BULK UNIVERSALITY HOLDS POINTWISE IN THE MEAN, FOR COMPACTLY SUPPORTED MEASURES

DORON S. LUBINSKY

Abstract. Let μ be a measure with compact support, with orthonormal polynomials $\{p_n\}$, and associated reproducing kernels $\{K_n\}$. We show that without any global assumptions on the measure, a weak local condition leads to the bulk universality limit in the mean. For example, if $\mu' \geq C > 0$ in some open interval J, then at each Lebesgue point ξ of J, and for each $r > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \sup_{|u|,|v|\leq r} \left| \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| = 0.$$

In particular, we don't assume regularity of the measure μ.

1. Introduction

Let μ be a finite positive Borel measure with compact support and infinitely many points in the support. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$, satisfying the orthonormality conditions

$$\int p_j p_k \, d\mu = \delta_{jk}.$$

Throughout we use μ' to denote the Radon-Nikodym derivative of μ. The nth reproducing kernel for μ is

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x) p_k(y), \tag{1.1}$$

and the normalized kernel is

$$\widetilde{K}_n(x,y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x,y). \tag{1.2}$$

In the theory of n by n random Hermitian matrices (the so-called unitary case), there arise probability distributions on the eigenvalues that are

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expressible as determinants of reproducing kernels [5, p. 112]:

$$P^{(n)}\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{n!} \det\left(\tilde{K}_n\left(x_i, x_j\right)\right)_{1 \leq i,j \leq n}.$$

One may use this to compute a host of statistical quantities - for example the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One important quantity is the $m-$point correlation function for $\mathcal{M}(n)$ [5, p. 112]:

$$\begin{aligned} &R_m\left(x_1, x_2, \ldots, x_m\right) \\ &\quad = \frac{n!}{(n-m)!} \int \cdots \int P^{(n)}\left(x_1, x_2, \ldots, x_n\right) dx_{m+1}\ dx_{m+2} \ldots dx_n \\ &\quad = \det\left(\tilde{K}_n\left(x_i, x_j\right)\right)_{1 \leq i,j \leq m}. \end{aligned}$$

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, and ξ in the interior of the support of μ, and real $a_1, a_2, \ldots, a_m$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\tilde{K}_n(\xi,\xi)^m} R_m\left(\xi + \frac{a_1}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi,\xi)}, \ldots, \xi + \frac{a_m}{\tilde{K}_n(\xi,\xi)}\right) \\ &\quad = \det\left(\frac{\sin \pi\left(a_i - a_j\right)}{\pi\left(a_i - a_j\right)}\right)_{1 \leq i,j \leq m}. \end{aligned}$$

Of course, when $a_i = a_j$, we interpret $\frac{\sin \pi(a_i - a_j)}{\pi(a_i - a_j)}$ as 1. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{1.3}$$

Thus, an assertion about the distribution of eigenvalues of random matrices reduces to a technical limit involving orthogonal polynomials. The adjective universal is justified: the limit on the right-hand side of (1.3) is independent of ξ, but more importantly is independent of the underlying measure.

Typically, the limit (1.3) is established uniformly for a, b in compact subsets of the real line, but if we remove the normalization from the outer K_n, we can also establish its validity for complex a, b, that is,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{1.4}$$

There is an extensive literature on the topic - an overview may be found in [1], [3], [4], [5], [6], [10]. In [13], we showed that universality holds in measure for compactly supported μ. More precisely, we showed:

Theorem 1.1. *Let μ be a measure with compact support and with infinitely many points in the support. Let $\varepsilon > 0$ and $r > 0$. Then as $n \to \infty$,*

$$meas\left\{\xi \in \{\mu' > 0\} : \sup_{|u|,|v| \le r} \left| \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| \ge \varepsilon \right\} \to 0. \quad (1.5)$$

Here *meas* denotes linear Lebesgue measure, while in the supremum, u, v are complex variables, and $\{\mu' > 0\} = \{x : \mu'(x) > 0\}$. Because convergence in measure implies convergence a.e. of subsequences, we deduced universality for subsequences.

The obvious drawback of this result is that universality holds only in measure. The strongest pointwise result to date, is due to to Totik [21], [22]. (See also [7], [11], [12], [16], [17].) A measure μ is called *regular* (in the sense of Stahl, Totik, Ullman) if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\text{cap}(\text{supp}[\mu])},$$

where cap(supp[μ]) denotes the logarithmic capacity of the support of μ. See [18] for a through exploration of this concept. Totik proved that if μ is a measure with compact support that is regular, and if in some interval I,

$$\int_I \log \mu' > -\infty,$$

then for a.e. $\xi \in I$, we have the universality limit (1.3). While regularity is a weak global condition, it is not yet clear whether it is necessary for a full pointwise result.

In this paper, we avoid any global assumptions on μ, other than compact support. We show that when μ satisfies some local regularity condition, then pointwise universality holds in the mean:

Theorem 1.2. *Let μ be a measure with compact support and with infinitely many points in the support. Assume that J is an open interval in which for some $C > 0$,*

$$\mu' \ge C \text{ a.e. in } J. \quad (1.6)$$

Let $\xi \in J$ be a Lebesgue point of μ. Then for each $r > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \sup_{|u|,|v| \le r} \left| \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| = 0. \quad (1.7)$$

In particular, this holds for a.e. $\xi \in J$.

Remarks . (i) By a Lebesgue point ξ of μ, we mean a point at which

$$\lim_{h\to 0+}\frac{1}{2h}\mu\left[\xi-h,\xi+h\right]=\mu'\left(\xi\right),$$

with $\mu'\left(\xi\right)$ finite. In particular, the singular part μ_s of μ satisfies

$$\lim_{h\to 0+}\frac{1}{2h}\mu_s\left[\xi-h,\xi+h\right]=0.$$

Of course if μ is absolutely continuous in a neighborhood of ξ, and μ' is continuous at ξ, then the Lebesgue point condition is satisfied at ξ.

(ii) An equivalent formulation is that universality holds outside a set of positive integers of density 0. That is, there exists a set $\mathcal{E}$ of integers of density 0, such that

$$\lim_{n\to\infty, n\notin\mathcal{E}}\frac{K_n\left(\xi+\frac{u}{\tilde{K}_n(\xi,\xi)},\xi+\frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n\left(\xi,\xi\right)}=\frac{\sin\pi\left(u-v\right)}{\pi\left(u-v\right)},$$

uniformly for u,v in compact subsets of $\mathbb{C}$. Here, recall that a set $\mathcal{E}$ of positive integers has density 0 if

$$\lim_{n\to\infty}\frac{1}{n}\#\left\{j:1\le j\le n\text{ and }j\in\mathcal{E}\right\}=0,$$

where $\#$ denotes cardinality. The set $\mathcal{E}$ depends on the particular ξ.

