Which Weights on $\mathbb{R}$ admit Jackson Theorems?

D.S. Lubinsky*
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
lubinsky@math.gatech.edu

Abstract

Let $W : \mathbb{R} \to (0, \infty)$ be continuous. Does W admit a classical Jackson Theorem? That is, does there exist a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 such that for $1 \leq p \leq \infty$,

$$\inf_{\deg(P) \leq n} \|(f - P)W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})}$$

for all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite? We show that such a theorem is true iff both

$$\lim_{x \to \infty} W(x) \int_0^x W^{-1} = 0$$

and

$$\lim_{x \to \infty} \left(\sup_{[0,x]} W^{-1} \right) \int_x^{\infty} W = 0,$$

with analogous limits as $x \to -\infty$. In particular $W(x) = \exp(-|x|)$ does not admit a Jackson theorem of this type.

*Research supported by NSF grant DMS-0400446

We also construct weights that admit an L_1 but not an L_∞ Jackson theorem (or conversely).
Keywords: Weighted approximation, polynomial approximation, Jackson-Bernstein theorems.

1 Introduction

Let $W : \mathbb{R} \to (0, \infty)$. In about 1910, S.N. Bernstein posed a problem that became known as Bernstein's approximation problem. When are the polynomials dense in the weighted space generated by W? That is, when is it true that for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} (fW)(x) = 0,$$

there exist a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \| (f - P_n) W\|_{L_\infty(\mathbb{R})} = 0?$$

This problem was resolved independently by Pollard, Mergelyan and Achieser in the 1950's.

For example [12, p. 153] Mergelyan showed that there is a positive answer to Bernstein's problem iff

$$\int_{-\infty}^{\infty} \frac{\log \Omega(t)}{1+t^2} dt = \infty,$$

where

$$\Omega(z) = \sup \left\{ |P(z)| : P \text{ a polynomial and } \sup_{t\in\mathbb{R}} \frac{|P(t) W(t)|}{\sqrt{1+t^2}} \leq 1 \right\}.$$

If $W \leq 1$ and is even, and $\ln 1/W(e^x)$ is even and convex, a simpler necessary and sufficient condition for density of the polynomials is [12, p. 170]

$$\int_0^{\infty} \frac{\ln 1/W(x)}{1+x^2} dx = \infty.$$

In particular, for

$$W_\alpha(x) = \exp\left(- |x|^\alpha\right), \tag{1}$$

the polynomials are dense iff $\alpha \geq 1$.

In the 1950's the search began for a quantitative form of Bernstein's Theorem. The first efforts in this direction were due to Dzrbasjan. In the 1960's and 1970's, Freud and Nevai made major strides in this topic [19], but efforts continue to this day, with many researchers involved. One obvious question was whether there are analogues of classical theorems of Jackson and Bernstein, dating back to the early 20th century, for the unweighted case. The latter independently proved that

$$\inf_{\deg(P)\leq n} \|f - P\|_{L_\infty[-1,1]} \leq \frac{C}{n} \|f'\|_{L_\infty[-1,1]},$$

with C independent of f and n, and the inf being over (algebraic) polynomials of degree at most n. The rate is best possible amongst absolutely continuous functions f on $[-1,1]$ whose derivative is bounded. Jackson also obtained general results involving moduli of continuity while Bernstein obtained tight forward and converse theorems for trigonometric polynomials. For ordinary polynomials, many of the problems were only resolved in the 1980's [5], [8].

For the weights W_α, where $\alpha > 1$, it is known that if $1 \leq p \leq \infty$,

$$\inf_{\deg(P)\leq n} \|(f - P) W_\alpha\|_{L_p(\mathbb{R})} \leq C n^{-1+\frac{1}{\alpha}} \|f' W_\alpha\|_{L_p(\mathbb{R})}, \tag{2}$$

with C independent of f and n [8, p. 185, (11.3.5)], [17, p. 81, (4.1.5a)]. Again the rate is best possible for the class of absolutely continuous functions f with $\|f' W_\alpha\|_{L_p(\mathbb{R})}$ finite. Freud proved these for $\alpha \geq 2$, and later E. Levin and the author provided the necessary technical estimates to extend this to all $\alpha > 1$. More general Jackson type theorems involving weighted moduli of continuity for various classes of weights were proved in [4], [6], [8], [9], [15], [17].

One particularly interesting case is $\alpha = 1$, namely $W_1(x) = \exp(-|x|)$. For this weight Bernstein's approximation problem has a positive solution, that is, the polynomials are dense. However, (2) suggests that there may not be an analogue of a Jackson theorem, because $n^{-1+\frac{1}{\alpha}}$ has limit 1. As a contra-indication, a result of Freud, Giroux and Rahman [10, p. 360] for L_1 suggests that possibly (2) is true with $n^{-1+\frac{1}{\alpha}}$ replaced by $\frac{1}{\log n}$. They used the modulus of continuity

$$\omega(f,\varepsilon)=\sup_{|h|\leq\varepsilon}\int_{-\infty}^{\infty}|(fW_1)(x+h)-(fW_1)(x)|\,dx+\varepsilon\int_{-\infty}^{\infty}|fW_1|$$

and proved that

$$\inf_{\deg(P)\leq n}\|(f-P)W_1\|_{L_1(\mathbb{R})}\leq C\left[\omega\left(f,\frac{1}{\log n}\right)+\int_{|x|\geq\sqrt{n}}|fW_1|(x)\,dx\right]. \quad (3)$$

Here C is independent of f and n. Ditzian, the author, Nevai and Totik later extended this result [7] to a characterization in L_1. Only recently has it been possible to establish the analogous results in L_p, $p>1$ [16].

One of the conclusions of this paper is that there is no Jackson type theorem like (2) for the weight W_1. More generally we answer the question: which weights admit a Jackson type theorem, of the form (2), with $\left\{n^{-1+1/\alpha}\right\}_{n=1}^{\infty}$ replaced by some sequence $\{\eta_n\}_{n=1}^{\infty}$ with limit 0? We give our characterization in the following theorem:

Theorem 1.1 *Let $W:\mathbb{R}\to(0,\infty)$ be continuous. The following are equivalent:*
(a) There exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 and with the following property. For each $1\leq p\leq\infty$, and for all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, we have

$$\inf_{\deg(P)\leq n}\|(f-P)W\|_{L_p(\mathbb{R})}\leq\eta_n\|f'W\|_{L_p(\mathbb{R})},\qquad n\geq 1. \quad (4)$$

(b) Both

$$\lim_{x\to\infty}W(x)\int_0^x W^{-1}=0 \quad (5)$$

and

$$\lim_{x\to\infty}\left(\min_{[0,x]}W\right)^{-1}\int_x^{\infty}W=0 \quad (6)$$

with analogous limits as $x\to-\infty$.

Corollary 1.2 *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous, with* $W = e^{-Q}$, *where* $Q(x)$ *is differentiable for large* $|x|$, *and*

$$\lim_{x\to\infty} Q'(x) = \infty \text{ and } \lim_{x\to-\infty} Q'(x) = -\infty. \tag{7}$$

Then there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *such that for each* $1 \leq p \leq \infty$, *and for all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbb{R})}$ *finite, we have (4).*

Corollary 1.3 *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous, with* $W = e^{-Q}$, *where* $Q(x)$ *is differentiable for large* $|x|$, *and* $Q'(x)$ *is bounded for large* $|x|$. *Then for both* $p = 1$ *and* $p = \infty$, *there does not exist a sequence* $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *satisfying (4) for all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbb{R})}$ *finite.*

Remarks

(a) The first condition (5) is necessary and sufficient for an L_∞ Jackson theorem, while the second (6) is necessary and sufficient for an L_1 Jackson theorem.
(b) For the case where Q is convex, and $p < \infty$, Corollary 1.2 was proved in [11]. It was used to relate asymptotic behavior of Sobolev and ordinary orthogonal polynomials. Our Corollary 1.2 allows one to relax the condition of convexity in Theorem 1.3 in [11].
(c) Of course $\{\eta_n\}_{n=1}^{\infty}$ may decay arbitrarily slowly to 0, though they are independent of p. The proof of Theorem 1.1 also shows that (5) and (6) are necessary even if we allow a different sequence $\{\eta_n\}_{n=1}^{\infty}$ for each different p.
(d) It may be possible that there is a modified Jackson theorem valid whenever the polynomials are dense in the relevant weighted space. The form we believe likely is

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})} + \|fW\|_{L_p(|x|\geq\xi_n)},$$

where $\{\xi_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers with limit ∞, independent of the particular function f. However, it cannot be established by the methods we use here.

