1 e^{-n^{C_4}} n^C \int_J\left(R W_1\right)^2 \leq e^{-n^C} \lambda_m\left(W_1^2, x_1\right).$$

Thus in all cases,

$$I_2 \le e^{-n^C}\left(1+\left\|h^{-1}\right\|^2_{L_\infty(I(x_0,\delta))}\right)\lambda_m\left(W_1^2,x_1\right).$$

Next, by (4.9) and (4.14),

$$\begin{aligned} I_3 &\le C_1 e^{-n^{C_4}}\left\|\max\left\{h^{-1},h^{-1}(x_0)\right\}\right\|^2_{L_\infty(\Delta_{2n}\setminus J)}\int_{\Delta_{2n}\setminus J}(RW_1)^2 \\ &\le C_1 e^{-2\delta n^{C_4}+n^{C_4/2}}\lambda_m\left(W_1^2,x_1\right), \end{aligned}$$

for n large enough. Next, by (2.10),

$$I_4 \le e^{-n^C}\int_{\Delta_{2n}}(RW)^2. \tag{4.18}$$

We now proceed to replace W by W_1. Firstly,

$$\begin{aligned} &\int_{\Delta_{2n}\setminus J}(RW)^2 \\ \le\ & \left\|W/W_1\right\|^2_{L_\infty(\Delta_{2n}\setminus J)}\int_{\Delta_{2n}\setminus J}(RW_1)^2 \\ \le\ & e^{O(n^r)}\lambda_m\left(W_1^2,x_1\right), \end{aligned} \tag{4.19}$$

for each $r>0$, by (4.14). Next, as above,

$$\int_J (RW)^2 \le Cn^{C_1}\left(1+\left\|h^{-1}\right\|^2_{L_\infty(I(x_0,\delta))}\right)\int_J (RW_1)^2.$$

Combining this, (4.18), and (4.19), we see that

$$I_4 \le e^{-n^C}\lambda_m\left(W_1^2,x_1\right).$$

Adding the estimates for I_1,I_2,I_3,I_4 gives for $n\ge n_0$,

$$\lambda_n\left(W^2,x_1\right)/\lambda_m\left(W_1^2,x_1\right) \le \left\|h^{-1}\right\|^2_{L_\infty(I(x_0,\delta))}\left(1+e^{-n^C}\right)+e^{-n^{C_5}}. \tag{4.20}$$

Then, using (4.15), and recalling that $m=n-[\eta n]\ge \frac{n}{2}$, we see that

$$\lambda_m\left(W^2,x_0\right)/\lambda_m\left(W_1^2,x_0\right) \le \left\|h^{-1}\right\|^2_{L_\infty(I(x_0,\delta))}(1+C\eta)+e^{-m^{C_5}}.$$

Finally, as n runs through the positive integers, so does $m=m(n)$ (for $m(n+1)-m(n)\le 1$), so choosing $\eta>0$ small enough, we obtain the result. ■

Proof of Theorem 4.1

Let $\eta\in\left(0,\frac{1}{2}\right)$. By uniform continuity of $\log h$ in $\mathcal{K}$, there exists $\delta>0$ such that

$$\left|\log h(s)-\log h(t)\right|\le\eta$$

for $|s-t|\le\delta$ and $\operatorname{dist}(s,\mathcal{K})\le\delta$ and $\operatorname{dist}(t,\mathcal{K})\le\delta$. Then for such s,t,

$$\left|\frac{h(s)}{h(t)}-1\right|\le e^\eta-1\le 2\eta$$

and so for $x_0\in J_n(\varepsilon)\cap\mathcal{K}$,

$$\left\|h^{\pm1}\right\|_{L_\infty(I(x_0,\delta))}/h^{\pm1}(x_0)\le 1+2\eta. \tag{4.21}$$

Moreover, for $n\ge n_0(A)$, we have $x_0+a\frac{\delta_n}{n}\in I(x_0,\delta/2)$, uniformly for $x_0\in\mathcal{K}\cap J_n(\varepsilon)$. Substituting these in Lemma 4.2, we obtain the result. ■

5. Localization

Throughout, we assume the hypotheses of Theorem 1.3, and the definitions (4.2), (4.3) of W^* and $W^{\#}$.