Theorem 1.2 is a special case of the following theorem, whose formulation involves maximal functions. For a finite positive measure ν on the real line, its maximal function is

$$\mathcal{M}\left[d\nu\right]\left(x\right)=\sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h}d\nu. \tag{1.8}$$

In the sequel, $\mathcal{M}\left[K_nd\mu\right]\left(x\right)$ denotes the maximal function for the measure $K_n\left(x,x\right)d\mu\left(x\right)$.

Theorem 1.3. *Let μ be a measure with compact support and with infinitely many points in the support. Let ξ be a Lebesgue point of μ with $\mu'\left(\xi\right)>0$. Assume that there exist C_1,C_2,C_3,C_4 with the following properties: given $r>0$, there exists $n_0=n_0\left(r\right)$ such that for $n\ge n_0$, both*

(I) *for all complex u,v with $\left|u\right|,\left|v\right|\le r$,*

$$\left|K_n\left(\xi+\frac{u}{n},\xi+\frac{v}{n}\right)\right|\le C_1ne^{C_2\left(\left|u\right|+\left|v\right|\right)}; \tag{1.9}$$

(II) *for all $s\in\left[-r,r\right]$,*

$$K_n\left(\xi+\frac{s}{n},\xi+\frac{s}{n}\right)\ge C_3n; \tag{1.10}$$

(III) *for $n\ge 1$,*

$$\mathcal{M}\left[K_nd\mu\right]\left(\xi\right)\le C_4n. \tag{1.11}$$

Then (1.7) *holds for all $r>0$.*

When μ satisfies a Szegő type condition $\int_J \log \mu' > -\infty$ in an interval J, then results of Totik [21], [22] indicate that both (1.9) and (1.10) hold at a.e. $\xi \in J$. However, it is not clear that (1.11) also follows. In [2], Avila, Last and Simon assumed similar conditions to (1.9), (1.10), but assumed instead of (1.11) an implicit limit condition, in proving pointwise universality.

This paper is structured as follows: in Section 2, we present the ideas of proof. In Section 3, we establish upper and lower bounds for K_n. In Section 4, we deduce normality of the normalized reproducing kernels, and establish properties of their subsequential limits, which are entire functions. In Section 5, we estimate averages of tail integrals using maximal functions, and then prove Theorems 1.2 and 1.3.

We close this section with some notation. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. We shall use calligraphic symbols such as $\mathcal{E}_n, \mathcal{F}_n, \mathcal{G}_n, \mathcal{H}_n \ldots$ to denote sets that typically have small measure. The nth Christoffel function for μ is

$$\lambda_n(x) = \frac{1}{K_n(x,x)} = \inf_{\deg(P) \leq n-1} \int \frac{P^2(t)}{P^2(x)} d\mu(t). \tag{1.12}$$

For $r > 0$, we define the tail integral

$$\Phi_n(x,r) = \frac{\int_{|t-x| \geq \frac{r}{n}} K_n(x,t)^2 \, d\mu(t)}{K_n(x,x)}. \tag{1.13}$$

Let

$$A_n(x) = p_{n-1}^2(x) + p_n^2(x). \tag{1.14}$$

For complex u, v, real ξ, and $r > 0$, we let

$$f_n(u,v,\xi) = \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}; \tag{1.15}$$

$$\begin{aligned} &\Gamma_n(u,v,\xi,r) \qquad\qquad (1.16) \\ &= \sup_{s \geq r \frac{\tilde{K}_n(\xi,\xi)}{n}} \left| f_n(u,v,\xi) - \int_{-s}^{s} f_n(u,t,\xi) f_n(v,t,\xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right|. \end{aligned}$$

In the integral in the right-hand side, t is the variable of integration. Also, let

$$I_n(\xi,r) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Gamma_n(u,v,\xi,r) \left(f_n(u,u,\xi) f_n(v,v,\xi)\right)^{-1/2} du \, dv. \tag{1.17}$$

For $\sigma > 0$, PW_σ denotes the *Paley-Wiener space*, consisting of entire functions of exponential type at most σ that are square integrable on the real axis, with the usual $L_2(\mathbb{R})$ norm. The reproducing kernel for PW_σ is $\frac{\sin\sigma(u-v)}{\pi(u-v)}$. Thus for $g \in PW_\sigma$, and all complex z [19, p. 95],

$$g(z) = \int_{-\infty}^{\infty} g(t) \frac{\sin\sigma(t-z)}{\pi(t-z)} dt.$$

The *Cartwright class* [9] consists of all entire functions g of exponential type such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |g(t)|}{1+t^2} dt < \infty, \tag{1.18}$$

where $\log^+ x = \max\{0, \log x\}$.

2. Ideas of Proof

Recall our notation

$$f_n(u,v,\xi) = \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}.$$

Our local hypotheses on μ in Theorem 1.2 give upper bounds on $K_n(t,t)$ for t in any compact subinterval of J. We can then use Bernstein's growth inequality in the plane to show that for $\xi \in J$ and all complex u, v,

$$\left| K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)\right| \le C_1 n e^{C_2(|u|+|v|)}.$$

Here C_1, C_2 depend on ε, but are independent of u, v, n, ξ. There is also a lower bound for $K_n(t,t)$ that holds for arbitrary measures. Thus the hypotheses of Theorem 1.2 imply those of Theorem 1.3. The latter give uniform boundedness of $\{f_n\}$ for all complex u, v,

$$|f_n(u,v,\xi)| \le C_1 e^{C_2(|u|+|v|)}.$$

One deduces that if $f(\cdot,\cdot,\xi)$ is a subsequential limit, it is entire of exponential type in each variable. Moreover, there exists $\sigma > 0$ such that for all real a, $f(a,\cdot,\xi)$ is of exponential type σ, and lies in Cartwright's class. Some assertions about the zeros of $f(0,\cdot,\xi)$ are then proved as in [11], [13].