(e) An equivalent way to state Theorem 1.1 is as a Jackson-Favard inequality

$$\inf_{\deg(P)\leq n} \| (f-P)\, W\|_{L_p(\mathbb{R})} \leq \eta_n \inf_{\deg(P)\leq n-1} \| (f'-P)\, W\|_{L_p(\mathbb{R})}.$$

An obvious question is the independence of the conditions (5) and (6). Does either imply the other? In fact they are independent. Moreover, we shall exhibit weights satisfying one but not the other, and also admitting an L_1 Jackson theorem but not an L_∞ Jackson theorem (or conversely). This is a highly unusual occurrence in weighted approximation — in fact the first occurrence of this phenomenon known to this author. Density of polynomials, and the degree of approximation is almost invariably the same for any L_p space (suitably weighted of course). We prove:

Theorem 1.4 *(a) There exists* $W : \mathbb{R} \rightarrow (0, \infty)$ *with*

$$1 \leq W(x) / \exp\left(-x^2\right) \leq 2\,(1+|x|)\,, \qquad x \in \mathbb{R}, \tag{8}$$

admitting an L_∞ *Jackson theorem , but not an* L_1 *Jackson theorem. That is, for* $p = \infty$*, there exist* $\{\eta_n\}_{n=1}^{\infty}$ *with limit* 0 *at* ∞ *satisfying (4), but there does not exist such a sequence for* $p = 1$*.*
(b) There exists $W : \mathbb{R} \rightarrow (0, \infty)$ *with*

$$1 \geq W(x) / \exp\left(-x^2\right) \geq 2/\,(1+|x|)\,, \qquad x \in \mathbb{R}, \tag{9}$$

admitting an L_1 *Jackson theorem, but not an* L_∞ *Jackson theorem. That is, for* $p = 1$*, there exist* $\{\eta_n\}_{n=1}^{\infty}$ *with limit* 0 *at* ∞ *satisfying (4), but there does not exist such a sequence for* $p = \infty$*.*

We note that the weights in Theorem 1.4 are equal to the Hermite weight $W_2(x) = \exp(-x^2)$ "most" of the time, with spikes upwards or downwards in small intervals. The weights we construct are not decreasing in $(0, \infty)$, though they can be made infinitely differentiable. We expect that with more work one can construct decreasing W in $(0, \infty)$ still satisfying the conclusions of Theorem 1.4.

This paper is organised as follows: we prove restricted range inequalities in the next section, and an estimate for the "tails"

$\|fW\|_{L_p(|x|\geq\lambda)}$ in Section 3. In Section 4, we prove Theorem 1.1 and Corollaries 1.2 and 1.3. In Section 5, we prove Theorem 1.4. Throughout $C, C_1, C_2, \ldots$ denote constants independent of n and x and polynomials P of degree $\leq n$. The same symbol may denote different constants in different occurrences.

2 Restricted range inequalities

Restricted range (or infinite-finite range) inequalities involve bounding the norm of a weighted polynomial over the whole real line in terms of the norm over a smaller interval depending only on the degree of the polynomial. They play a major role in analysis of weighted polynomials, orthogonal polynomials, and weighted potential theory. A key example is the Mhaskar-Saff identity [18]

$$\|PW_\alpha\|_{L_\infty(\mathbb{R})} = \|PW_\alpha\|_{L_\infty(-C_\alpha n^{1/\alpha}, C_\alpha n^{1/\alpha})},$$

valid for all polynomials of degree $\leq n$ and $\alpha > 0$. The constant C_α is "smallest possible" and depends only on α. For further orientation on this topic see [13], [17], [21]. Unfortunately, although there are restricted range inequalities for very general weights, none of them are applicable to the weights we use here. In this section we prove two inequalities, that we may apply under the forward and converse hypotheses of Theorem 1.1:

Theorem 2.1 *Let $W : \mathbb{R} \to (0, \infty)$ be continuous and assume that both*

$$\lim_{x\to\infty} W(x) \int_0^x W^{-1} = 0 \tag{10}$$

and

$$\lim_{x\to\infty} \left(\min_{[0,x]} W\right)^{-1} \int_x^\infty W = 0 \tag{11}$$

with analogous limits as $x \to -\infty$. Then there exists an increasing sequence of positive numbers $\{q_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \frac{q_n}{n} = 0, \tag{12}$$

and such that for $1 \leq p \leq \infty, n \geq 1$*, and all polynomials* P *of degree* $\leq n$*,*

$$\|PW\|_{L_p(|x| \geq q_n)} / \|PW\|_{L_p(\mathbb{R})} \leq 2^{-n} C_1, \tag{13}$$

where C_1 *is independent of* n, p, P*.*

Theorem 2.2 *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous,* $1 \leq p \leq \infty$*, and assume that for each* $n \geq 0$*,*

$$\|x^n W(x)\|_{L_p(\mathbb{R})} < \infty. \tag{14}$$

Then there exists an increasing sequence of positive numbers $\{\xi_n\}_{n=1}^{\infty}$ *such that for* $n \geq 1$ *and all polynomials* P *of degree* $\leq n$*,*

$$\|PW\|_{L_p(|x| \geq \xi_n)} \leq C_1 2^{-n} \|PW\|_{L_p(-1,1)}, \tag{15}$$

where C_1 *is independent of* n, p, P*.*

We shall prove one lemma, then Theorem 2.1, and then prove the far easier Theorem 2.2.

Lemma 2.3 *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous.*
(a) Assume that $\delta : [0, \infty) \to (0, \infty)$ *is a decreasing function with limit* 0 *at* ∞ *such that for* $x > 0$*,*

$$W(x) \int_0^x W^{-1} \leq \delta(x). \tag{16}$$

For large enough n*, let* $\ell = \ell(n)$ *denote the smallest integer satisfying*

$$\frac{\ell}{\delta(\ell)^{1/2}} \geq 4n. \tag{17}$$

Then for large enough n*,*

$$\sup_{x \geq \ell(n)} x^n W(x) \leq \delta(\ell)^{1/2} \sup_{x \in \left[\frac{1}{2}\ell(n), \ell(n)\right]} x^n W(x). \tag{18}$$

Moreover,

$$\lim_{n\to\infty}\frac{\ell(n)}{n}=0. \tag{19}$$

(b) In addition, assume that $\varepsilon:[0,\infty)\to(0,\infty)$ *is a decreasing function, such that for* $x>0$*,*

$$\left(\min_{[0,x]}W\right)^{-1}\int_x^\infty W\leq\varepsilon(x). \tag{20}$$

Then for $x\geq 2u>0$*,*

$$W^{-1}(u)\,W(x)\leq\frac{4\delta(0)\,\varepsilon(0)}{x^2}. \tag{21}$$

Proof

(a) Note first that (19) follows easily from the fact that δ has limit 0 at ∞. Fix n and let $\ell=\ell(n)$. If $x\geq\ell$,

$$\int_{\frac{1}{2}\ell}^x t^n dt=\frac{x^{n+1}}{n+1}\left[1-\left(\frac{\ell}{2x}\right)^{n+1}\right]\geq\frac{x^{n+1}}{2(n+1)}.$$

Then

$$\begin{aligned}
x^nW(x) &\leq \frac{2(n+1)}{x}\left(\int_{\frac{1}{2}\ell}^x t^n dt\right)W(x)\\
&\leq \frac{2(n+1)}{x}\left(\sup_{t\in\left[\frac{1}{2}\ell,x\right]}t^nW(t)\right)W(x)\int_{\frac{1}{2}\ell}^x W^{-1}(t)\,dt\\
&\leq \frac{4n}{\ell}\delta(\ell)\sup_{t\in[\frac{1}{2}\ell,\infty)}t^nW(t),
\end{aligned}$$

by (16). Using our choice (17) of ℓ, we continue this as

$$\sup_{x\in[\ell,\infty)}x^nW(x)\leq\delta(\ell)^{1/2}\sup_{t\in[\frac{1}{2}\ell,\infty)}t^nW(t).$$

If n is so large that $\delta(\ell)<1$, this last inequality implies that $t^nW(t)$ cannot attain its sup in $[\frac{1}{2}\ell,\infty)$ in $[\ell,\infty)$. Hence

$$\sup_{x\in[\ell,\infty)}x^nW(x)\leq\delta(\ell)^{1/2}\sup_{t\in[\frac{1}{2}\ell,\infty,\ell)}t^nW(t)$$

so we have (18).
(b) By Cauchy-Schwarz, for $x > 0$,

$$\frac{x}{2} \leq \left(\int_{x/2}^{x} W \right)^{1/2} \left(\int_{x/2}^{x} W^{-1} \right)^{1/2}.$$

Then

$$W(x) \leq \frac{\delta(x)}{\int_0^x W^{-1}} \leq \frac{\delta(x)}{\int_{x/2}^x W^{-1}} \leq \frac{\delta(x)}{(x/2)^2} \int_{x/2}^{x} W,$$

so

$$W^{-1}(u) W(x) \leq \frac{\delta(x) \varepsilon(u)}{(x/2)^2} \frac{\int_{x/2}^x W}{\int_u^\infty W} \leq \frac{4\delta(x) \varepsilon(u)}{x^2},$$

since $u \leq x/2$. □

We note that (18) implies for each $n \geq 0$,

$$\lim_{x \to \infty} x^n W(x) = 0$$

and hence for every polynomial P,

$$\lim_{x \to \infty} P(x) W(x) = 0.$$

Proof of Theorem 2.1
Our approach is similar to that in [14]. Let P be a polynomial of degree $k \leq n$, say

$$P(z) = c \prod_{j=1}^{k} (z - x_j).$$

We assume $c \neq 0$, and split the zeros into “small" and “large" zeros: we assume that

$$\begin{aligned} |x_j| &\leq \ell(2n), & j &\leq i; \\ |x_j| &> \ell(2n), & j &> i. \end{aligned}$$