Theorem 5.1
Let $A > 0$. Then as $n \to \infty$,

$$\sup_{a,b\in[-A,A],x_0\in J_n(\varepsilon)\cap\mathcal{K}} \left|\left(\tilde{K}_n^h - \tilde{K}_n^*\right)\left(x_0 + a\frac{\delta_n}{n}, x_0 + b\frac{\delta_n}{n}\right)\right| / \tilde{K}_n^h\left(x_0,x_0\right) \to 0. \tag{5.1}$$

Remark
We emphasize that $\tilde{K}_n^*$ depends on the specific x_0, and τ, although the limit is uniform in $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$ (for a given τ).
Proof
Recall that $W^h = W^* = W^{\#}$ in $I(x_0,\tau)$, and

$$W^*, W^h \le W^{\#} \text{ in } \mathbb{R}. \tag{5.2}$$

The idea is to estimate the L_2 norm of $K_n^{\#} - K_n^h$ over $\mathbb{R}$, and then to use Christoffel function estimates, and to develop an analogous estimate for $K_n^{\#} - K_n^*$. Now

$$\begin{aligned} & \int_{\mathbb{R}} \left(K_n^{\#}(x,t) - K_n^h(x,t)\right)^2 \left(W^h(t)\right)^2 dt \\ = & \int_{\mathbb{R}} \left(K_n^{\#}(x,t)\right)^2 \left(W^h(t)\right)^2 dt - 2\int_{\mathbb{R}} K_n^{\#}(x,t) K_n^h(x,t) \left(W^h(t)\right)^2 dt + \int_{\mathbb{R}} \left(K_n^h(x,t)\right)^2 \left(W^h(t)\right)^2 dt \\ = & \int_{\mathbb{R}} \left(K_n^{\#}(x,t)\right)^2 \left(W^h(t)\right)^2 dt - 2K_n^{\#}(x,x) + K_n^h(x,x), \end{aligned}$$

by the reproducing kernel property. In view of (5.2), we also have

$$\int_{\mathbb{R}} \left(K_n^{\#}(x,t)\right)^2 \left(W^h(t)\right)^2 dt \le \int_{\mathbb{R}} \left(K_n^{\#}(x,t)\right)^2 \left(W^{\#}(t)\right)^2 dt = K_n^{\#}(x,x).$$

So

$$\begin{aligned} & \int_{\mathbb{R}} \left(K_n^{\#}(x,t) - K_n^h(x,t)\right)^2 \left(W^h(t)\right)^2 dt \\ \le \; & K_n^h(x,x) - K_n^{\#}(x,x). \end{aligned} \tag{5.3}$$

Next for any polynomial P of degree $\le n-1$, we have by definition of the Christoffel functions,

$$|P(y)| \le K_n^h(y,y)^{1/2} \left(\int_{\mathbb{R}} \left(PW^h\right)^2\right)^{1/2}. \tag{5.4}$$

Applying this to $P(t) = K_n^{\#}(x,t) - K_n^h(x,t)$ and using (5.3) gives

$$\begin{aligned} & \left|K_n^{\#}(x,y) - K_n^h(x,y)\right| \\ \le \; & K_n^h(y,y)^{1/2} \left[K_n^{\#}(x,x) - K_n^h(x,x)\right]^{1/2} \end{aligned}$$

so for all $x, y \in \mathbb{R}$,

$$\begin{aligned} & \left|K_n^{\#}(x,y) - K_n^h(x,y)\right| / K_n^h(x,x) \\ \le \; & \left(\frac{K_n^h(y,y)}{K_n^h(x,x)}\right)^{1/2} \left[1 - \frac{K_n^h(x,x)}{K_n^{\#}(x,x)}\right]^{1/2}. \end{aligned}$$