The most difficult step is to show that

$$f(u,v,\xi) = \frac{\sin\pi(u-v)}{\pi(u-v)}. \tag{2.1}$$

We adopt an indirect approach, based on a uniqueness theorem proved in [13]. The essential feature there, is that the relation

$$f(a,b,\xi) = \int_{-\infty}^{\infty} f(a,t,\xi) f(b,t,\xi) dt, \tag{2.2}$$

for all complex a, b, together with $f(0,0,\xi) = 1$, and some other restrictions on zeros of $f(0,\cdot)$, yields (2.1).

To establish (2.2), we estimate averages of the tail integrals

$$\Phi_n(x,r) = \frac{\int_{|t-x|\geq \frac{r}{n}} K_n(x,t)^2 \, d\mu(t)}{K_n(x,x)}.$$

Using maximal functions, we show in Section 5, that for $|y-x| \leq \frac{r}{4m}$,

$$\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \leq \frac{8C_0}{r^{1/2}} \left(\frac{K_{2m}(x,x)}{K_m(x,x)} \right)^{1/2} \left(m\mathcal{M}[K_{2m} d\mu](y) \right)^{1/2},$$

where

$$C_0 = \sup_n \frac{\gamma_{n-1}}{\gamma_n}.$$

We can then deduce estimates for averages of

$$I_n(\xi,r) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Gamma_n(u,v,\xi,r) \left(f_n(u,u,\xi) f_n(v,v,\xi) \right)^{-1/2} du \; dv,$$

where

$$\begin{aligned} &\Gamma_n(u,v,\xi,r) \\ &= \sup_{s \geq r \frac{\tilde{K}_n(\xi,\xi)}{n}} \left| f_n(u,v,\xi) - \int_{-s}^{s} f_n(u,t,\xi) f_n(v,t,\xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right|. \end{aligned}$$

More precisely, we show that for some C independent of r and m,

$$\frac{1}{m} \sum_{n=m}^{2m-1} I_n(\xi,r)^{1/2} \leq \frac{C}{r^{1/2}}.$$

This leads to (2.2), and hence (2.1), for subsequential limits f that avoid a thin set of integers. Using the fact that $\{f_n\}$ are uniformly bounded in compact sets, we then obtain (1.7).

3. Bounds for K_n

We show that the hypotheses of Theorem 1.2 imply those of Theorem 1.3.

Lemma 3.1. *Assume the hypotheses of Theorem 1.2. Let J_1 be a compact subinterval of J. There exist C_1, C_2 and C_3 with the following properties:*

(a) *Given $r > 0$, there exists $n_0 = n_0(r)$ such that for $n \geq n_0$, $\xi \in J_1$, and for all complex u, v with $|u|, |v| \leq r$,*

$$\left| K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n} \right) \right| \leq C_1 n e^{C_2(|u|+|v|)}; \tag{3.1}$$

(b) *Let $\xi \in J$ be a Lebesgue point of μ. Then for $n \geq 1$,*

$$\mathcal{M}[K_n d\mu](\xi) \leq C_3 n. \tag{3.2}$$

Proof. (a) This follows from the assumed lower bounds on μ' and Bernstein's growth inequality for polynomials, by using standard methods. Here are some details: let ω denote the Legendre measure for the interval J, so that $\omega' = 1$ there. By (1.6) and monotonicity of Christoffel functions,

$$\lambda_n(\mu, x) \geq C\lambda_n(\omega, x).$$

Standard estimates for the Christoffel function for the Legendre weight [15, p. 108, Lemma 5] give that for $n \geq 1$ and $x \in J_1$,

$$\lambda_n(\omega, x) \geq C_1/n.$$

Thus for $n \geq 1$ and $x \in J_1$,

$$K_n(x, x) = \lambda_n^{-1}(x) \leq (CC_1)^{-1} n.$$

By Cauchy-Schwarz, for $n \geq 1$, and $x, y \in J_1$,

$$|K_n(x, y)| \leq (CC_1)^{-1} n.$$

We now apply Bernstein's growth inequality

$$|P(z)| \leq \left|z + \sqrt{z^2 - 1}\right|^n \|P\|_{L_\infty[-1,1]},$$

valid for all complex z, and polynomials P of degree $\leq n$. We reformulate this for the interval J, and estimate in a standard fashion to obtain (3.1). See [11, Lemmas 5.1 and 5.2, pp. 383–384].

(b) Choose $\eta > 0$ so that $\xi \pm \eta \in J_1$, a compact subinterval of J. In J_1, (a) implies that $K_n(x, x) \leq C_1 n$. Then for $\xi \in J$ and $0 < h < \eta$,

$$\frac{1}{2h}\int_{\xi-h}^{\xi+h} K_n(t, t)\, d\mu(t) \leq C_1 n \frac{1}{2h}\mu[\xi - h, \xi + h].$$

As ξ is a Lebesgue point of μ,

$$\lim_{h\to 0+} \frac{1}{2h}\mu[\xi - h, \xi + h] = \mu'(\xi) < \infty.$$

Hence there exists $C_2 > 0$ such that for $h > 0$,

$$\frac{1}{2h}\mu[\xi - h, \xi + h] \leq C_2.$$

This yields the desired estimate for $0 < h < \eta$. For $h \geq \eta$, we use the trivial estimate

$$\frac{1}{2h}\int_{\xi-h}^{\xi+h} K_n(t, t)\, d\mu(t) \leq \frac{1}{2\eta}\int K_n(t, t)\, d\mu(t) = \frac{n}{2\eta}.$$

□

Lemma 3.2. *Let μ be a measure with compact support, and with infinitely many points in its support. For each Lebesgue point $\xi$of μ, there exists*

$C = C(\xi)$ with the following property: let $T > 0$. Then there exists n_0 such that for $n \geq n_0$,

$$\inf_{s\in[-T,T]} K_n\left(\xi + \frac{s}{n}, \xi + \frac{s}{n}\right) \geq Cn. \tag{3.3}$$

Moreover, this also holds at every point $\xi \notin \operatorname{supp}[\mu]$.