For $|u| \leq \frac{1}{2}\ell(2n), x \geq \ell(2n)$ and $i < j \leq k$,

$$\left| \frac{x - x_j}{u - x_j} \right| \leq \frac{1 + x/|x_j|}{1 - |u|/|x_j|} \leq 2\left(1 + \frac{x}{\ell(2n)}\right) \leq 4\frac{x}{\ell(2n)}.$$

Then for such x, u

$$\left|\frac{P(x)}{P(u)}\right| \leq \left(\prod_{j=1}^{i} \frac{2x}{|u-x_j|}\right)\left(4\frac{x}{\ell(2n)}\right)^{k-i}.$$

We now apply a famous lemma of Cartan:

$$\left|\prod_{j=1}^{i}(u-x_j)\right| \geq \varepsilon^i$$

for u outside a set of linear measure at most $4e\varepsilon$ [1, p. 175], [3, p. 350]. Choosing $\varepsilon = \frac{\ell(2n)}{100}$, we obtain

$$\left|\frac{P(x)}{P(u)}\right| \leq \left(\frac{200x}{\ell(2n)}\right)^k \leq \left(\frac{200x}{\ell(2n)}\right)^n,$$

for $x \geq \ell(2n)$, $u \in \left[0, \frac{1}{2}\ell(2n)\right] \setminus \mathcal{S}$, where

$$meas(\mathcal{S}) \leq \frac{4e}{100}\ell(2n) < \frac{1}{8}\ell(2n).$$

Here and in the sequel, $meas$ denotes linear Lebesgue measure. Then for such u,

$$\int_{400\ell(2n)}^{\infty} |P(x)W(x)|^p\,dx \leq \left(\frac{200}{\ell(2n)}\right)^{np} |P(u)|^p \int_{400\ell(2n)}^{\infty} x^{np}W^p(x)\,dx. \qquad (22)$$

Next

$$\int_{400\ell(2n)}^{\infty} x^{np}W^p(x)\,dx$$
$$\leq \left(\sup_{x\geq 400\ell(2n)} x^{2n}W(x)\right)^p (400\ell(2n))^{-np+2p} \int_{400\ell(2n)}^{\infty} x^{-2p}dx$$
$$\leq \left(\delta(\ell(2n))^{1/2} \sup_{x\in\left[\frac{1}{2}\ell(2n),\ell(2n)\right]} x^{2n}W(x)\right)^p (400\ell(2n))^{-np+1}$$

$$\leq \left(\delta\left(\ell\left(2n\right)\right)^{1/2} \sup_{x\in\left[\frac{1}{2}\ell(2n),\ell(2n)\right]} W\left(x\right)\right)^{p} 400^{-np+1}\ell\left(2n\right)^{np+1},$$

by Lemma 2.3(a). Moreover, $\left[0,\frac{1}{4}\ell\left(2n\right)\right]\setminus\mathcal{S}$ has measure at least $\frac{1}{8}\ell\left(2n\right)$, so we may choose u in this set such that

$$\begin{aligned}\left|P\left(u\right)\right|^{p} &\leq \frac{W\left(u\right)^{-p}}{meas\left(\left[0,\frac{1}{4}\ell\left(2n\right)\right]\setminus\mathcal{S}\right)}\int_{\left[0,\frac{1}{4}\ell(2n)\right]\setminus\mathcal{S}}\left|PW\right|^{p} \\ &\leq \frac{8W\left(u\right)^{-p}}{\ell\left(2n\right)}\int_{\mathbb{R}}\left|PW\right|^{p}.\end{aligned}$$

Finally, from Lemma 2.3(b), for $x \in \left[\frac{1}{2}\ell\left(2n\right),\ell\left(2n\right)\right]$, and $u \in \left[0,\frac{1}{4}\ell\left(2n\right)\right]\setminus\mathcal{S}$,

$$W^{-1}\left(u\right)W\left(x\right) \leq \frac{16\delta\left(0\right)\varepsilon\left(0\right)}{\ell\left(2n\right)^{2}}.$$

Putting these last 3 estimates in (22) (and dropping a factor of $\delta\left(\ell\left(2n\right)\right)^{p/2}\ell\left(n\right)^{-2p+1}=o\left(1\right)$) gives

$$\int_{400\ell(2n)}^{\infty}\left|P\left(x\right)W\left(x\right)\right|^{p}dx/\int_{\mathbb{R}}\left|PW\right|^{p} \leq 2^{-np}\ell\left(2n\right)^{-2p}C^{p},$$

where C is independent of n,p,P. A similar inequality holds over $\left(-\infty,-400\ell\left(2n\right)\right)$ and so, taking pth roots,

$$\left\|PW\right\|_{L_{p}(|x|\geq 400\ell(2n))}/\left\|PW\right\|_{L_{p}(\mathbb{R})} \leq 2^{-n}C_{1},$$

with C_1 independent of n,p,P. Letting $p\to\infty$ gives the result for $p=\infty$ also. Thus we may take

$$q_{n}=400\ell\left(2n\right)$$

in Theorem 2.1. Note that (12) follows from (19) in Lemma 2.3. Although this choice does not guarantee monotonicity of $\left\{q_{n}\right\}_{n=1}^{\infty}$, we can easily modify the sequence to be monotone increasing. $\square$

Proof of Theorem 2.2
Write

$$P(x) = \sum_{j=0}^{n} a_j x^j.$$

Here by Bernstein's inequality [5, Cor. 1.2, p. 98], followed by Nikolskii's inequality [5, Thm. 2.6, p. 102],

$$\begin{aligned} |a_j| &= \left| \frac{P^{(j)}(0)}{j!} \right| \leq \frac{n^j}{j!} \|P\|_{L_\infty[-1,1]} \\ &\leq \frac{n^j}{j!} \left((p+1) n^2 \right)^{1/p} \|P\|_{L_p[-1,1]} \\ &\leq \frac{n^j}{j!} e n^2 \|W^{-1}\|_{L_\infty[-1,1]} \|PW\|_{L_p[-1,1]}. \end{aligned}$$

Then for $\lambda \geq n$,

$$\begin{aligned} &\|PW\|_{L_p(|x| \geq \lambda)} \\ &\quad \leq e n^2 \|W^{-1}\|_{L_\infty[-1,1]} \|PW\|_{L_p[-1,1]} \sum_{j=0}^{n} \frac{n^j}{j!} \|x^j W(x)\|_{L_p(|x| \geq \lambda)} \\ &\quad \leq e n^2 \|W^{-1}\|_{L_\infty[-1,1]} \|PW\|_{L_p[-1,1]} \|x^{2n} W(x)\|_{L_p(|x| \geq \lambda)} \sum_{j=0}^{n} \frac{n^{2j-2n}}{j!} \\ &\quad \leq C \|PW\|_{L_p[-1,1]} \|x^{2n} W(x)\|_{L_p(|x| \geq \lambda)}, \end{aligned}$$

where C depends only on W (not on n, p, P). The finiteness (14) of the norms for all monomials, shows that for large enough λ,

$$C \|x^{2n} W(x)\|_{L_p(|x| \geq \lambda)} \leq 2^{-n}.$$

We may choose ξ_n to be this λ. By an obvious process, we may also ensure that $\{\xi_n\}_{n=1}^{\infty}$ is increasing. $\square$

3 A Bound for Tails

The result of this section is:

Theorem 3.1 *Assume that $W:\mathbb{R}\to(0,\infty)$ is continuous.*
(a) Assume W satisfies (5) and (6), with analogous limits at $-\infty$. Then there exists a decreasing positive function $\eta:[0,\infty)\to(0,\infty)$ with limit 0 at ∞ such that for $1\le p\le\infty$ and $\lambda\ge 0$,

$$\|fW\|_{L_p(\mathbb{R}\setminus[-\lambda,\lambda])}\le\eta(\lambda)\|f'W\|_{L_p(\mathbb{R})} \tag{23}$$

for all absolutely continuous functions $f:\mathbb{R}\to\mathbb{R}$ for which $f(0)=0$ and the right-hand side is finite.
(b) Conversely assume that (23) holds for $p=1$ and for $p=\infty$, for large enough λ. Then the limits (5) and (6) in Theorem 1.1 are valid, with analogous limits at $-\infty$.

Note that (5) is alone necessary and sufficient for the conclusion of Theorem 3.1 to hold for $p=\infty$, and (6) alone is necessary and sufficient for the conclusion to hold for $p=1$.