Now we set $x = x_0 + a\frac{\delta_n}{n}$ and $y = x_0 + b\frac{\delta_n}{n}$, where $a, b \in [-A, A]$. By Theorem 4.1, uniformly for $x \in J_n(\varepsilon) \cap \mathcal{K}$, and $|a|, |b| \leq A$,

$$\frac{K_n^h(x,x)}{K_n^{\#}(x,x)} = 1 + o(1).$$

Moreover, by Theorem 4.1, Lemma 2.2 (a), (d), (e), and the uniform continuity of $\log h$ (compare (4.21)),

$$\frac{K_n^h(y,y)}{K_n^h(x,x)} \leq C\frac{(hW)^2(x)\,\sigma_n(y)}{(hW)^2(y)\,\sigma_n(x)} \leq C.$$

Similarly,

$$\frac{K_n^h(x,x)}{K_n^h(x_0,x_0)} \leq C.$$

So,

$$\begin{aligned}&\sup_{a,b\in[-A,A]}\left|\left(K_n^{\#} - K_n^h\right)\left(x_0 + a\frac{\delta_n}{n}, x_0 + b\frac{\delta_n}{n}\right)\right| / K_n^h(x_0,x_0)\\ = \;& o(1).\end{aligned}$$

The estimate holds uniformly for $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$. The exact same proof shows that

$$\begin{aligned}&\sup_{a,b\in[-A,A]}\left|\left(K_n^{\#} - K_n^*\right)\left(x_0 + a\frac{\delta_n}{n}, x_0 + b\frac{\delta_n}{n}\right)\right| / K_n^*(x_0,x_0)\\ = \;& o(1).\end{aligned}$$

Theorem 4.1 shows that $K_n^h(x_0,x_0)/K_n^*(x_0,x_0) = 1 + o(1)$ uniformly for $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$. Then we may combine the last two estimates, giving uniformly for $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$,

$$\begin{aligned}&\sup_{a,b\in[-A,A]}\left|\left(K_n^h - K_n^*\right)\left(x_0 + a\frac{\delta_n}{n}, x_0 + b\frac{\delta_n}{n}\right)\right| / K_n^h(x_0,x_0)\\ = \;& o(1).\end{aligned}$$

Finally, by Lemma 2.2(d), uniformly for $x_0 \in J_n(\varepsilon)$ and $|a|, |b| \leq A$,

$$W\left(x_0 + a\frac{\delta_n}{n}\right)/W(x_0) \sim 1 \sim W\left(x_0 + b\frac{\delta_n}{n}\right)/W(x_0).$$

so

$$\begin{aligned}&\sup_{a,b\in[-A,A]}\left|\left(\tilde{K}_n^h - \tilde{K}_n^*\right)\left(x_0 + a\frac{\delta_n}{n}, x_0 + b\frac{\delta_n}{n}\right)\right| / \tilde{K}_n^h(x_0,x_0)\\ = \;& \sup_{a,b\in[-A,A]}\frac{(Wh)\left(x_0 + a\frac{\delta_n}{n}\right)(Wh)\left(x_0 + b\frac{\delta_n}{n}\right)}{(Wh)^2(x_0)}\left|\left(K_n^h - K_n^*\right)\left(x_0 + a\frac{\delta_n}{n}, x_0 + b\frac{\delta_n}{n}\right)\right| / K_n^h(x_0,x_0)\\ = \;& o(1).\end{aligned}$$

■

6. Proof of Theorem 1.3

In this section, we prove Theorem 1.3, whose hypotheses we assume throughout. We also assume the definition (4.2) and (4.3) of W^* and $W^{\#}$.

Theorem 6.1
Let $A > 0$, $\eta \in \left(0, \frac{1}{4}\right)$. *There exist* $C, \tau > 0$ *and* n_0 *such that for* $n \geq n_0$, and $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$,

$$\sup_{a,b \in [-A,A]} \left| \left(\tilde{K}_n^* - \tilde{K}_n\right)\left(x_0 + \frac{a\delta_n}{n}, x_0 + \frac{b\delta_n}{n}\right)\right| / \tilde{K}_n(x_0, x_0) \leq C\eta^{1/2}, \tag{6.1}$$

where C *is independent of* η, τ, n, x_0.
Proof
Choose $\tau > 0$ such that

$$\frac{1}{1+\eta} \leq \frac{h(t)}{h(s)} \leq 1+\eta \text{ for } s \in I(t,\tau) \text{ and } t \in \mathcal{K}. \tag{6.2}$$

This is possible because of the uniform continuity of $\log h$ in $\mathcal{K}$. Fix $x_0 \in J_n(\varepsilon) \cap \mathcal{K}$ and let $W^{\blacksquare}$ be the scaled weight

$$W^{\blacksquare}(x) = h(x_0) W(x) \text{ in } \mathbb{R}.$$

Note that $p_n\left(W^{\blacksquare 2}, x\right) = \frac{1}{h(x_0)} p_n\left(W^2, x\right)$, and hence,

$$K_n^{\blacksquare}(x,y) = \frac{1}{h^2(x_0)} K_n(x,y). \tag{6.3}$$

Observe that (4.2) and (6.2) imply that

$$(1+\eta)^{-1} \leq \frac{W^*}{W^{\blacksquare}} \leq 1+\eta \text{ in } \mathbb{R}. \tag{6.4}$$

Then, much as in the previous section,

$$\begin{aligned}
& \int_{\mathbb{R}} \left(K_n^*(x,t) - K_n^{\blacksquare}(x,t)\right)^2 W^{\blacksquare 2}(t)\, dt \\
= & \int_{\mathbb{R}} K_n^{*2}(x,t) W^{\blacksquare 2}(t)\, dt - 2\int_{\mathbb{R}} K_n^*(x,t) K_n^{\blacksquare}(x,t) W^{\blacksquare 2}(t)\, dt + \int_{\mathbb{R}} K_n^{\blacksquare 2}(x,t) W^{\blacksquare 2}(t)\, dt \\
= & \int_{\mathbb{R}} K_n^{*2}(x,t) W^{*2}(t)\, dt + \int_{I(x_0,\tau)} K_n^{*2}(x,t)\left(W^{\blacksquare 2} - W^{*2}\right)(t)\, dt - 2K_n^*(x,x) + K_n^{\blacksquare}(x,x) \\
= & K_n^{\blacksquare}(x,x) - K_n^*(x,x) + \int_{I(x_0,\tau)} K_n^{*2}(x,t)\left(W^{\blacksquare 2} - W^{*2}\right)(t)\, dt,
\end{aligned}$$

recall that $W^* = W^{\blacksquare} = h(x_0) W$ in $\mathbb{R} \backslash I(x_0,\tau)$. By (6.4),

$$\int_{I(x_0,\tau)} K_n^{*2}(x,t)\left(W^{\blacksquare 2} - W^{*2}\right)(t)\, dt \leq 3\eta \int_{I(x_0,\tau)} K_n^{*2}(x,t) W^{*2}(t)\, dt \leq 3\eta K_n^*(x,x).$$

So

$$\int_{\mathbb{R}} \left(K_n^*(x,t) - K_n^{\blacksquare}(x,t)\right)^2 W^{\blacksquare 2}(t)\, dt \leq K_n^{\blacksquare}(x,x) - (1-3\eta) K_n^*(x,x).$$

Applying an obvious analogue of (5.4) to $P(t) = K_n^*(x,t) - K_n^{\blacksquare}(x,t)$ gives for $x, y \in \mathbb{R}$,

$$\begin{aligned} & \left| K_n^*(x,y) - K_n^{\blacksquare}(x,y) \right| \\ \leq \ & K_n^{\blacksquare}(y,y)^{1/2} \left[K_n^{\blacksquare}(x,x) - (1-3\eta) K_n^*(x,x) \right]^{1/2} \end{aligned}$$

so

$$\begin{aligned} & \left| K_n^*(x,y) - K_n^{\blacksquare}(x,y) \right| / K_n^{\blacksquare}(x,x) \\ \leq \ & \left(\frac{K_n^{\blacksquare}(y,y)}{K_n^{\blacksquare}(x,x)} \right)^{1/2} \left[1 - (1-3\eta) \frac{K_n^*(x,x)}{K_n^{\blacksquare}(x,x)} \right]^{1/2}. \end{aligned}$$

In view of (6.4), we also have

$$\frac{K_n^*(x,x)}{K_n^{\blacksquare}(x,x)} = \frac{\lambda_n^{\blacksquare}(x)}{\lambda_n^*(x)} \geq \frac{1}{(1+\eta)^2},$$

so for all $x, y \in \mathbb{R}$,

$$\begin{aligned} & \left| K_n^*(x,y) - K_n^{\blacksquare}(x,y) \right| / K_n^{\blacksquare}(x,x) \\ \leq \ & \left(\frac{K_n^{\blacksquare}(y,y)}{K_n^{\blacksquare}(x,x)} \right)^{1/2} \left[1 - \frac{1-3\eta}{(1+\eta)^2} \right]^{1/2} \\ \leq \ & \sqrt{6\eta} \left(\frac{K_n^{\blacksquare}(y,y)}{K_n^{\blacksquare}(x,x)} \right)^{1/2} \\ = \ & \sqrt{6\eta} \left(\frac{K_n(y,y)}{K_n(x,x)} \right)^{1/2}. \end{aligned} \tag{6.5}$$

Here we have used (6.3). That relation also implies that

$$\tilde{K}_n^{\blacksquare}(x,y) = \tilde{K}_n(x,y).$$

Then for $x, y \in I(x_0, \tau)$,

$$\begin{aligned} & \left| \tilde{K}_n^*(x,y) - \tilde{K}_n(x,y) \right| / \tilde{K}_n(x,x) \\ = \ & \left| \tilde{K}_n^*(x,y) - \tilde{K}_n^{\blacksquare}(x,y) \right| / \tilde{K}_n^{\blacksquare}(x,x) \\ = \ & \frac{W(y)}{W(x)} \left| \frac{h(y)h(x)}{h(x_0)^2} K_n^*(x,y) - K_n^{\blacksquare}(x,y) \right| / K_n^{\blacksquare}(x,x) \\ \leq \ & \frac{W(y)}{W(x)} \left| \frac{h(y)h(x)}{h(x_0)^2} - 1 \right| \left| K_n^*(x,y) \right| / K_n^{\blacksquare}(x,x) \\ & + \frac{W(y)}{W(x)} \left| K_n^*(x,y) - K_n^{\blacksquare}(x,y) \right| / K_n^{\blacksquare}(x,x). \end{aligned} \tag{6.6}$$

Here by Cauchy-Schwarz and (6.4),

$$\begin{aligned}
&\frac{W(y)}{W(x)}\left|K_n^*(x,y)\right|/K_n^{\blacksquare}(x,x)\\
\leq\ &\frac{W(y)}{W(x)}\left(\frac{K_n^*(x,x)}{K_n^{\blacksquare}(x,x)}\frac{K_n^*(y,y)}{K_n^{\blacksquare}(x,x)}\right)^{1/2}\\
\leq\ &(1+\eta)^2\frac{W(y)}{W(x)}\left(\frac{K_n^{\blacksquare}(y,y)}{K_n^{\blacksquare}(x,x)}\right)^{1/2}\\
=\ &(1+\eta)^2\left(\frac{\tilde{K}_n(y,y)}{\tilde{K}_n(x,x)}\right)^{1/2}.
\end{aligned}$$

Then (6.2), (6.5), (6.6) and the above two inequalities give

$$\begin{aligned}
&\left|\tilde{K}_n^*(x,y)-\tilde{K}_n(x,y)\right|/\tilde{K}_n(x,x)\\
\leq\ &\left(\frac{\tilde{K}_n(y,y)}{\tilde{K}_n(x,x)}\right)^{1/2}\left\{(1+\eta)^2\left[(1+\eta)^2-1\right]+\sqrt{6\eta}\right\}.
\end{aligned}$$

Now we set $x=x_0+\frac{a\delta_n}{n}$ and $y=x_0+\frac{b\delta_n}{n}$, where $a,b\in[-A,A]$. Applying Lemma 2.2(a), (d), (e), we obtain

$$\sup_{a,b\in[-A,A]}\left|\left(\tilde{K}_n^*-\tilde{K}_n\right)\left(x_0+\frac{a\delta_n}{n},x_0+\frac{b\delta_n}{n}\right)\right|/\tilde{K}_n(x_0,x_0)\leq C\sqrt{\eta},$$

uniformly for $x_0\in J_n(\varepsilon)\cap\mathcal{K}$. ■

Proof of Theorem 1.3

Let $A,\varepsilon>0$. By Lemma 2.2(a), (b) and Theorem 4.1, uniformly for $n\geq 1$ and $x_0\in J_n(\varepsilon)\cap\mathcal{K}$,

$$\tilde{K}_n(x_0,x_0)\sim\frac{n}{\delta_n}\sim\tilde{K}_n^h(x_0,x_0). \tag{6.7}$$

Combining Theorem 5.1 and Theorem 6.1, we see that uniformly for $n\geq n_0$ and $x_0\in J_n(\varepsilon)\cap\mathcal{K}$,

$$\sup_{a,b\in[-A,A]}\left|\left(\tilde{K}_n^h-\tilde{K}_n\right)\left(x_0+\frac{a\delta_n}{n},x_0+\frac{b\delta_n}{n}\right)\right|/\tilde{K}_n(x_0,x_0)\leq C\eta^{1/2}. \tag{6.8}$$

Here C is independent of η, but n_0 may depend on η. As the left-hand side is independent of η, we deduce that as $n\to\infty$,

$$\sup_{a,b\in[-A,A]}\left|\left(\tilde{K}_n^h-\tilde{K}_n\right)\left(x_0+\frac{a\delta_n}{n},x_0+\frac{b\delta_n}{n}\right)\right|/\tilde{K}_n(x_0,x_0)\to 0$$

uniformly for $x_0\in J_n(\varepsilon)\cap\mathcal{K}$. It follows (because of the uniformity in a,b above, and by (6.7)) that also

$$\sup_{a,b\in[-A,A]}\left|\left(\tilde{K}_n^h-\tilde{K}_n\right)\left(x_0+\frac{a}{\tilde{K}_n(x_0,x_0)},x_0+\frac{b}{\tilde{K}_n(x_0,x_0)}\right)\right|/\tilde{K}_n(x_0,x_0)=o(1).$$

Then Theorem 1.2 gives
(6.9)
$$\tilde{K}_n^h\left(x_0+\frac{a}{\tilde{K}_n\left(x_0,x_0\right)},x_0+\frac{b}{\tilde{K}_n\left(x_0,x_0\right)}\right)/\tilde{K}_n\left(x_0,x_0\right)=\frac{\sin\pi\left(a-b\right)}{\pi\left(a-b\right)}+o\left(1\right),$$

uniformly for $x_0 \in J_n(\varepsilon)\cap\mathcal{K}$. To replace $\tilde{K}_n(x_0,x_0)$ by $\tilde{K}_n^h(x_0,x_0)$ in the left-hand side, we use the fact that

$$\tilde{K}_n\left(x_0,x_0\right)/\tilde{K}_n^h\left(x_0,x_0\right)=1+o\left(1\right)$$

uniformly for $x_0 \in J_n(\varepsilon)\cap\mathcal{K}$, by Theorem 4.1. We also use the uniformity in a, b in (6.9). ■

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