Proof. See, for example, [13, Lemmas 3.1 and 3.2]. A far more precise asymptotic lower bound was proved in [20]. □

4. Normal Family Estimates

Recall the definition (1.15) of f_n. In this section, we prove:

Theorem 4.1. *Assume that μ and ξ are as in Theorem 1.3. There exist $C_1, C_2 > 0$ with the following properties:*

(a) *For all complex u, v,*

$$|f_n(u, v, \xi)| \leq C_1 e^{C_2(|u|+|v|)}. \tag{4.1}$$

(b) *Let $f(\cdot, \cdot, \xi)$ be the limit of some subsequence $\{f_n\}_{n\in\mathcal{T}}$ of $\{f_n\}_{n\geq 1}$. Then*

  (i) *$f(\cdot, \cdot, \xi)$ is entire in each variable, and with C_1, C_2 as in (a), for all complex u, v,*

$$|f(u, v, \xi)| \leq C_1 e^{C_2(|u|+|v|)}. \tag{4.2}$$

  (ii) *For each complex u,*

$$\int_{-\infty}^{\infty} |f(u, s, \xi)|^2 \, ds \leq f(u, \bar{u}, \xi) < \infty. \tag{4.3}$$

  (iii) *$f(0, \cdot, \xi)$ has infinitely many real simple zeros $\{\rho_j\}_{j\neq 0}$ where*

$$\cdots < \rho_{-2} < \rho_{-1} < 0 < \rho_1 < \rho_2 < \cdots$$

  and no other zeros. Let $\rho_0 = 0$. For $j \neq 0$, $f(\rho_j, \cdot, \xi)$ has zeros $\{\rho_k\}_{k\in\mathbb{Z}\setminus\{j\}}$ and no other zeros.

  (iv) *There exists $C_0 > 0$ such that for all real t,*

$$f(t, t, \xi) \geq C_0, \tag{4.4}$$

  and $f(0, 0) = 1$.

  (v) *There exists $\sigma > 0$ such that for each real a, $f(a, \cdot, \xi)$ is an entire function of exponential type σ.*

Remark. C_0, C_1, and C_2 are independent of n, u, v, and the particular subsequential limit f.

Proof of Theorem 4.1(a). From Lemmas 3.1 and 3.2, we deduce that for Lebesgue points $\xi \in J_1$, and all complex u, v, we have

$$\left| \frac{K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)}{K_n(\xi, \xi)} \right| \leq C_1 e^{C_2(|u|+|v|)}.$$

Here C_1, C_2 are independent of n, u, v. Since also

$$\frac{n}{\tilde{K}_n(\xi,\xi)} \leq C$$

in J_1, some C (from Lemma 3.2), we obtain for (different) C_1, C_2,

$$\left| \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} \right| \leq C_1 e^{C_2(|u|+|v|)}.$$

$\square$

Proof of Theorem 4.1(b).

(i) From (i), $\{f_n\}_{n\geq 1}$ is a normal family in compact subsets of $\mathbb{C}^2$. If f denotes some subsequential limit, say as $n \to \infty$ through $\mathcal{T}$, then (a) gives the bound

$$|f(u,v,\xi)| \leq C_1 e^{C_2(|u|+|v|)},$$

for all complex u, v.

(ii) Next, let $u \in \mathbb{C}$, and $U = \xi + \frac{u}{\tilde{K}_n(\xi,\xi)}$, and use the reproducing kernel relation

$$1 = \int \frac{\left|K_n^2(U,t)\right|}{K_n\left(U,\bar{U}\right)} d\mu(t).$$

We drop most of the integral and make the substitution $t = \xi + \frac{s}{\tilde{K}_n(\xi,\xi)}$:

$$\begin{aligned} 1 &\geq \int_{\xi - \frac{r}{\tilde{K}_n(\xi,\xi)}}^{\xi + \frac{r}{\tilde{K}_n(\xi,\xi)}} \frac{\left|K_n^2(U,t)\right|}{K_n\left(U,\bar{U}\right)} d\mu(t) \\ &= \int_{-r}^{r} \frac{|f_n(u,s,\xi)|^2}{f_n(u,\bar{u},\xi)} \frac{d\mu\left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)}. \end{aligned}$$

As we assumed that ξ is a Lebesgue point of μ, and we may assume that as $n \to \infty$ through $\mathcal{T}$, $f_n \to f$ locally uniformly, we obtain

$$1 \geq \int_{-r}^{r} \frac{|f(u,s,\xi)|^2}{f(u,\bar{u},\xi)} ds.$$

Now let $r \to \infty$.

(iii) Now for each fixed real ξ, with $(p_{n-1}p_n)(\xi) \neq 0$, the function

$$\begin{aligned} L_n(t,\xi) &= (t-\xi) K_n(t,\xi) \\ &= \frac{\gamma_{n-1}}{\gamma_n}\left(p_n(t)\, p_{n-1}(\xi) - p_{n-1}(t)\, p_n(\xi)\right) \end{aligned}$$

has simple zeros that interlace those of p_n. See, for example [8, p. 19 ff.]. More precisely $L_n(\cdot,\xi)$ has a simple zero in $(x_{jn}, x_{j-1,n})$ for $2 \leq j \leq n$, and one zero outside $(x_{nn,} x_{1n})$. When $(p_{n-1}p_n)(\xi) = 0$, then L_n is a multiple of p_{n-1} or p_n. It follows that in all cases $L_n(\cdot,\xi)$ has a zero in $[x_{jn}, x_{j-1,n})$, $2 \leq j \leq n$, and at most one other zero, outside $[x_{nn}, x_{1n})$. Let $\{t_{jn}\}_{j\neq 0} = \{t_{jn}(\xi)\}_{j\neq 0}$ denote these zeros of $K_n(\xi, t)$, and $t_{0n}(\xi) = \xi$. We order the zeros as

$$\cdots < t_{-1n}(\xi) < t_{0n}(\xi) < t_{1n}(\xi) < t_{2n}(\xi) < \cdots$$

Then $f_n(0, \cdot, \xi)$ has simple zeros

$$\rho_{jn} = \tilde{K}_n(\xi, \xi)(t_{jn} - \xi), \quad j \neq 0,$$

and no other zeros. Let $\rho_{0n} = 0$. Note that

$$\cdots < \rho_{-1,n} < \rho_{0n} = 0 < \rho_{1n} < \rho_{2n} < \cdots$$

Now as $n \to \infty$ through $\mathcal{T}$, we have

$$\lim_{n\to\infty, n\in\mathcal{T}} f_n(0, u, \xi) = f(0, u, \xi)$$

uniformly for u in compact subsets of the plane. Moreover, $f(0, 0, \xi) = \lim_{n\to\infty, n\in\mathcal{T}} f_n(0, 0, \xi) = 1$, so f is not identically 0. By Hurwitz' theorem, each zero of $f(0, \cdot, \xi)$ is a limit of zeros of $f_n(0, \cdot, \xi)$.