We shall prove Theorem 3.1(a) for $p=\infty$, then $p=1$, and then use interpolation to do the case $1<p<\infty$. The converse will be proved at the end of this section. The ideas of proof of Theorem 3.1(a) go back at least to G. Freud, and are elegantly presented, in the setting of Freud weights, in Mhaskar's monograph [17]. They are similar to the ideas of proof of Hardy's inequality. Note that there is no simple analogue of this result for $p<1$.

Proof of Theorem 3.1(a) for $p=\infty$
For $x>0$,

$$(fW)(x)=W(x)\int_0^x(f'W)(t)W^{-1}(t)\,dt. \tag{24}$$

Hence for $\lambda\ge 0$,

$$\sup_{x\ge\lambda}|fW|(x)\le\|f'W\|_{L_\infty(\mathbb{R})}\sup_{x\ge\lambda}W(x)\int_0^x W^{-1}(t)\,dt.$$

A similar inequality holds for $x\le-\lambda$. Now apply the limit (5).□

Proof of Theorem 3.1(a) for $p=1$
For $\lambda\ge 0$,

$$\int_\lambda^\infty|fW|(x)\,dx=\int_\lambda^\infty\left|\left(\int_0^\lambda+\int_\lambda^x\right)(f'W)(t)W^{-1}(t)\,dt\right|W(x)\,dx$$

$$
\begin{aligned}
&\leq \int_0^\lambda |f'W|(t)\,dt \left(\min_{[0,\lambda]} W\right)^{-1} \int_\lambda^\infty W(x)\,dx \\
&\quad + \int_\lambda^\infty |f'W|(t)\,W^{-1}(t) \left(\int_t^\infty W(x)\,dx\right) dt \\
&\leq \left(\int_0^\lambda |f'W|(t)\,dt\right) \left(\min_{[0,\lambda]} W\right)^{-1} \int_\lambda^\infty W(x)\,dx \\
&\quad + \left(\int_\lambda^\infty |f'W|(t)\,dt\right) \sup_{t\geq\lambda} \left[W^{-1}(t) \left(\int_t^\infty W(x)\,dx\right)\right]. \quad (25)
\end{aligned}
$$

A similar inequality holds over $(-\infty,-\lambda)$. Now apply the limit (6). □

Proof of Theorem 3.1(a) for $1 < p < \infty$

The above implies that for $p = 1$ and $p = \infty$, there exists a decreasing positive function $\eta : [0,\infty) \to (0,\infty)$ with limit 0 at ∞ such that for $\lambda \geq 0$,

$$\|fW\|_{L_p(\mathbb{R}\setminus[-\lambda,\lambda])} \leq \eta(\lambda)\,\|f'W\|_{L_p(\mathbb{R})},$$

for all absolutely continuous f with $f(0) = 0$ and for which the norm on the right-hand side is finite. Let us fix $\lambda \geq 0$ and set $g = f'$, and define the linear operator

$$L[g](x) = \chi_{\mathbb{R}\setminus[-\lambda,\lambda]}(x) \int_0^x g(t)\,dt,$$

where $\chi_{\mathbb{R}\setminus[-\lambda,\lambda]}(x)$ is the characteristic function of $\mathbb{R}\setminus[-\lambda,\lambda]$. We see that we have proved

$$\|L[g]\,W\|_{L_p(\mathbb{R})} \leq \eta(\lambda)\,\|gW\|_{L_p(\mathbb{R})}$$

for $p = 1,\infty$ and for all measurable g with $gW \in L_p(\mathbb{R})$. The Riesz-Thorin Theorem [2, p. 196] then shows this is true for all $1 < p < \infty$. Substituting back $g = f'$ gives the result as stated. □

Proof of Theorem 3.1(b): the case $p = \infty$

Next, let us fix $\lambda > 0$ and define

$$f(x) = \begin{cases} 0, & x < 0, \\ \int_0^x W^{-1}, & x \in [0,\lambda], \\ \int_0^\lambda W^{-1}, & x > \lambda. \end{cases}$$

Then $f' = W^{-1}$ in $(0, \lambda)$ and $f' = 0$ in $\mathbb{R} \backslash [0, \lambda]$. We see from (23) that

$$\eta(\lambda) = \eta(\lambda) \|f'W\|_{L_\infty(\mathbb{R})} \geq (fW)(\lambda) = W(\lambda) \int_0^\lambda W^{-1}.$$

So we obtain (5). The analogous limit at $-\infty$ is similar. □

Proof of Theorem 3.1(b): the case $p = 1$

First note that if we take f to be an absolutely continuous function with bounded derivative that is 0 at 0 and equal to 1 outside $[-1, 1]$, our hypothesis (23) gives

$$\lim_{\lambda \to \infty} \|W\|_{L_1(\mathbb{R} \backslash [-\lambda, \lambda])} = 0. \tag{26}$$

Next, let us fix $\lambda > 0$, and choose $t_0 \in [0, \lambda]$ such that

$$W(t_0) = \min_{[0,\lambda]} W.$$

For large enough λ, (26) shows that $t_0 > 0$. Choose $\alpha \in \left(0, \frac{t_0}{2}\right)$ and define

$$f(x) = \begin{cases} 0, & x < t_0 - \alpha, \\ \frac{1}{\alpha} \int_{t_0-\alpha}^{x} W^{-1}, & x \in [t_0 - \alpha, t_0], \\ \frac{1}{\alpha} \int_{t_0-\alpha}^{t_0} W^{-1}, & x > t_0. \end{cases}$$

Then $f' = \frac{1}{\alpha} W^{-1}$ in $(t_0 - \alpha, t_0)$ and $f' = 0$ in $\mathbb{R} \backslash [t_0 - \alpha, t_0]$. We see from (23) that

$$\eta(\lambda) = \eta(\lambda) \|f'W\|_{L_1(\mathbb{R})} \geq \|fW\|_{L_1(\lambda, \infty)}.$$

Moreover, as $f' \geq 0$, we see that

$$\begin{aligned} \|fW\|_{L_1(\lambda,\infty)} &= \int_\lambda^\infty \left| \int_{t_0-\alpha}^{t_0} \frac{1}{\alpha} W^{-1}(t)\, dt \right| W(x)\, dx \\ &\geq \left(\max_{[t_0-\alpha, t_0]} W \right)^{-1} \int_\lambda^\infty W(x)\, dx. \end{aligned}$$

Thus

$$\eta(\lambda) \geq \left(\max_{[t_0-\alpha, t_0]} W \right)^{-1} \int_\lambda^\infty W(x)\, dx.$$

Since α may be made arbitrarily small and W is continuous, we obtain

$$\eta(\lambda) \geq W(t_0)^{-1} \int_\lambda^\infty W(x)\,dx = \left(\min_{[0,\lambda]} W\right)^{-1} \int_\lambda^\infty W(x)\,dx.$$

So we obtain (6). The analogous limit at $-\infty$ is similar. □

4 Proof of Theorem 1.1 and its Corollaries

We shall prove the sufficiency part of Theorem 1.1 after two lemmas. Throughout this section, we use special notation. We shall use integers $n \geq 4$ and $1 \leq m \leq \frac{n}{4}$, as well as parameters

$$1 < \lambda \leq \frac{1}{2} q_m, \tag{27}$$

where $\{q_n\}_{n=1}^\infty$ is as in Theorem 2.1. We denote by $\rho(m)$ an increasing function that depends on m and W, while $\sigma(\lambda)$ denotes a function increasing in λ. These functions change in different occurrences. The main feature is that σ is independent of m, n, p and functions f, while ρ is independent of λ, p and functions f. At the end, we choose m to grow slowly enough as a function of n, and then $\lambda \to \infty$ sufficiently slowly.

Lemma 4.1 *Let $W : \mathbb{R} \to (0, \infty)$ be continuous and satisfy (5), (6) with analogous limits at $-\infty$.*
(a) There is an increasing function $\sigma : [0, \infty) \to [0, \infty)$ with the following properties: let $m \geq 1$ and $\lambda \geq 1$. For $1 \leq p \leq \infty$ and all absolutely continuous f with $f'W \in L_p(\mathbb{R})$, there exists a polynomial R_m of degree $\leq m$ such that

$$\|(f - R_m)W\|_{L_p[-2\lambda, 2\lambda]} \leq \frac{\sigma(\lambda)}{m} \|f'W\|_{L_p(\mathbb{R})}. \tag{28}$$

(b) Moreover there is an increasing function $\rho : \mathbb{Z}_+ \to (0, \infty)$ depending only on W such that

$$\|R_m W\|_{L_p(\mathbb{R})} + \|R_m W\|_{L_\infty(\mathbb{R})} \leq \rho(m) \left(\|fW\|_{L_p(\mathbb{R})} + \|f'W\|_{L_p(\mathbb{R})}\right). \tag{29}$$

Proof
(a) By the classical form of Jackson's Theorem [5, (6.4), Theorem 6.2, p. 219], translated from $[-1,1]$ to $[-\lambda,\lambda]$, there exists R_m of degree $\leq m$ with

$$\|f - R_m\|_{L_p[-2\lambda,2\lambda]} \leq \frac{\pi\lambda}{m+1}\|f'\|_{L_p[-2\lambda,2\lambda]}. \tag{30}$$