Next, (i) shows that $f(0, \cdot, \xi)$ is of exponential type at most type C_2, while from (ii), $\int_{-\infty}^{\infty} f(0, s, \xi)^2 \, ds < \infty$. A well known bound [9, p. 149] asserts that

$$|f(0, x + iy, \xi)|^2 \leq \frac{2}{\pi} e^{2C_2(|y|+1)} \int_{-\infty}^{\infty} f(0, s, \xi)^2 \, ds \tag{4.5}$$

for all complex $x + iy$. In particular, then $f(0, \cdot, \xi)$ is bounded on the real axis and so satisfies (1.18) and lies in the Cartwright class. It is also real valued on the real axis. Then [9, p. 130], if $\{\rho_j\}$ are the zeros of $f(0, \cdot, \xi)$,

$$f(0, z, \xi) = \lim_{R\to\infty} \prod_{|\rho_j|<R} \left(1 - \frac{z}{\rho_j}\right).$$

It follows that f has infinitely many zeros $\{\rho_j\}$, and these are then necessarily the limits of the zeros $\{\rho_{j,n}\}$ of $f_n(0, \cdot, \xi)$. Since each $\rho_{j,n}$ is a simple zero of f_n, ρ_j is a simple zero of $f(0, \cdot, \xi)$ unless $\rho_j = \rho_{j-1}$ or ρ_{j+1}.

Next, we note that for $j \neq k$,

$$K_n(t_{jn}, t_{kn}) = 0.$$

Indeed, it follows from the Christoffel-Darboux formula that both t_{jn} and t_{kn} are roots of the equation

$$p_n(t) p_{n-1}(\xi) - p_{n-1}(t) p_n(\xi) = 0.$$

Then for $j \neq k$,

$$f_n(\rho_{jn}, \rho_{kn}, \xi) = 0$$

and because of the locally uniform convergence,

$$f(\rho_j, \rho_k, \xi) = 0.$$

Moreover, because of Hurwitz' theorem, $f(\rho_j, \cdot, \xi)$ has no other zeros. We still have to show the simplicity of the zeros.

(iv) We know from Lemma 3.2, that there exists $C > 0$, such that given $T > 0$, there exists $n_0 = n_0(T)$ such that for $n \geq n_0$,

$$\inf_{s\in[-T,T]} K_n\left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right) \geq Cn,$$

where C is independent of T. Also, we have the upper bound (3.1) for $K_n(\xi,\xi)$. Thus

$$\inf_{s\in[-T,T]} f_n(s,s,\xi) \geq C.$$

As C is independent of T, we obtain

$$\inf_{t\in\mathbb{R}} f(t,t,\xi) \geq C.$$

This also shows that $f(\rho_j,\rho_j,\xi) > 0$, so necessarily $\rho_{j\pm1} \neq \rho_j$, and all zeros of $f(0,\cdot,\xi)$ are simple.

(v) As above, the zeros of $L_n(t,\xi) = (t-\xi)K_n(t,\xi)$ interlace those of p_n. Let $m > k$. It follows that whatever is ξ, the number j of zeros of $K_n(t,\xi)$ in $[x_{mn}, x_{kn}]$ satisfies

$$|j - (m-k)| \leq 1.$$

Now let $N(g,r)$ denote the number of zeros of a function g in $[-r,r]$. It follows from this last estimate that for any real a, b, and $r > 0$, and $n \geq 1$, we have

$$|N(f_n(a,\cdot,\xi),r) - N(f_n(b,\cdot,\xi),r)| \leq 2.$$

Letting $n \to \infty$ through the appropriate subsequence of integers gives for each $r > 0$,

$$|N(f(a,\cdot,\xi),r) - N(f(b,\cdot,\xi),r)| \leq 4. \tag{4.6}$$

Since $f(a,\cdot,\xi)$ has only real zeros, and lies in Cartwright's class, as follows from (i) and (ii), so,

$$\lim_{r\to\infty} \frac{N(f(a,\cdot),r)}{2\pi r} = \sigma_a,$$

where σ_a is the exponential type of $f(a,\cdot,\xi)$, see [9, p. 127, eqn. (5)]. It follows from (4.6) that $\sigma_a = \sigma$ is independent of a. We must still show that $\sigma > 0$. To do this, we use the bound (4.5) with $C_2 = \sigma$:

$$|f(0,x+iy,\xi)|^2 \leq \frac{2}{\pi} e^{2\sigma(|y|+1)} \int_{-\infty}^{\infty} |f(0,t,\xi)|^2 \, dt.$$

If $\sigma = 0$, this implies that $f(0,\cdot,\xi)$ is bounded and hence constant, contradicting its square integrability over the real line.

□

5. PROOF OF THEOREMS 1.2 AND 1.3

We begin by estimating the tail integral Φ_n using maximal functions. It is really these estimates that allow us to avoid the hypothesis that μ is regular. Recall our notation (1.13)–(1.17). A version of Lemma 5.1(a) was already proved and used in [14].

In the sequel, we let

$$C_0 = \sup_n \frac{\gamma_{n-1}}{\gamma_n}. \tag{5.1}$$

Lemma 5.1. (a) *Let μ be a measure on the real line with infinitely many points in its support. Let $r > 0$ and $m \geq 1$. Let $|y-x| \leq \frac{r}{4m}$. Then*

$$\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \leq \frac{8C_0}{r^{1/2}} \left(\frac{K_{2m}(x,x)}{K_m(x,x)}\right)^{1/2} \left(m\mathcal{M}[K_{2m}d\mu](y)\right)^{1/2}. \tag{5.2}$$

(b) *Let $0 < A \leq r/4$. Then*

$$\begin{aligned} &\int_{\xi-\frac{A}{m}}^{\xi+\frac{A}{m}} \left(\sum_{n=m}^{2m-1} \Phi_n(t,r)^{1/2}\right) dt \\ &\leq \frac{8C_0}{r^{1/2}} \left(m\mathcal{M}[K_{2m}d\mu](\xi)\right)^{1/2} \int_{\xi-\frac{A}{m}}^{\xi+\frac{A}{m}} \left(\frac{K_{2m}(t,t)}{K_m(t,t)}\right)^{1/2} dt. \end{aligned} \tag{5.3}$$

(c) *Assume in addition, that μ and ξ satisfy the hypotheses of Theorem 1.3. There exist $C > 0$ and m_1 such that for $m \geq m_1$ and all $r > 0$,*

$$\int_{\xi-\frac{A}{m}}^{\xi+\frac{A}{m}} \left(\sum_{n=m}^{2m-1} \Phi_n(t,r)^{1/2}\right) dt \leq \frac{C}{\sqrt{r}}. \tag{5.4}$$

Here C is independent of m, r, but depends on A, ξ.

Proof. (a) Observe that

$$\begin{aligned} |K_n(x,t)| &= \frac{\gamma_{n-1}}{\gamma_n} \left| \frac{p_n(x)\,p_{n-1}(t) - p_{n-1}(x)\,p_n(t)}{x-t} \right| \\ &\leq \frac{\gamma_{n-1}}{\gamma_n} \frac{A_n(x)^{1/2} A_n^{1/2}(t)}{|x-t|}, \end{aligned}$$

by Cauchy-Schwarz. Then for $2m-1 \geq n \geq m$,

$$\Phi_n(x,r) \leq C_0^2 \frac{A_n(x)}{K_m(x,x)} \int_{|t-x| \geq \frac{r}{2m}} \frac{A_n(t)}{(t-x)^2} d\mu(t).$$

Using Cauchy-Schwarz, we obtain

$$
\begin{aligned}
&\sum_{n=m}^{2m-1} \Phi_n(x,r)^{1/2} \\
&\leq \frac{C_0}{K_m(x,x)^{1/2}} \sum_{n=m}^{2m-1} A_n(x)^{1/2} \left(\int_{|t-x| \geq \frac{r}{2m}} \frac{A_n(t)}{(t-x)^2} d\mu(t) \right)^{1/2} \\
&\leq \frac{C_0}{K_m(x,x)^{1/2}} \left(\sum_{n=m}^{2m-1} A_n(x) \right)^{1/2} \left(\sum_{n=m}^{2m-1} \int_{|t-x| \geq \frac{r}{2m}} \frac{A_n(t)}{(t-x)^2} d\mu(t) \right)^{1/2} \\
&\leq \frac{2C_0}{K_m(x,x)^{1/2}} \left(K_{2m}(x,x) \right)^{1/2} \left(\int_{|t-x| \geq \frac{r}{2m}} \frac{K_{2m}(t,t)}{(t-x)^2} d\mu(t) \right)^{1/2}. \qquad (5.5)
\end{aligned}
$$

We assumed that $|y-x| \leq \frac{r}{4m}$. Then for $j \geq -1$, and $|t-x| \leq 2^{j+1}\frac{r}{2m}$,

$$|t-y| \leq 2^{j+1}\frac{r}{2m} + |x-y| \leq 2^{j+2}\frac{r}{2m}$$

so using the definition of the maximal function, we see that

$$
\begin{aligned}
\int_{2^j \frac{r}{2m} \leq |t-x| \leq 2^{j+1} \frac{r}{2m}} K_m(t,t)\, d\mu(t) &\leq \int_{|t-y| \leq 2^{j+2} \frac{r}{2m}} K_m(t,t)\, d\mu(t) \\
&\leq 2^{j+2} \frac{r}{m} \mathcal{M}[K_{2m} d\mu](y).
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{|t-x| \geq \frac{r}{2m}} \frac{K_{2m}(t,t)}{(t-x)^2} d\mu(t) &\leq \sum_{j=-1}^{\infty} \int_{2^j \frac{r}{m} \leq |t-x| \leq 2^{j+1} \frac{r}{2m}} \frac{K_{2m}(t,t)}{(2^j r/m)^2} d\mu(t) \\
&\leq \sum_{j=-1}^{\infty} \frac{m^2}{2^{2j} r^2} 2^{j+2} \frac{r}{m} \mathcal{M}[K_{2m} d\mu](y) \\
&= \frac{16m}{r} \mathcal{M}[K_{2m} d\mu](y).
\end{aligned}
$$

Substituting this into (5.5) yields (5.2).

(b) This follows directly from (a) as $|t-\xi| \leq \frac{A}{m} \Rightarrow |t-\xi| \leq \frac{r}{4m}$.

(c) This follows directly from (b) and our hypotheses (1.9), (1.10), (1.11).

□

We can now deduce estimates for Γ_n and I_n, defined respectively by (1.16) and (1.17).

Lemma 5.2. *Assume that μ and ξ are as in Theorem 1.3. There exists $\delta > 0$ with the following properties:*

(a) *For* $r>0$ *and* $|u|,|v|\le\frac{r\delta}{2}$*,*

$$\Gamma_n(u,v,\xi,r)\le\left[f_n(u,u,\xi)\,\Phi_n\left(\xi+\frac{u}{\tilde K_n(\xi,\xi)},\frac{r}{2}\right)\right]^{1/2}$$
$$\times\left[f_n(v,v,\xi)\,\Phi_n\left(\xi+\frac{v}{\tilde K_n(\xi,\xi)},\frac{r}{2}\right)\right]^{1/2}. \tag{5.6}$$

(b) *For* $r\ge 4/\delta$*,*

$$\frac{1}{m}\sum_{n=m}^{2m-1}I_n(\xi,r)^{1/2}\le\frac{C_1}{r^{1/2}}. \tag{5.7}$$

Here C_1 *is independent of* m,r*.*

Proof. We use the fact that for some $\delta\in(0,1)$,

$$\delta n\le\tilde K_n(\xi,\xi)\le\delta^{-1}n,\quad n\ge 1. \tag{5.8}$$

This follows from Lemmas 3.1 and 3.2.

(a) This is as in [13]. Let

$$U=\xi+\frac{u}{\tilde K_n(\xi,\xi)};\quad V=\xi+\frac{v}{\tilde K_n(\xi,\xi)},$$

and let $s\ge r$. From the reproducing kernel relation,

$$\frac{K_n(U,V)}{K_n(\xi,\xi)}-\int_{|y-\xi|\le\frac{s}{n}}\frac{K_n(U,y)}{K_n(\xi,\xi)}\frac{K_n(V,y)}{K_n(\xi,\xi)}\tilde K_n(\xi,\xi)\frac{d\mu(y)}{\mu'(\xi)}$$
$$=\int_{|y-\xi|>\frac{s}{n}}\frac{K_n(U,y)}{\sqrt{K_n(\xi,\xi)}}\frac{K_n(V,y)}{\sqrt{K_n(\xi,\xi)}}d\mu(y).$$