Then

$$\begin{aligned}&\|\left(f - R_m\right) W\|_{L_p[-2\lambda,2\lambda]} \\ &\quad \leq \frac{\pi\lambda}{m}\|W\|_{L_\infty[-2\lambda,2\lambda]}\|\frac{1}{W}\|_{L_\infty[-2\lambda,2\lambda]}\|f'W\|_{L_p(\mathbb{R})}.\end{aligned}$$

So we can take

$$\sigma\left(\lambda\right) = \pi\lambda\|W\|_{L_\infty[-2\lambda,2\lambda]}\|\frac{1}{W}\|_{L_\infty[-2\lambda,2\lambda]}.$$

(b) By the restricted range inequalities in Theorem 2.1, for some C independent of f,p,m,

$$\begin{aligned}&\|R_mW\|_{L_p(\mathbb{R})} + \|R_mW\|_{L_\infty(\mathbb{R})} \\ &\quad \leq C\left(\|R_mW\|_{L_p[-q_m,q_m]} + \|R_mW\|_{L_\infty[-q_m,q_m]}\right) \\ &\quad \leq C\|R_m\|_{L_\infty[-q_m,q_m]}\left(\|W\|_{L_p(\mathbb{R})} + \|W\|_{L_\infty(\mathbb{R})}\right).\end{aligned}$$

Here simple estimation shows that

$$\|W\|_{L_p(\mathbb{R})} \leq \left(1 + \|W\|_{L_\infty(\mathbb{R})}\right)\left(1 + \|W\|_{L_1(\mathbb{R})}\right).$$

So for some C_1 independent of f,p,m,

$$\|R_mW\|_{L_p(\mathbb{R})} + \|R_mW\|_{L_\infty(\mathbb{R})} \leq C_1\|R_m\|_{L_\infty[-q_m,q_m]}. \tag{31}$$

Recall the Chebyshev inequality [5, Proposition 2.3, p. 101], valid for polynomials P of degree $\leq m$:

$$\left|P\left(x\right)\right| \leq \left|T_m\left(x\right)\right| \|P\|_{L_\infty[-1,1]}, \qquad |x| > 1.$$

Here T_m is the classical Chebyshev polynomial of the first kind. By dilating this, and using the bound

$$\left|T_m\left(x\right)\right| \leq \left(2\left|x\right|\right)^m, \qquad |x| > 1,$$

we obtain

$$\|R_m\|_{L_\infty[-q_m,q_m]} \leq \left(\frac{q_m}{\lambda}\right)^m \|R_m\|_{L_\infty[-2\lambda,2\lambda]}.$$

Using Nikolskii inequalities [5, Theorem 2.6, p. 102], we continue this as

$$\begin{aligned}
&\leq \left(\frac{q_m}{\lambda}\right)^m \left(\frac{(p+1)\,m^2}{2\lambda}\right)^{1/p} \|R_m\|_{L_p[-2\lambda,2\lambda]} \\
&\leq e q_m^m m^{2/p} \left(\|f\|_{L_p[-2\lambda,2\lambda]} + \frac{\pi\lambda}{m}\|f'\|_{L_p[-2\lambda,2\lambda]}\right),
\end{aligned}$$

by the fact that $\lambda \geq 1$ and by (30). Using our bound (27) on λ, we continue this as

$$\begin{aligned}
\|R_m\|_{L_\infty[-q_m,q_m]} &\leq e q_m^m m^2 \|W^{-1}\|_{L_\infty[-q_m,q_m]} \\
&\quad \times (1+\pi q_m)\left(\|fW\|_{L_p[-2\lambda,2\lambda]} + \|f'W\|_{L_p[-2\lambda,2\lambda]}\right).
\end{aligned}$$

Combining this and (31) gives the result, with

$$\rho(m) = C_1 e q_m^m m^2 \|W^{-1}\|_{L_\infty[-q_m,q_m]}(1+\pi q_m). \qquad \square$$

Next we construct polynomials that approximate the characteristic function of $[-\lambda,\lambda]$ in a suitable sense:

Lemma 4.2 *There exists $C > 0$ such that for $n \geq 8$, and for $1 \leq \lambda \leq \frac{1}{2}q_n$, there are nonnegative polynomials V_n of degree $\leq 3n/4$ such that*
(a)

$$|1 - V_n(x)| \leq C\frac{q_n}{n\lambda}, \qquad x \in [-\lambda,\lambda]; \tag{32}$$

$$0 \leq V_n(x) \leq C, \qquad |x| \in [\lambda, 2\lambda]; \tag{33}$$

$$\begin{aligned}
&0 \leq V_n(x) \\
&\leq C\left(\frac{q_n}{n\lambda}\right)^2 \exp\left(-\left[\frac{n}{16q_n}\,||x| - 2\lambda|\right]^{1/2}\right), \quad |x| \in [2\lambda, q_n].
\end{aligned} \tag{34}$$

Here C is independent of n, λ and x.
(b) Moreover, for $1 \leq p \leq \infty$,

$$\|1 - V_n\|_{L_p[-\lambda,\lambda]} \leq C_1 \frac{q_n}{n}, \tag{35}$$

and

$$\|V_n\|_{L_p[-q_n,q_n]\setminus[-2\lambda,2\lambda]} \leq C_1 \frac{q_n}{n}, \tag{36}$$

with C_1 independent of n, λ, and p.

Proof
(a) Let us fix n and set

$$r = \frac{\lambda}{q_n} \in \left[0, \frac{1}{2}\right].$$

Step 1 Approximate via Jackson's Theorem
Define a piecewise linear function $h_n : [-1,1] \to [0,1]$ by

$$h_n(x) = \begin{cases} 1, & |x| \leq r; \\ 2 - \frac{|x|}{r}, & r < |x| \leq 2r; \\ 0, & 2r < |x| \leq 1. \end{cases}$$

Then h_n is continuous with piecewise constant derivative. By Jackson's Theorem [5, (6.4), p. 219], there is a polynomial U_n of degree $\leq \frac{n}{8}$ such that

$$\|h_n - U_n\|_{L_\infty[-1,1]} \leq \frac{\pi}{2(n/8)} \|h_n'\|_{L_\infty[-1,1]} = \frac{4\pi q_n}{n\lambda}. \tag{37}$$

Step 2 Use fast decreasing polynomials to damp near $[-1,1]$
Nevai and Totik [20, Corollary 2, p. 117] showed that there exist polynomials P_n of degree $\leq n/8$ such that for $x \in [-1,1]$,

$$|P_n(x) - sign(x)| \leq C \exp\left(-\left[\frac{n}{8}|x|\right]^{1/2}\right).$$

The constant C is independent of n and x. For $a \in [0,1]$, we set

$$S_{n,a}(x) = \frac{1}{2}\left(1 + P_n\left(\frac{x-a}{1+a}\right)\right).$$

Since $x \in [-1,1] \Rightarrow \frac{x-a}{1+a} \in [-1,1]$, and since (except at $x = a$),

$$\chi_{(a,\infty)}(x) = \frac{1}{2}\left(1 + sign\left(\frac{x-a}{1+a}\right)\right)$$

we have

$$\begin{aligned}\left|S_{n,a}(x) - \chi_{(a,\infty)}(x)\right| &\leq C \exp\left(-\left[\frac{n}{8}\left|\frac{x-a}{1+a}\right|\right]^{1/2}\right) \\ &\leq C \exp\left(-\left[\frac{n}{16}|x-a|\right]^{1/2}\right).\end{aligned}$$

For $a \in [-1, 0)$, we set

$$S_{n,a}(x) = 1 - S_{n,-a}(-x),$$

and see that it admits a similar estimate. Now we set

$$R_n(x) = S_{n,-2r}(x)\left(1 - S_{n,2r}(x)\right),$$

a polynomial of degree $\leq n/2$, and use

$$\chi_{[-2r,2r]}(x) = \chi_{(-2r,\infty)}(x)\left(1 - \chi_{(2r,\infty)}(x)\right)$$

(except at $x = -2r$) to deduce that for $x \in [-1,1]$,

$$\begin{aligned}\left|R_n(x) - \chi_{[-2r,2r]}(x)\right| &\leq C \exp\left(-\left[\frac{n}{16}|x-2r|\right]^{1/2}\right) \\ &\quad + C \exp\left(-\left[\frac{n}{16}|x+2r|\right]^{1/2}\right) \\ &\leq C \exp\left(-\left[\frac{n}{16}\left||x|-2r\right|\right]^{1/2}\right). \qquad (38)\end{aligned}$$

Step 3 Combine U_n and R_n

We set

$$V_n(x) = U_n^2\left(\frac{x}{q_n}\right) R_n^2\left(\frac{x}{q_n}\right),$$

a nonnegative polynomial of degree at most $\frac{3n}{4}$. Note here that

$$\frac{q_n}{n\lambda} \leq \frac{q_n}{n} = o(1).$$

From (37) and (38), we see that for $x \in [-\lambda, \lambda] = [-q_n r, q_n r]$,

$$|1 - V_n(x)| \leq C_1 \left[\frac{q_n}{n\lambda} + \exp\left(-\left[\frac{n\lambda}{16 q_n} \right]^{1/2} \right) \right],$$

with C_1 independent of n, x, λ. Then (32) follows. It is easy to deduce (33). Next in $[2\lambda, q_n]$, (37) and (38) give

$$0 \leq V_n(x) \leq \left(\frac{C_0 q_n}{n\lambda} \right)^2 \exp\left(-\left[\frac{n}{16 q_n} \left| |x| - 2\lambda \right| \right]^{1/2} \right).$$

(b) The first inequality (35) is immediate from (32) and the fact that $\lambda^{-1+1/p} \leq 1$. For the second, we use (34): if $p < \infty$,

$$\begin{aligned} \|V_n\|^p_{L_p[2\lambda, q_n]} &\leq \left(\frac{C_0 q_n}{n\lambda} \right)^{2p} \int_{2\lambda}^{q_n} \exp\left(-\left[\frac{p^2 n}{16 q_n} |x - 2\lambda| \right]^{1/2} \right) dx \\ &\leq \left(\frac{C_0 q_n}{n\lambda} \right)^{2p} \frac{16 q_n}{p^2 n} \int_0^\infty \exp\left(-s^{1/2} \right) ds. \end{aligned}$$

Since $p \geq 1$ and q_n is independent of p and is $o(n)$, we obtain for $p < \infty$ and some C_1 independent of n, p,

$$\|V_n\|_{L_p[2\lambda, q_n]} \leq C_1 \left(\frac{q_n}{n\lambda} \right)^2 \leq C_1 \frac{q_n}{n}.$$

Letting $p \to \infty$ gives the result for $p = \infty$ also. $\square$

Proof of the sufficiency part of Theorem 1.1

We may assume that $f(0) = 0$. (If not, replace f by $f - f(0)$ and absorb the constant $f(0)$ into the approximating polynomial.) We choose $n \geq 8$ and $1 \leq m \leq n/4$, and let λ satisfy $1 \leq \lambda \leq \frac{1}{2} q_m$. Let R_m and V_n denote the polynomials of Lemma 4.1 and 4.2 respectively, and let

$$P_n = R_m V_n.$$

Then P_n is a polynomial of degree $\leq n$, and

$$\begin{aligned}\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})} &\leq \|(f-P_n)W\|_{L_p(\mathbb{R})} \\ &\leq \|(f-P_n)W\|_{L_p[-q_n,q_n]} \\ &\quad + \|fW\|_{L_p(\mathbb{R}\backslash[-q_n,q_n])} + \|P_nW\|_{L_p(\mathbb{R}\backslash[-q_n,q_n])} \\ &\leq \|(f-P_n)W\|_{L_p[-q_n,q_n]} \\ &\quad + \|fW\|_{L_p(\mathbb{R}\backslash[-\lambda,\lambda])} + C2^{-n}\|P_nW\|_{L_p[-q_n,q_n]}, \end{aligned} \tag{39}$$

by Theorem 2.1 and as $q_n > \lambda$. Next,

$$\begin{aligned}\|(f-P_n)W\|_{L_p[-q_n,q_n]} &\leq \|(f-P_n)W\|_{L_p[-\lambda,\lambda]} + \|fW\|_{L_p(\mathbb{R}\backslash[-\lambda,\lambda])} \\ &\quad + \|P_nW\|_{L_p([-q_n,q_n]\backslash[-\lambda,\lambda])} \\ &=: T_1 + T_2 + T_3. \end{aligned} \tag{40}$$

Firstly

$$\begin{aligned} T_1 &\leq \|(f-R_m)W\|_{L_p[-\lambda,\lambda]} + \|R_m(1-V_n)W\|_{L_p[-\lambda,\lambda]} \\ &\leq \|(f-R_m)W\|_{L_p[-\lambda,\lambda]} + \|R_mW\|_{L_\infty[-\lambda,\lambda]}\|1-V_n\|_{L_p[-\lambda,\lambda]} \\ &\leq \frac{\sigma(\lambda)}{m}\|f'W\|_{L_p(\mathbb{R})} + \rho(m)\Big(\|fW\|_{L_p(\mathbb{R})} \\ &\quad + \|f'W\|_{L_p(\mathbb{R})}\Big)\|1-V_n\|_{L_p[-\lambda,\lambda]} \\ &\leq \frac{\sigma(\lambda)}{m}\|f'W\|_{L_p(\mathbb{R})} + C_1\rho(m)\frac{q_n}{n}\|f'W\|_{L_p(\mathbb{R})}, \end{aligned} \tag{41}$$

by Lemmas 4.1, 4.2(b) and Theorem 3.1. Here C_1 is independent of f, m, n, ρ, λ. Since $f(0) = 0$, Theorem 3.1 gives

$$\|fW\|_{L_p(\mathbb{R})} \leq \eta(0)\|f'W\|_{L_p(\mathbb{R})}.$$

The crucial thing in (41) is that σ and ρ are independent of f, n, p. Next, if η is as in Theorem 3.1,

$$T_2 \leq \eta(\lambda)\|f'W\|_{L_p(\mathbb{R})}. \tag{42}$$

Of course this estimate also applies to the middle term in the right-hand side of (39). Next,

$$T_3 \leq \|P_nW\|_{L_p(\lambda\leq|x|\leq 2\lambda)} + \|P_nW\|_{L_p(2\lambda\leq|x|\leq q_n)} =: T_{31} + T_{32}.$$

Here

$$\begin{aligned} T_{31} &\leq \|R_m W\|_{L_p(\lambda\leq|x|\leq 2\lambda)} \|V_n\|_{L_\infty(\lambda\leq|x|\leq 2\lambda)} \\ &\leq C \left(\| \left(R_m - f\right) W\|_{L_p(\lambda\leq|x|\leq 2\lambda)} + \|fW\|_{L_p(\lambda\leq|x|\leq 2\lambda)} \right) \\ &\leq C \left(\frac{\sigma\left(\lambda\right)}{m} \|f'W\|_{L_p(\mathbb{R})} + \eta\left(\lambda\right) \|f'W\|_{L_p(\mathbb{R})} \right), \end{aligned} \tag{43}$$

by Lemmas 4.1, 4.2 and Theorem 3.1. Next,

$$\begin{aligned} T_{32} &\leq \|R_m W\|_{L_\infty(2\lambda\leq|x|\leq q_n)} \|V_n\|_{L_p(2\lambda\leq|x|\leq q_n)} \\ &\leq \rho\left(m\right) \left(\|f'W\|_{L_p(\mathbb{R})} \right) C_1 \frac{q_n}{n}. \end{aligned}$$

by Lemmas 4.1, 4.2 and another application of Theorem 3.1. Combining this and the estimates in (40) to (43) gives for $n \geq 8$,

$$\begin{aligned} &\| \left(f - P_n\right) W\|_{L_p[-q_n, q_n]} \\ &\qquad \leq \|f'W\|_{L_p(\mathbb{R})} C \left\{ \frac{\sigma\left(\lambda\right)}{m} + \rho\left(m\right) \frac{q_n}{n} + \eta\left(\lambda\right) \right\}. \end{aligned} \tag{44}$$

Then using this estimate and Theorem 3.1, we deduce that

$$\|P_n W\|_{L_p[-q_n, q_n]} \leq \|f'W\|_{L_p(\mathbb{R})} C \left\{ \frac{\sigma\left(\lambda\right)}{m} + \rho\left(m\right) \frac{q_n}{n} + 1 \right\}.$$

Combining this estimate, (39) and (44) gives

$$\begin{aligned} &\inf_{\deg(P)\leq n} \| \left(f - P\right) W\|_{L_p(\mathbb{R})} \\ &\qquad \leq \|f'W\|_{L_p(\mathbb{R})} C \left\{ \frac{\sigma\left(\lambda\right)}{m} + \rho\left(m\right) \frac{q_n}{n} + \eta\left(\lambda\right) + 2^{-n} \right\}, \end{aligned}$$

with C independent of $n, m, \lambda, \rho, \sigma$. The functions σ and ρ obey the conventions listed at the beginning of this section, and are independent of f, n, m, p, as is the constant C. For a given large enough $n \geq 8$, we choose $m = m\left(n\right)$ to be the largest integer $\leq n/2$ such that

$$\rho\left(m\right) \frac{q_n}{n} \leq \left(\frac{q_n}{n} \right)^{1/2}.$$

Since (by Theorem 2.1) $q_n/n \to 0$ as $n \to \infty$, while ρ is increasing and finite valued, necessarily $m = m(n)$ approaches ∞ as $n \to \infty$. Next, for the given $m = m(n)$, we choose the largest $\lambda = \lambda(n) \leq m$ such that

$$\sigma(\lambda) \leq \sqrt{m}$$

As σ is finite valued, necessarily $\lambda(n) \to \infty$, so $\eta(\lambda(n)) \to 0$, $n \to \infty$. Then for some sequence $\{\eta_n\}_{n=1}^{\infty}$ with limit 0, and which is independent of f, p,

$$\inf_{\deg(P) \leq n} \|(f - P) W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})}.$$

For the remaining finitely many n, say for $n \leq n_1$, we use Theorem 3.1 to deduce that

$$\inf_{\deg(P) \leq n} \|(f - P) W\|_{L_p(\mathbb{R})} \leq \|fW\|_{L_p(\mathbb{R})} \leq \eta(0) \|f'W\|_{L_p(\mathbb{R})}.$$

We choose $\eta_n = \eta(0)$ for $n \leq n_1$. □

Proof of the necessity part of Theorem 1.1

We assume that (4) is true for every absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, where $p = 1$ or $p = \infty$. In particular, if we choose f to be 0 outside $[-1, 1]$, and not a polynomial in $[-1, 1]$, we obtain for some sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials with degrees tending to ∞,

$$\|P_n W\|_{L_p(|x| \geq 1)} \to 0, n \to \infty.$$

As P_n behaves for large $|x|$ like its leading term, this forces

$$\sup_{n \geq 1} \|x^n W(x)\|_{L_p(\mathbb{R})} < \infty.$$

Then the hypothesis (14) of Theorem 2.2 is fulfilled, and consequently there exist $\{\xi_n\}_{n=1}^{\infty}$ such that (15) holds for all polynomials P_n of degree $\leq n$. Let us now consider an absolutely continuous f with $f(0) = 0$ and $\|f'W\|_{L_p(\mathbb{R})}$ finite. Our hypothesis asserts that there are for large n polynomials $\{P_n\}_{n=1}^{\infty}$ of degree $\leq n$ with

$$\begin{aligned} &\|(f - P_n) W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})} \\ &\Rightarrow \|fW\|_{L_p(|x| \geq \xi_n)} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})} + \|P_n W\|_{L_p(|x| \geq \xi_n)}. \end{aligned}$$

Here by Theorem 2.2, and then our hypothesis on $\{P_n\}_{n=1}^{\infty}$,

$$\begin{aligned}\|P_nW\|_{L_p(|x|\geq\xi_n)} &\leq C2^{-n}\|P_nW\|_{L_p[-1,1]}\\ &\leq C2^{-n}\left(\|fW\|_{L_p[-1,1]}+\eta_n\|f'W\|_{L_p(\mathbb{R})}\right).\end{aligned}$$

Here if $p=1$,

$$\begin{aligned}\|fW\|_{L_p[0,1]} &\leq \|W\|_{L_\infty[0,1]}\int_0^1\left|\int_0^x f'(t)\,dt\right|dx\\ &\leq \|W\|_{L_\infty[0,1]}\int_0^1 |f'(t)|\,dt\\ &\leq \|W\|_{L_\infty[0,1]}\|W^{-1}\|_{L_\infty[0,1]}\int_0^1 |(f'W)(t)|\,dt.\end{aligned}$$

A similar inequality holds over $[-1,0]$ and hence

$$\|fW\|_{L_1[-1,1]} \leq 2\|W\|_{L_\infty[-1,1]}\|W^{-1}\|_{L_\infty[-1,1]}\|f'W\|_{L_1[-1,1]}.$$

The case $p=\infty$ is easier. Combining all the above inequalities gives

$$\|fW\|_{L_p(|x|\geq\xi_n)} \leq \eta_n^*\|f'W\|_{L_p(\mathbb{R})},$$

where $\{\eta_n^*\}_{n=1}^{\infty}$ has limit 0 and is independent of f. The same inequality then holds for the L_p norm of fW over $|x|\geq\lambda$, where $\lambda\in\left[\xi_n,\xi_{n+1}\right]$. If $\lambda\leq\xi_1$, we can just use Theorem 3.1 and the result for $\lambda\geq\xi_1$. It follows that there is a positive decreasing function η with limit 0 at ∞ such that (23) holds for absolutely continuous f with $f(0)=0$ and $\|f'W\|_{L_p(\mathbb{R})}$ finite. Then Theorem 3.1 gives the limits (5) and (6). □

Proof of Corollary 1.2

We must prove the limits (5) and (6). We do the second first. Let $A>0$. Choose $R>0$ such that

$$u\geq R\Rightarrow Q'(u)\geq A.$$

Then for $x\geq R$ and $t\in[x,\infty)$,

$$\begin{aligned}W(x)^{-1}W(t) &= e^{Q(x)-Q(t)}\\ &= e^{-Q'(\zeta)(t-x)}\leq e^{-A(t-x)},\end{aligned}$$

where ζ lies between t and x. Then for $u \geq R$ and $x \in [R, u]$

$$W^{-1}(x) \int_u^\infty W(t)\, dt \leq \int_u^\infty e^{-A(t-x)} dt \leq \frac{1}{A}.$$

For $x \in [0, R]$, this last inequality gives

$$W^{-1}(x) \int_u^\infty W(t)\, dt \leq W^{-1}(x) \frac{W(u)}{A}$$

Hence for such u,

$$\left(\min_{[0,u]} W\right)^{-1} \int_u^\infty W \leq \max \left\{ \left(\min_{[0,R]} W\right)^{-1} \frac{W(u)}{A}, \frac{1}{A} \right\}.$$

Since our hypothesis forces Q to have limit ∞ at ∞, we obtain

$$\limsup_{u \to \infty} \left(\min_{[0,u]} W\right)^{-1} \int_u^\infty W \leq \frac{1}{A}.$$

As A may be arbitrarily large, we have (6). Next, if $x \geq R$, we similarly see that

$$\begin{aligned} W(x) \int_0^x W^{-1}(t)\, dt &\leq W(x) \int_0^R W^{-1}(t)\, dt + \int_R^x e^{-A(x-t)} dt \\ &\leq W(x) \int_0^R W^{-1}(t)\, dt + \frac{1}{A}. \end{aligned}$$

Then

$$\limsup_{x \to \infty} W(x) \int_0^x W^{-1}(t)\, dt \leq \frac{1}{A}.$$

Again (5) follows as $A > 0$ is arbitrary. $\square$

Proof of Corollary 1.3

Let us choose fixed $B > 0$ and $R > 0$ such that

$$u \geq R \Rightarrow Q'(u) \leq B.$$

Then for $x \geq R$ and $t \in [x, \infty)$,

$$W(x)^{-1} W(t) = e^{Q(x) - Q(t)} = e^{-Q'(\zeta)(t-x)} \geq e^{-B(t-x)},$$

so

$$W(x)^{-1} \int_x^\infty W(t)\, dt \geq \int_x^\infty e^{-B(t-x)} dt = \frac{1}{B}.$$

Hence (6) fails. Similarly (5) fails. $\square$

5 Proof of Theorem 1.4

In this section, we let

$$W_2(x) = \exp\left(-x^2\right), \qquad x \in \mathbb{R},$$

denote the Hermite weight. We choose intervals

$$[j - \alpha_j, j + \alpha_j], \qquad j \geq 2$$

where $\alpha_j \in (0,1)$, and decays rapidly to 0 as $j \to \infty$. We set

$$W(x) = W_2(x), \qquad x \in \mathbb{R} \backslash \bigcup_{j=2}^{\infty} (j - \alpha_j, j + \alpha_j). \tag{45}$$

For Theorem 1.4(a), where we want an L_∞ Jackson theorem, but not an L_1 Jackson theorem, we set

$$W(j) = W_2(j)/j, \qquad j \geq 2. \tag{46}$$

For Theorem 1.4 (b), we set

$$W(j) = W_2(j)\, j, \qquad j \geq 2. \tag{47}$$

In both cases we then define W so that W/W_2 is linear in $[j - \alpha_j, j]$ and in $[j, j + \alpha_j]$. This ensures that W is continuous in $\mathbb{R}$. (Of course we could ensure it is C^∞ by smoothing at j and $j \pm \alpha_j$.) It also implies under (46) that,

$$1 \geq W(x)/W_2(x) \geq \frac{1}{1+|x|}, \qquad x \in \mathbb{R}, \tag{48}$$

and under (47),

$$1 \leq W(x)/W_2(x) \leq 1 + |x|, \qquad x \in \mathbb{R}. \tag{49}$$

In proving the Jackson theorem for W in the relevant L_p, we shall need a restricted range inequality. This does not follow from Theorem 2.1, since at least one of the conditions there is not fulfilled. Using classical results for the Hermite weight, we prove:

Lemma 5.1 *Let W be as above, satisfying either (48) or (49). Let $1 \leq p \leq \infty$. Then there exist $C_1 > 0$ and $C_2 > 0$ such that for $n \geq 1$ and polynomials P of degree $\leq n$,*

$$\|PW\|_{L_p\left(|x|\geq 2\sqrt{n}\right)} / \|PW\|_{L_p(\mathbb{R})} \leq C_1 e^{-nC_2}. \tag{50}$$

Proof
For both cases, we have for $|x| \geq 2$,

$$\frac{1}{1+|x|} \leq W(x) / W_2(x) \leq 3\,|1+x|\,, \tag{51}$$

so for all n,

$$\|PW\|_{L_p\left(|x|\geq 2\sqrt{n}\right)} \leq 2\|P(x)(1+x)\,W_2(x)\|_{L_p\left(|x|\geq 2\sqrt{n}\right)}.$$

Since $P(x)(1+x)$ is of degree $\leq n+1$, we obtain from classical inequalities [18], [21, Thm. VI.5.1, p. 334], that we can continue this as

$$\begin{aligned}
&\leq C_1 e^{-nC_2}\|P(x)(1+x)\,W_2(x)\|_{L_p\left(|x|\leq \frac{3}{2}\sqrt{n+1}\right)} \\
&\leq Cne^{-nC_2}\|P(x)\,\frac{W_2(x)}{1+|x|}\|_{L_p\left(|x|\leq \frac{3}{2}\sqrt{n+1}\right)} \\
&\leq Ce^{-nC_3}\|P(x)\,W(x)\|_{L_p\left(|x|\leq \frac{3}{2}\sqrt{n+1}\right)},
\end{aligned}$$

by (51). Then (50) follows. □

Next, we show that the weights W defined above satisfy exactly one of the limit relations in Theorem 1.1:

Lemma 5.2 *(a) Let W satisfy (45), (46) and (48). If we choose the sequence $\{\alpha_j\}$ to decay sufficently rapidly to 0, then the limit (5) in Theorem 1.1 is true, while the limit (6) fails.*
(b) Let W satisfy (45), (47) and (49). If we choose the sequence $\{\alpha_j\}$ to decay sufficently rapidly to 0, then the limit (6) in Theorem 1.1 is true, while the limit (5) fails.

Proof
We choose the $\{\alpha_j\}_{j=1}^{\infty}$ so small that for large enough $x > 0$,

$$\frac{1}{2} \leq \int_0^x W^{-1} / \int_0^x W_2^{-1} \leq 2 \tag{52}$$

and

$$\frac{1}{2} \leq \int_x^\infty W / \int_x^\infty W_2 \leq 2. \tag{53}$$

For example, this will be true if

$$\alpha_j < (2 + j)^{-1} 2^{-j-3} W_2 (j + 1), \qquad j \geq 1. \tag{54}$$

Indeed, using our bound (51), this implies

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| W^{-1} - W_2^{-1} \right| &= \sum_{j=2}^{\infty} \int_{j-\alpha_j}^{j+\alpha_j} \left| W^{-1} - W_2^{-1} \right| \\
&\leq \sum_{j=2}^{\infty} 6\alpha_j (1 + j + \alpha_j) W_2^{-1} (j + 1) \\
&\leq \sum_{j=2}^{\infty} 2^{-j} = \frac{1}{2} < \frac{1}{2} \int_0^1 W_2^{-1}.
\end{aligned}$$

Then (52) follows easily. Next,

$$\begin{aligned}
\int_x^\infty |W - W_2| &\leq \sum_{j \geq x} \int_{j-\alpha_j}^{j+\alpha_j} |W - W_2| \\
&\leq \sum_{j \geq x} 6\alpha_j (1 + j + \alpha_j) W_2 (j - 1) \\
&\leq \sum_{j \geq x} 2^{-j} W_2 (j + 1) \leq 2^{1-x} \int_x^\infty W_2,
\end{aligned}$$

so (53) follows.
(a) Now from (48) and (52) for large enough x,

$$W(x) \int_0^x W^{-1} \leq W_2(x) \int_0^x W^{-1} \leq 2 W_2(x) \int_0^x W_2^{-1}.$$

It is easily checked that the last right-hand side approaches 0 (or follows from the well known fact that the Hermite weight admits a Jackson Theorem in all L_p, together with Theorem 1.1). So (5) is true. Since $W = W_2$ in $(-\infty, 0)$, the limit at $-\infty$ is immediate. On the other hand, by (46) and (53), for large enough j,

$$\begin{aligned} W^{-1}(j) \int_j^\infty W &\geq j W_2^{-1}(j) \frac{1}{2} \int_j^\infty W_2 \\ &\geq \frac{1}{8} W_2^{-1}(j) \int_j^{2j} W_2(s)\, 2s\, ds \geq \frac{1}{16}, \end{aligned}$$

for j large enough. So (6) fails.
(b) Now from (49) and (53) for large enough x,

$$\left(\min_{[0,x]} W\right)^{-1} \int_x^\infty W \leq W_2^{-1}(x) \int_x^\infty W \leq 2W_2^{-1}(x) \int_x^\infty W_2^{-1}.$$

Again it is easily checked that the last right-hand side approaches 0. So (6) is true. On the other hand by (47) and (52), for large enough j,

$$\begin{aligned} W(j) \int_0^j W^{-1} &\geq j W_2(j) \frac{1}{2} \int_0^j W_2^{-1} \\ &\geq W_2(j) \frac{1}{4} \int_0^j W_2^{-1}(s)\, 2s\, ds \\ &= \frac{1}{4}(1 - W_2(j)). \end{aligned}$$

So (5) fails. □

Proof of Theorem 1.4(a)

Now W of (45), (46), (48) admits the restricted range inequality Lemma 5.1. Also the limit (5) is true, together with its analogue at $-\infty$. Then the exact proof that we used before in the proof of Theorem 1.1(a) gives an L_∞ Jackson theorem. Note that we can take $q_n = 2\sqrt{n}$, so $q_n = o(n)$.

On the other hand, we have shown in Lemma 5.2 that the limit (6) fails, and then the proof of Theorem 1.1(b) shows that there is no L_1 Jackson theorem. □

Proof of Theorem 1.4(b)
Now W of (45), (47), (49) admits the restricted range inequality Lemma 5.1. Also the limit (6) is true, together with its analogue at $-\infty$. Then the exact proof that we used before in the proof of Theorem 1.1(a) gives an L_1 Jackson theorem. Note again that we can take $q_n = 2\sqrt{n}$.

On the other hand, we have shown that the limit (5) fails, so the proof of Theorem 1.1(b) shows that there is no L_∞ Jackson theorem. □

References

[1] G.A. Baker, Essentials of Padé Approximants, Academic Press, New York, 1975.

[2] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, Orlando, 1988.

[3] P. Borwein and T. Erdelyi, Polynomials and Polynomial Inequalities, Springer, New York, 1993.

[4] S.B. Damelin and D.S. Lubinsky, Jackson Theorems for Erdös Weights in $L_p\,(0 < p \leq \infty)$, J. Approx. Theory, 94(1998), 333–382.

[5] R. DeVore and G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.

[6] Z. Ditzian and D. Lubinsky, Jackson and Smoothness Theorems for Freud Weights in $L_p(0 < p \leq \infty)$, Constructive Approximation, 13(1997), 99–152.

[7] Z. Ditzian, D.S. Lubinsky, P. Nevai, V. Totik, Polynomial Approximation with Exponential Weights, Acta Math. Hung., 50(1987), 165–175.

[8] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer, New York, 1987.

[9] G. Freud, On Markov-Bernstein Type Inequalities and Their Applications, J. Approx. Theory, 19(1977), 22–37.

[10] G. Freud, A. Giroux, Q.I. Rahman, Sur l' approximation polynomiale avec poids $\exp(-|x|)$, Canadian J. Math., 30(1978), 358–372.

[11] J.S. Geronimo, D.S. Lubinsky and F. Marcellan, Asymptotics for Sobolev Orthogonal Polynomials for Exponential Weights, to appear in Constructive Approximation.

[12] P. Koosis, The Logarithmic Integral I, Cambridge University Press, Cambridge, 1988.

[13] E. Levin and D.S. Lubinsky, Orthogonal Polynomials for Exponential Weights, Springer, New York, 2001.

[14] D.S. Lubinsky, A Weighted Polynomial Inequality, Proc. Amer. Math. Soc., 92(1984), 263–267.

[15] D.S. Lubinsky, Forward and Converse Theorems of Polynomial Approximation for Exponential Weights on $[-1,1]$ I, J. Approx. Theory, 91(1997), 1–47.

[16] D.S. Lubinsky, Jackson and Bernstein Theorems for the Weight $\exp(-|x|)$ on $\mathbb{R}$, to appear in Israel Journal of Mathematics.

[17] H.N. Mhaskar, Introduction to the Theory Of Weighted Polynomial Approximation, World Scientific, Singapore, 1996.

[18] H.N. Mhaskar and E.B. Saff, Extremal Problems Associated with Exponential Weights, Trans. Amer. Math. Soc., 285(1984), 223–234.

[19] P. Nevai, Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study, J. Approx. Theory, 48(1986), 3–167.

[20] P. Nevai, V. Totik, Weighted Polynomial Inequalities, Constructive Approximation, 2(1986), 113–127.

[21] E.B. Saff and V. Totik, Logarithmic Potentials with External Fields, Springer, New York, 1997.