We now make the substitution $y=\xi+\frac{t}{\tilde K_n(\xi,\xi)}$, in the first integral only, recasting the last equation as

$$f_n(u,v,\xi)-\int_{-s\frac{\tilde K_n(\xi,\xi)}{n}}^{s\frac{\tilde K_n(\xi,\xi)}{n}}f_n(u,t,\xi)\,f_n(v,t,\xi)\,\frac{d\mu\left(\xi+\frac{t}{\tilde K_n(\xi,\xi)}\right)}{\mu'(\xi)}$$
$$=f_n(u,u,\xi)^{1/2}f_n(v,v,\xi)^{1/2}\int_{|y-\xi|>\frac{s}{n}}\frac{K_n(U,y)}{\sqrt{K_n(U,U)}}\frac{K_n(V,y)}{\sqrt{K_n(V,V)}}d\mu(y). \tag{5.9}$$

Next, observe that for $\xi\in J$, (5.8) gives for $s\ge r$,

$$|y-\xi|\ge\frac{s}{n}\Rightarrow|y-U|\ge|y-\xi|-\frac{|u|}{n}\frac{n}{\tilde K_n(\xi,\xi)}$$
$$\ge\frac{s}{n}-\frac{|u|}{\delta n}\ge\frac{s}{2n}\ge\frac{r}{2n},$$

as $|u| \leq \frac{\delta r}{2} \leq \frac{\delta s}{2}$. Now use Cauchy-Schwarz on the right-hand side of (5.9), and the fact that $s \geq r$:

$$\begin{aligned}
&\Gamma_n(u,v,\xi,r) \\
&= \sup_{s \geq r} \left| f_n(u,v,\xi) - \int_{-s\frac{\tilde{K}_n(\xi,\xi)}{n}}^{s\frac{\tilde{K}_n(\xi,\xi)}{n}} f_n(u,t,\xi)\, f_n(v,t,\xi)\, \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right| \\
&\leq \left[f_n(u,u,\xi)\, f_n(v,v,\xi) \int_{|y-U|>\frac{r}{2n}} \frac{K_n^2(U,y)}{K_n(U,U)} d\mu(y) \int_{|y-V|>\frac{r}{2n}} \frac{K_n^2(V,y)}{K_n(V,V)} d\mu(y) \right]^{1/2}.
\end{aligned}$$

We obtain (5.6), on taking account of the definition (1.13) of Φ_n.

(b) Using (a) and integrating, gives

$$\begin{aligned}
I_n(\xi,r) &= \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Gamma_n(u,v,\xi,r) \left(f_n(u,u,\xi)\, f_n(v,v,\xi)\right)^{-1/2} du\ dv \\
&\leq \left(\frac{1}{2} \int_{-1}^{1} \Phi_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \frac{r}{2}\right)^{1/2} du \right)^2 \\
&= \left(\frac{\tilde{K}_n(\xi,\xi)}{2} \int_{|t-\xi| \leq \frac{1}{\tilde{K}_n(\xi,\xi)}} \Phi_n\left(t, \frac{r}{2}\right)^{1/2} dt \right)^2 \\
&\leq \left(\frac{n}{2\delta} \int_{|t-\xi| \leq \frac{1}{n\delta}} \Phi_n\left(t, \frac{r}{2}\right)^{1/2} dt \right)^2,
\end{aligned}$$

by (5.8). Adding for $m \leq n \leq 2m-1$, gives

$$\begin{aligned}
\frac{1}{m} \sum_{n=m}^{2m-1} I_n(\xi,r)^{1/2} &\leq \delta^{-1} \int_{|t-\xi| \leq \frac{1}{m\delta}} \left[\sum_{n=m}^{2m-1} \Phi_n(t,r)^{1/2} \right] dt \\
&\leq \frac{C}{r^{1/2}},
\end{aligned}$$

by Lemma 5.1(c). Here we need $\frac{1}{\delta} \leq \frac{r}{4}$.

□

We need:

Definition 5.3. For a given (ξ,r), we say a positive integer n is (ξ,r) **bad** if

$$I_n(\xi,r) \geq r^{-1/2}.$$

We denote by $\mathcal{B}(\xi,r)$ the set of all (ξ,r) bad integers, and for $k \geq 1$, we let

$$\mathcal{D}_k(\xi) = \bigcup_{j=k}^{\infty} \mathcal{B}\left(\xi, 2^j\right). \tag{5.10}$$

Lemma 5.4. *Let δ be as in Lemma 5.2. For $n \geq 1$ and $k \geq \log_2(4/\delta)$,*

$$\frac{1}{n}\#\left(\mathcal{D}_k(\xi)\cap[1,n]\right) \leq C_2 2^{-k/4}. \tag{5.11}$$

Here C_2 is independent of n and k.

Proof. From Lemma 5.2(b), provided $r \geq 4/\delta$, we have

$$\begin{aligned} C_1 r^{-1/2} &\geq \frac{1}{m}\sum_{n=m}^{2m-1} I_n(\xi,r)^{1/2} \\ &\geq \frac{1}{m} r^{-1/4}\#\left(\mathcal{B}(\xi,r)\cap[m,2m-1]\right). \end{aligned}$$

That is,

$$\#\left(\mathcal{B}(\xi,r)\cap[m,2m-1]\right) \leq C_1 m r^{-1/4}.$$

Then for $\ell \geq 1$, and $2^k \geq 4/\delta$,

$$\begin{aligned} \#\left(\mathcal{D}_k(\xi)\cap\left[2^\ell,2^{\ell+1}-1\right]\right) &\leq \sum_{j=k}^{\infty}\#\left(\mathcal{B}(\xi,2^j)\cap\left[2^\ell,2^{\ell+1}-1\right]\right) \\ &\leq C_1\sum_{j=k}^{\infty} 2^\ell/2^{j/4} \leq C_2 2^{\ell-k/4}. \end{aligned}$$

Then

$$\begin{aligned} \#\left(\mathcal{D}_k(\xi)\cap[1,n]\right) &\leq \sum_{\ell=0}^{[\log_2 n]+1}\#\left(\mathcal{D}_k(\xi)\cap\left[2^\ell,2^{\ell+1}-1\right]\right) \\ &\leq C_2\sum_{\ell=0}^{[\log_2 n]+1} 2^{\ell-k/4} \leq C_3 n 2^{-k/4}. \end{aligned}$$

□

We need a characterization of the sinc kernel:

Lemma 5.5. *Let $\sigma > 0$ and $F : \mathbb{C}^2 \to \mathbb{C}$ be an entire function in each variable with the following properties:*

(i) *For each real a, $F(a,\cdot)$ is an entire function of exponential type σ, that is real on the real axis, with*

$$\int_{-\infty}^{\infty}|F(a,s)|^2\,ds < \infty.$$

(ii) *Let $\rho_0 = 0$ and $F(0,\cdot)$have distinct simple zeros $\{\rho_j\}_{j\in\mathbb{Z}\setminus\{0\}}$, ordered in increasing size, and no other zeros. Assume that for $j \neq 0$, $F(\rho_j,\cdot)$ has zeros $\{\rho_k\}_{k\in\mathbb{Z}\setminus\{j\}}$ and no other zeros.*

(iii) *There exists* $C > 0$ *such that for all real* t,

$$F(t,t) \geq C,$$

and $F(0,0) = 1$.

(iv) *For all complex* a, b,

$$F(a,b) = \int_{-\infty}^{\infty} F(a,s)\, F(b,s)\, ds.$$

Then for all complex u, v,

$$F(u,v) = \frac{\sin \pi (u-v)}{\pi (u-v)}.$$

Proof. See Theorem 6.1 in [13]. $\square$

Lemma 5.6. *Let* δ *be as in Lemma* 5.2. *Let* $k \geq \log_2(4/\delta)$. *Let* μ *and* $\xi \in J$ *be as in Theorem* 1.3. *Then uniformly for* u, v*in compact subsets of* $\mathbb{C}$,

$$\lim_{n\to\infty, n\notin \mathcal{D}_k(\xi)} f_n(u,v,\xi) = \frac{\sin \pi (u-v)}{\pi (u-v)}.$$

Proof. We know that $\{f_n(\cdot,\cdot,\xi)\}_{n\geq 1}$ is a normal family. Suppose that $\mathcal{S}$ is a subsequence of positive integers that does not intersect $\mathcal{D}_k(\xi)$. By passing to a further subsequence (and keeping the same notation for the sequence), we can assume that $f_n \to f$ as $n \to \infty$ through $\mathcal{S}$, uniformly in compact subsets of $\mathbb{C}^2$. Now if $n \in \mathcal{S}$, then $n \notin \mathcal{B}(\xi, 2^j)$ for all $j \geq k$. It follows that for fixed such j,

$$I_n(\xi, 2^j) < 2^{-j/2}.$$

That is, taking account of (1.17) and the uniform boundedness above and below of $f_n(u,u,\xi)$, we have for each fixed $s \geq 2^j \frac{\tilde{K}_n(\xi,\xi)}{n}$, and hence for $s \geq \frac{4}{\delta} 2^j$,

$$\int_{-1}^{1}\int_{-1}^{1} \left| f_n(u,v,\xi) - \int_{-s}^{s} f_n(u,t,\xi)\, f_n(v,t,\xi)\, \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right| du\, dv$$
$$\leq C 2^{-j/2}.$$

The constant C is independent of both n and j. Letting $n \to \infty$ through $\mathcal{S}$, and using that ξ is a Lebesgue point, gives

$$\int_{-1}^{1}\int_{-1}^{1} \left| f(u,v,\xi) - \int_{-s}^{s} f(u,t,\xi)\, f(v,t,\xi)\, dt \right| du\, dv \leq C 2^{-j/2}.$$

Letting first $s \to \infty$, and then $j \to \infty$, we obtain

$$\int_{-1}^{1}\int_{-1}^{1} \left| f(u,v,\xi) - \int_{-\infty}^{\infty} f(u,t,\xi)\, f(v,t,\xi)\, dt \right| du\, dv = 0.$$

This is permissible in view of (4.3). Thus, for a.e. $(u,v) \in [-1,1] \times [-1,1]$, we have

$$f(u,v,\xi) = \int_{-\infty}^{\infty} f(u,t,\xi)\, f(v,t,\xi)\, dt.$$

As both sides are entire, this equation holds for all complex u,v. Then Lemma 5.5 shows that

$$f(u,v,\xi) = \frac{\sin \pi (u-v)}{\pi (u-v)}.$$

Indeed, all the remaining hypotheses of Lemma 5.5 were proved in Theorem 4.1. As every subsequence of positive integers outside $\mathcal{D}_k(\xi)$ has a subsequence converging locally uniformly to the sinc kernel, it follows that the full sequence outside $\mathcal{D}_k(\xi)$ converges to the sinc kernel. □

Lemma 5.4 shows that $\mathcal{D}_k(\xi)$ is a set of density at most $C2^{-k/4}$. This is small for large k, but not 0. We now turn to the

Proof of Theorem 1.3. Let δ be as in Lemma 5.2. Given $k \geq \log_2(4/\delta)$, and $r > 0$, there exists n_k such that for $n \geq n_k$ and $n \notin \mathcal{D}_k(\xi)$,

$$\sup_{|u|,|v|\leq r} \left| f_n(u,v,\xi) - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| \leq \frac{1}{k}. \tag{5.12}$$

Moreover, because of the uniform boundedness proved in Theorem 4.1, there exists $C(r)$ depending only on r, such that for $n \geq 1$,

$$\sup_{|u|,|v|\leq r} \left| f_n(u,v,\xi) - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| \leq C(r). \tag{5.13}$$

Then

$$\begin{aligned} &\frac{1}{m}\sum_{n=1}^{m} \sup_{|u|,|v|\leq r} \left| f_n(u,v,\xi) - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| \\ &\leq \frac{1}{m}\left(\sum_{n_k \leq n \leq m, n \notin \mathcal{D}_k(\xi)} \frac{1}{k} + \sum_{n \leq n_k \text{ or } n_k < n \leq m, n \in \mathcal{D}_k(\xi)} C(r) \right) \\ &\leq \frac{1}{k} + \frac{n_k}{m} C(r) + \left(C_2 2^{-k/4} \right) C(r). \end{aligned}$$

Here we have used Lemma 5.4, and it is crucial that both $C(r)$ and C_2 are independent of k,m. We first take lim sup's as $m \to \infty$, and then let $k \to \infty$, to obtain (1.7). □

Remark. The proof actually shows that if $\Psi : [0,\infty) \to [0,\infty)$ is an increasing function with $\lim_{t\to 0+} \Psi(t) = 0$, then

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \Psi\left(\sup_{|u|,|v|\leq r} \left| f_n(u,v,\xi) - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| \right) = 0.$$

Proof of Theorem 1.2. The hypotheses of Theorem 1.2 were shown to imply those of Theorem 1.3 in Lemmas 3.1 and 3.2. □

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu