WHICH WEIGHTS ON $\mathbb{R}$ ADMIT L_p JACKSON THEOREMS?

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Abstract. Let $1 \leq p \leq \infty$ and $W : \mathbb{R} \to (0, \infty)$ be continuous. Does W admit a Jackson Theorem in L_p? That is, does there exist a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 such that

$$\inf_{\deg(P) \leq n} \| (f - P)W \|_{L_p(\mathbb{R})} \leq \eta_n \| f'W \|_{L_p(\mathbb{R})}$$

for all absolutely continuous f with $\| f'W \|_{L_p(\mathbb{R})}$ finite? We show that such a theorem is true iff

$$\lim_{x \to \infty} \left\| W^{-1} \right\|_{L_q[0,x]} \|W\|_{L_p[x,\infty)} = 0,$$

where q is the conjugate parameter of p. In an earlier paper, we considered weights admitting a Jackson theorem for all $1 \leq p \leq \infty$.

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1. Introduction

Let $W : \mathbb{R} \to (0, \infty)$. Bernstein's approximation problem addresses the following question: when are the polynomials dense in the weighted space generated by W? That is, when is it true that for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x| \to \infty} (fW)(x) = 0,$$

there exist a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n \to \infty} \| (f - P_n) W \|_{L_\infty(\mathbb{R})} = 0?$$

This problem was resolved independently by Pollard, Mergelyan and Achieser in the 1950's [6]. If $W \leq 1$, is even, and $\ln 1/W(e^x)$ is even and convex, a necessary and sufficient condition for density of the polynomials is [6, p. 170]

$$\int_0^{\infty} \frac{\ln 1/W(x)}{1 + x^2} dx = \infty.$$

In particular, for $W_\alpha(x) = \exp(-|x|^\alpha)$, the polynomials are dense iff $\alpha \geq 1$.

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In the 1950's the search began for a quantitative form of Bernstein's Theorem. One obvious question is whether there are weighted analogues of classical theorems of Jackson and Bernstein, namely

$$\inf_{\deg(P)\leq n} \| f-P \|_{L_\infty[-1,1]} \leq \frac{C}{n} \| f' \|_{L_\infty[-1,1]},$$

with C independent of f and n, and the inf being over (algebraic) polynomials of degree at most n. For the weights W_α, where $\alpha > 1$, it is known that if $1 \leq p \leq \infty$,

$$\inf_{\deg(P)\leq n} \| (f-P) W_\alpha \|_{L_p(\mathbb{R})} \leq C n^{-1+\frac{1}{\alpha}} \| f'W \|_{L_p(\mathbb{R})}, \tag{1}$$

with C independent of f and n [5, p. 185, (11.3.5)] [11, p. 81, (4.1.5a)]. This inequality is also often formulated in Jackson-Favard form,

$$\inf_{\deg(P)\leq n} \| (f-P) W_\alpha \|_{L_p(\mathbb{R})} \leq C n^{-1+\frac{1}{\alpha}} \inf_{\deg(P)\leq n-1} \| \left(f'-P\right) W_\alpha \|_{L_p(\mathbb{R})} .$$

More general Jackson type theorems involving weighted moduli of continuity for various classes of weights were proved in [4], [5], [11].

In a recent paper [10], the author showed that the weight W_1 does not admit a Jackson estimate like (1), even though the polynomials are dense in the weighted space generated by W_1. The author also characterized weights that admit Jackson theorems in L_p for all $1 \leq p \leq \infty$. The main result there was:

Theorem 1.1
Let $W : \mathbb{R} \to (0,\infty)$ be continuous. The following are equivalent:
(a) There exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 and with the following property. For each $1 \leq p \leq \infty$, and for all absolutely continuous f with $\| f'W \|_{L_p(\mathbb{R})}$ finite, we have

$$\inf_{\deg(P)\leq n} \| (f-P) W \|_{L_p(\mathbb{R})} \leq \eta_n \| f'W \|_{L_p(\mathbb{R})}, n \geq 1. \tag{2}$$

(b) Both

$$\lim_{x\to\infty} W(x) \int_0^x W^{-1} = 0 \tag{3}$$

and

$$\lim_{x\to\infty} W(x)^{-1} \int_x^\infty W = 0 \tag{4}$$

with analogous limits as $x \to -\infty$.

As a corollary it was shown that if $W = e^{-Q}$, where Q' exists for large $|x|$, then there is a Jackson theorem in L_p for all $1 \leq p \leq \infty$, when $\pm Q'(x) \to \infty$ as $x \to \pm\infty$ and there is no Jackson theorem if $Q'(x)$ is bounded for large $|x|$.

In this paper, we focus on just a single L_p space and ask which weights admit Jackson theorems in that space. We prove:

Theorem 1.2
Let $W : \mathbb{R} \to (0, \infty)$ be continuous. Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:
(a) There exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 such that for all absolutely continuous f with $\| f'W \|_{L_p(\mathbb{R})}$ finite, we have

$$\inf_{\deg(P) \le n} \| (f - P) W \|_{L_p(\mathbb{R})} \le \eta_n \| f'W \|_{L_p(\mathbb{R})}, n \ge 1. \tag{5}$$

(b)

$$\lim_{x \to \infty} \|W\|_{L_p[x,\infty]} \left\|W^{-1}\right\|_{L_q[0,x]} = 0, \tag{6}$$

with an analogous limit as $x \to -\infty$.

Remarks
(a) Thus there is a Jackson type theorem in a specific L_p space iff (6) holds. In fact, we shall show in Section 3 that (6) is necessary and sufficient for the existence of a decreasing function $\eta : (0, \infty) \to (0, \infty)$ with limit 0 at ∞, such that

$$\left\|f'W\right\|_{L_p[a,\infty)} \le \eta(a) \|fW\|_{L_p[0,\infty)}$$

for all absolutely continuous f with $f(0) = 0$. This is a "shifting" weighted Hardy inequality.
(b) Theorem 1.2 actually implies Theorem 1.1. For the condition (6) for $p = 1$ is equivalent to (4) and for $p = \infty$ is equivalent to (3). Interpolation then gives (2) for $1 < p < \infty$. Of course, Theorem 1.1 does not imply Theorem 1.2.
(b) It was shown in [10] that there is a weight W admitting an L_1 Jackson theorem, but not an L_∞ one (and conversely). Here we show:

Theorem 1.3
Let $1 \le p, r \le \infty$ with $p \ne r$. There exists $W : \mathbb{R} \to (0, \infty)$ such that

$$\frac{1}{1 + x^2} \le W(x) / \exp\left(-x^2\right) \le 1 + x^2, \quad x \in \mathbb{R},$$

and W admits an L_r Jackson theorem , but not an L_p Jackson theorem. That is, there exist $\{\eta_n\}_{n=1}^{\infty}$ with limit 0 at ∞ satisfying (5) in the L_r norm, but there does not exist such a sequence satisfying (5) in the L_p norm.

Theorem 1.3 shows that not only rate of decay, but also regularity, of W is necessary for a Jackson theorem. After all, the Hermite weight $\exp\left(-x^2\right)$ admits a Jackson theorem in L_p for all $1 \le p \le \infty$, but W is close to W_2, yet admits a Jackson theorem in L_r but not L_p.

This paper is organised as follows: we prove restricted range inequalities in the next section, and an estimate for the "tails" $\| fW \|_{L_p(|x| \ge \lambda)}$ in Section 3. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3.

Throughout $C, C_1, C_2, \ldots$ denote constants independent of n and x and polynomials P of degree $\leq n$. The same symbol may denote different constants in different occurrences. If (c_n) and (d_n) are sequences of real numbers, we write

$$c_n \sim d_n$$

if there exist $C_1, C_2 > 0$ such that

$$C_1 \leq c_n/d_n \leq C_2, n \geq 1.$$

Similar notation is used for functions. The linear measure of a set $B \subset \mathbb{R}$ is denoted by $meas\,(B)$. The set of all polynomials of degree $\leq n$ is denoted P_n.

2. Restricted range inequalities

Restricted range (or infinite-finite range) inequalities are a crucial ingredient in weighted approximation on the real line [8], [11], [12], [14]. However, none of the standard ones cover our class of weights. The methods used to prove the form we need, are similar to, but not the same, as in [10]. In this section, we fix $1 \leq p \leq \infty$, and let

$$\widetilde{W}\,(x) = \left\|W^{-1}\right\|_{L_q[0,x]}^{-1}, x \in (0, \infty)\,, \tag{7}$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

Theorem 2.1
Assume that for $x \in [0, \infty)$,

$$\|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[0,x]} \leq \psi\,(x)\,, \tag{8}$$

where ψ *is decreasing in* $[0, \infty)$ *and*

$$\lim_{x\to\infty} \psi\,(x) = 0, \tag{9}$$

with a similar relation in $(-\infty, 0]$*. There exists* $q_n > 0, n \geq 1$*, such that*

$$q_n = o\,(n)\,, n \to \infty, \tag{10}$$

and for $n \geq 1$*, and all polynomials* P *of degree* $\leq n$*,*

$$\|PW\|_{L_p(|x|\geq q_n)} \leq C4^{-n}\,\|PW\|_{L_p(\mathbb{R})}\,. \tag{11}$$

Here C *is independent of* n *and* P*.*

In the rest of this section, ψ is the function specified in Theorem 2.1. For $n \geq 1$, we choose $A_n > 0$ such that

$$\|x^n W\,(x)\|_{L_p[A_n,2A_n]} = \max_{u\geq 1} \|x^n W\,(x)\|_{L_p[u,2u]} =: \Lambda_n.$$

(We show below that A_n exists).

Lemma 2.2
(i) For $n \geq 0$,

$$\|x^n W(x)\|_{L_p[1,\infty)}$$

is finite.
(ii) For $n \geq 1$, A_n *exists, is finite and positive, and*

$$\lim_{n\to\infty} A_n = \infty. \tag{12}$$

(iii) For $n \geq 1$,

$$(2A_{n+2})^{-2}\Lambda_{n+2} \leq \|x^n W(x)\|_{L_p[1,\infty)} \leq (2A_{n+2}^{-2p} + 2^{2p+1})^{1/p}\Lambda_{n+2}. \tag{13}$$

(iv)

$$A_n = o(n), n \to \infty. \tag{14}$$

(v) If $\mathcal{B} \subset [0, 2A_{n+2}]$ *has linear Lebesgue measure at least* 1, *then*

$$\|W\|_{L_p(\mathcal{B})} \geq \psi(1)^{-1} (2A_{2n+2})^{-(2n+2)} \Lambda_{2n+2}.$$

Proof
Observe that (8) implies

$$\|W\|_{L_p[x,\infty)} \leq \psi(x)\widetilde{W}(x), x > 0, \tag{15}$$

and by Hölder's inequality, for $x \geq 1$,

$$1 \leq \|W\|_{L_p[x-1,x]} \left\|W^{-1}\right\|_{L_q[x-1,x]} \leq \|W\|_{L_p[x-1,x]} \left\|W^{-1}\right\|_{L_q[0,x]},$$

so that

$$\widetilde{W}(x) \leq \|W\|_{L_p[x-1,x]}, x \geq 1. \tag{16}$$

(i) If $p = \infty$, this was established in Lemma 2.3(a) in [10]. Suppose now $p < \infty$. Let $0 \leq a < b < \infty$. We see using (15) and (16) that

$$\int_a^b x^{np} \left(\int_x^\infty W^p(t)\,dt\right) dx \leq \int_a^b x^{np}\psi^p(x)\widetilde{W}^p(x)\,dx$$

$$\Rightarrow \int_a^\infty W^p(t)\left[\int_a^{\min\{t,b\}} x^{np}\,dx\right] dt \leq \psi^p(a)\int_a^b x^{np}\left[\int_{x-1}^x W^p(t)\,dt\right] dx$$

$$\Rightarrow \int_a^b W^p(t)\frac{t^{np+1}-a^{np+1}}{np+1}dt \leq \psi^p(a)\int_{a-1}^b W^p(t)\left[\int_{\max\{t,a\}}^{\min\{t+1,b\}} x^{np}\,dx\right] dt$$

$$\leq \psi^p(a)\int_{a-1}^b (t+1)^{np}W^p(t)\,dt.$$

If $t \geq a2^{\frac{1}{np+1}}$, then $t^{np+1} \geq 2a^{np+1}$, and if $a \geq 2$, in the integral on the right-hand side,

$$(t+1)^{np} = t^{np}\left(1+\frac{1}{t}\right)^{np} \leq t^{np}\left(1+\frac{2}{a}\right)^{np} \leq t^{np}e^{\frac{2np}{a}}.$$

Thus

$$\int_{a2^{\frac{1}{np+1}}}^{b} t^{np+1}W^{p}(t)\,dt \leq 2\psi^{p}(a)(np+1)e^{\frac{2np}{a}}\int_{a-1}^{b} t^{np}W^{p}(t)\,dt. \tag{17}$$

As $a \geq 2$, $t^{np} \leq t^{np+1}$ in the integral on the right, so

$$\begin{aligned} &\int_{a2^{\frac{1}{np+1}}}^{b} t^{np+1}W^{p}(t)\,dt\left[1-2\psi^{p}(a)(np+1)e^{\frac{2np}{a}}\right] \\ \leq\ & 2\psi^{p}(a)(np+1)e^{\frac{2np}{a}}\int_{a-1}^{a2^{\frac{1}{np+1}}} x^{np}W^{p}(x)\,dx. \end{aligned}$$

If a is so large that $a \geq 2np$ and

$$2\psi^{p}(a)(np+1)e \leq \frac{1}{2}, \tag{18}$$

this gives

$$\int_{a2^{\frac{1}{np+1}}}^{b} t^{np+1}W^{p}(t)\,dt \leq \int_{a-1}^{a2^{\frac{1}{np+1}}} x^{np}W^{p}(x)\,dx.$$

Letting $b \to \infty$ gives the finiteness of the norm $\|x^{n}W(x)\|_{L_p[1,\infty)}$.

(ii) The existence of $A_n \in (0,\infty)$ follows as the norm in (i) is finite, and $u \to \|x^{n}W(x)\|_{L_p[u,2u]}$ is a continuous function of u, with limit 0 as $u \to 0+$ and $u \to \infty$. (In the case $p=\infty$, this follows from the finiteness of $\left\|x^{n+1}W(x)\right\|_{L_p[1,\infty)}$). Next, for fixed $u>0$,

$$\Lambda_n \geq \|x^{n}W(x)\|_{L_p[u,2u]} \geq u^{n}\|W\|_{L_p[u,2u]}$$

so

$$\liminf_{n\to\infty} \Lambda_n^{1/n} \geq u,$$

and hence

$$\lim_{n\to\infty} \Lambda_n^{1/n} = \infty.$$

If a subsequence of $\{A_n\}$ remained bounded, we see that the corresponding subsequence of $\{\Lambda_n\}$ cannot admit the growth just proven.

(iii) If $p=\infty$, the right-hand inequality in (13) is immediate. Suppose now that $p<\infty$. Choose j_0 such that

$$2^{j_0} \leq A_{n+2} \leq 2^{j_0+1}.$$

We see that

$$\begin{aligned} \int_{1}^{A_{n+2}} x^{np}W^{p}(x)\,dx &\leq \sum_{j=0}^{j_0}\int_{A_{n+2}/2^{j+1}}^{A_{n+2}/2^{j}} x^{np}\left(\frac{x}{A_{n+2}/2^{j+1}}\right)^{2p}W^{p}(x)\,dx \\ &\leq A_{n+2}^{-2p}\sum_{j=0}^{j_0} 2^{(j+1)2p}\Lambda_{n+2}^{p} \\ &\leq A_{n+2}^{-2p}2^{(j_0+1)2p+1}\Lambda_{n+2}^{p} \leq 2^{2p+1}\Lambda_{n+2}^{p}. \end{aligned}$$

Also

$$\int_{A_{n+2}}^{\infty} x^{np} W^p(x)\,dx \leq \sum_{j=0}^{\infty} \int_{A_{n+2}2^j}^{A_{n+2}2^{j+1}} x^{np} \left(\frac{x}{A_{n+2}2^j}\right)^{2p} W^p(x)\,dx$$

$$\leq A_{n+2}^{-2p} \left(\sum_{j=0}^{\infty} 2^{-2jp}\right) \Lambda_{n+2}^p \leq 2A_{n+2}^{-2p} \Lambda_{n+2}^p, \tag{19}$$

for large n. Then the upper bound in (13) follows. The lower bound follows from

$$\begin{aligned} \|x^n W(x)\|_{L_p[1,\infty)} &\geq \|x^n W(x)\|_{L_p[A_{n+2},2A_{n+2}]} \\ &\geq (2A_{n+2})^{-2} \left\|x^{n+2} W(x)\right\|_{L_p[A_{n+2},2A_{n+2}]} \\ &= (2A_{n+2})^{-2} \Lambda_{n+2}. \end{aligned}$$

(iv) If $p = \infty$, this follows from (19) of Lemma 2.3(a) in [10]. (There $\ell(n)$ plays a role similar to A_n). Suppose now $p < \infty$. If we choose $a = a_n := A_{n+2}2^{-\frac{1}{np+1}}$, and $b = 2A_{n+2}$, (17) gives for large enough n,

$$\int_{A_{n+2}}^{2A_{n+2}} t^{np+1} W^p(t)\,dt \leq 2\psi^p(a_n)(np+1)\, e^{\frac{2np}{a_n}} \int_{a_n-1}^{b} t^{np} W^p(t)\,dt.$$

Here by (iii),

$$\begin{aligned} \int_{a_n-1}^{b} t^{np} W^p(t)\,dt &\leq (a_n-1)^{-2p} \int_{a_n-1}^{b} t^{(n+2)p} W^p(t)\,dt \\ &\leq C A_{n+2}^{-2p} \Lambda_{n+2}^p, \end{aligned}$$

with C independent of n. Combining the above two inequalities gives

$$\begin{aligned} \Lambda_{n+2}^p &= \int_{A_{n+2}}^{2A_{n+2}} t^{(n+2)p} W^p(t)\,dt \\ &\leq (2A_{n+2})^{2p-1} \int_{A_{n+2}}^{2A_{n+2}} t^{np+1} W^p(t)\,dt \\ &\leq (2A_{n+2})^{2p-1}\, 2\psi^p(a_n)(np+1)\, e^{\frac{2np}{a_n}} C A_{n+2}^{-2p} \Lambda_{n+2}^p \\ &\leq C_1 \frac{n\psi^p(a_n)}{a_n} e^{\frac{2np}{a_n}} \Lambda_{n+2}^p. \end{aligned}$$

Here C_1 is independent of n. If we write $a_n = \delta_n n$, we can recast this as

$$\frac{1}{\psi^p(a_n)} \leq C_1 \frac{1}{\delta_n} e^{\frac{2p}{\delta_n}}.$$

Since ψ has limit 0 at ∞, and $a_n = A_{n+2}2^{-\frac{1}{np+1}} \to \infty$, $n \to \infty$, it follows that necessarily $\delta_n = o(1)$ and so $a_n = o(n)$. That is

$$A_{n+2} = o(n).$$

(v) Exactly as above, Hölder's inequality gives

$$1 \leq \|W\|_{L_p(\mathcal{B})} \left\|W^{-1}\right\|_{L_q(\mathcal{B})} \leq \|W\|_{L_p(\mathcal{B})} \left\|W^{-1}\right\|_{L_q[0,A_{2n+2}]} .$$

Using (15), we can continue this as

$$\begin{aligned} \|W\|_{L_p(\mathcal{B})} &\geq \widetilde{W}(A_{2n+2}) \\ &\geq \psi(A_{2n+2})^{-1} \|W\|_{L_p[A_{2n+2},\infty)} \\ &\geq \psi(1)^{-1}(2A_{2n+2})^{-(2n+2)} \left\|x^{2n+2}W(x)\right\|_{L_p[A_{2n+2},2A_{2n+2}]} \\ &= \psi(1)^{-1}(2A_{2n+2})^{-(2n+2)} \Lambda_{2n+2}. \end{aligned}$$

■

Lemma 2.3
There exists $C_2 > 0$ such that for $n \geq 1$ and all polynomials P of degree $\leq n$,

$$\|PW\|_{L_p[1600A_{2n+2},\infty)} \leq C_2 4^{-n} \|PW\|_{L_p[0,\infty)} .$$

Proof
Our approach is similar to that in [9]. Let P be a polynomial of degree $k \leq n$, say

$$P(z) = c\prod_{j=1}^{k}(z - x_j).$$

We assume $\rho > 8, c \neq 0$, and split the zeros into "small" and "large" zeros: we assume that

$$\begin{aligned} |x_j| &\leq \rho, \quad j \leq i; \\ |x_j| &> \rho, \quad j > i. \end{aligned}$$

For $|u| \leq \frac{1}{2}\rho, x \geq \rho$ and $i < j \leq k$,

$$\left|\frac{x - x_j}{u - x_j}\right| \leq \frac{1 + x/|x_j|}{1 - |u|/|x_j|} \leq 2\left(1 + \frac{x}{\rho}\right) \leq 4\frac{x}{\rho}.$$

Then for such x, u

$$\left|\frac{P(x)}{P(u)}\right| \leq \left(\prod_{j=1}^{i}\frac{2x}{|u - x_j|}\right)\left(4\frac{x}{\rho}\right)^{k-i}.$$

We now apply a famous lemma of Cartan:

$$\left|\prod_{j=1}^{i}(u - x_j)\right| \geq \varepsilon^i$$

for u outside a set of linear measure at most $4e\varepsilon$ [1, p. 175], [2, p. 350]. Choosing $\varepsilon = \frac{\rho}{100}$, we obtain

$$\left|\frac{P(x)}{P(u)}\right| \leq \left(\frac{200x}{\rho}\right)^k \leq \left(\frac{200x}{\rho}\right)^n,$$

for $x \geq \rho, u \in \left[0, \frac{1}{2}\rho\right] \backslash \mathcal{S}$, where

$$meas\,(\mathcal{S}) \leq \frac{4e}{100}\rho < \frac{1}{8}\rho.$$

Recall that *meas* denotes linear Lebesgue measure. Then for such u,

$$\|PW\|_{L_p[400\rho,\infty)} \leq \left(\frac{200}{\rho}\right)^n |P(u)|\, \|x^n W(x)\|_{L_p[400\rho,\infty)}. \tag{20}$$

Moreover, $\left[0, \frac{1}{4}\rho\right] \backslash \mathcal{S}$ has measure at least $\frac{1}{8}\rho \geq 1$, so we may find $\mathcal{B} \subset \left[0, \frac{1}{4}\rho\right] \backslash \mathcal{S}$ with linear measure at least 1 and hence

$$\|PW\|_{L_p[400\rho,\infty)} \|W\|_{L_p(\mathcal{B})} \leq \left(\frac{200}{\rho}\right)^n \|PW\|_{L_p(\mathcal{B})} \|x^n W(x)\|_{L_p[400\rho,\infty)}.$$

Now we choose $\rho = 4A_{2n+2}$, at least for n so large that $4A_{2n+2} > 8$. Then $\left[0, \frac{1}{4}\rho\right] \backslash \mathcal{S} \subset [0, A_{2n+2}]$. By the previous lemma,

$$\|W\|_{L_p(\mathcal{B})} \geq \psi(1)^{-1} (2A_{2n+2})^{-(2n+2)} \Lambda_{2n+2}.$$

Combining the above inequalities, and (v) of the above lemma, gives if P is not identically 0,

$$\begin{aligned}
& \|PW\|_{L_p[400\rho,\infty)} / \|PW\|_{L_p[0,\infty)} \\
\leq\; & \left(\frac{200}{\rho}\right)^n \|x^n W(x)\|_{L_p[400\rho,\infty)} / \left[\psi(1)^{-1} (2A_{2n+2})^{-(2n+2)} \Lambda_{2n+2}\right] \\
\leq\; & \left(\frac{1}{2\rho^2}\right)^n \left\|x^{2n} W(x)\right\|_{L_p[400\rho,\infty)} / \left[\psi(1)^{-1} (2A_{2n+2})^{-(2n+2)} \Lambda_{2n+2}\right] \\
\leq\; & C8^{-n} A_{2n+2}^2,
\end{aligned}$$

by (iii) of the previous lemma. Here C is independent of n and P, and $A_{2n+2} = o(n)$, so the result follows. For the remaining finitely many n, for which $4A_{2n+2} < 8$, a simple compactness argument gives the result, if C_2 is large enough. ■

Proof of Theorem 2.1

This follows from Lemma 2.3, its analogue in $(-\infty, 0]$, and the fact that $A_n = o(n)$. ■

We also record:

Lemma 2.4

Let $W:\mathbb{R}\to(0,\infty)$ *be continuous,* $1\le p\le\infty$*, and assume that for each* $n\ge 0$,

$$\| x^n W(x) \|_{L_p(\mathbb{R})} < \infty. \tag{21}$$

Then there exists an increasing sequence of positive numbers $\{\xi_n\}_{n=1}^{\infty}$ *such that for* $n\ge 1$ *and all polynomials* P *of degree*$\le n$,

$$\| PW \|_{L_p(|x|\ge\xi_n)} \le C_1 2^{-n} \| PW \|_{L_p(-1,1)}, \tag{22}$$

where C_1 *is independent of* n,p,P.

Proof

See Theorem 2.2 in [10]. ■

3. Tail Estimates

We prove a "shifting" weighted Hardy inequality, involving the function

$$\phi(x) = \|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[0,x]}, x\ge 0.$$

Theorem 3.1

Let $W:\mathbb{R}\to(0,\infty)$ *be continuous. Let* $1\le p\le\infty$ *and* $\frac{1}{q}+\frac{1}{p}=1$*. The following are equivalent:*

(I) There exists a decreasing function $\eta:(0,\infty)\to(0,\infty)$ *with limit* 0 *at* ∞ *such that*

$$\|fW\|_{L_p(|x|\ge a)} \le \eta(a) \left\|f'W\right\|_{L_p[0,\infty)}, \tag{23}$$

for all $a>0$ *and every absolutely continuous function* $f:\mathbb{R}\to\mathbb{R}$ *with* $f(0)=0$.

(II)

$$\lim_{a\to\infty}\phi(a) = \lim_{a\to\infty}\|W\|_{L_p[a,\infty)}\left\|W^{-1}\right\|_{L_q[0,a]} = 0, \tag{24}$$

with a similar limit as $a\to-\infty$.

Lemma 3.2

Let $a>0$*. Then*

$$\|fW\|_{L_p[a,\infty)} \le p^{\frac{1}{p}} q^{\frac{1}{q}} \left(\sup_{x\ge a}\phi(x)\right) \left\|f'W\right\|_{L_p[a,\infty)},$$

for every absolutely continuous function $f:[a,\infty)\to\mathbb{R}$ *with* $f(a)=0$*. Here if* $p=\infty$ or $p=1$*, we interpret* $p^{\frac{1}{p}}q^{\frac{1}{q}}$ *as* 1.

Proof

Let

$$B = \sup_{x\in(a,\infty)} \|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[a,x]}.$$

The classical weighted Hardy inequality asserts that for every f as above,

$$\|fW\|_{L_p[a,\infty)} \le p^{\frac{1}{p}} q^{\frac{1}{q}} B \left\|f'W\right\|_{L_p[a,\infty)}.$$

(See [13, p. 13, Thm. 1.14] for the proof when $1 < p < \infty$. Take $q = p$ there and $w = v = W^p$. For $p = 1$ or $p = \infty$, see [13, Lemma 5.4, p. 49]. An alternative reference is [7].) Since

$$B \leq \sup_{x \in (a,\infty)} \|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[0,x]} = \sup_{x \geq a} \phi(x),$$

the result follows. ■

Lemma 3.3
Let $a > 0$. Then

$$\|fW\|_{L_p[a,\infty)} \leq \left(1 + p^{\frac{1}{p}} q^{\frac{1}{q}}\right) \left(\sup_{x \geq a} \phi(x)\right) \left\|f'W\right\|_{L_p[0,\infty)},$$

for every absolutely continuous function $f : [0,\infty) \to \mathbb{R}$ with $f(0) = 0$.
Proof
Write for $x \geq a$,

$$f(x) = \int_0^a f' + \int_a^x f' =: C + f_1(x).$$

Then

$$\|fW\|_{L_p[a,\infty)} \leq \|CW\|_{L_p[a,\infty)} + \|f_1 W\|_{L_p[a,\infty)}. \tag{25}$$

Here by Hölder's inequality, applied to C,

$$\begin{aligned} \|CW\|_{L_p[a,\infty)} &\leq \left\|f'W\right\|_{L_p[0,a)} \left\|W^{-1}\right\|_{L_q[0,a]} \|W\|_{L_p[a,\infty)} \\ &= \left\|f'W\right\|_{L_p[0,a)} \phi(a). \end{aligned}$$

Moreover by Lemma 3.2, as $f_1(a) = 0$,

$$\|f_1 W\|_{L_p[a,\infty)} \leq p^{\frac{1}{p}} q^{\frac{1}{q}} \left(\sup_{x \geq a} \phi(x)\right) \left\|f'W\right\|_{L_p[a,\infty)}.$$

Combining the above three inequalities gives the result. ■

Proof of Theorem 3.1
Sufficiency of (24) and its analogous limit at $-\infty$
This follows directly from Lemma 3.3. We can choose

$$\eta_+(a) = \left(1 + p^{1/p} q^{1/q}\right) \sup_{x \geq a} \phi(x), \ a > 0,$$

with a similar function η_- to handle $(-\infty, 0)$, and then set $\eta = \max\left\{\eta_-, \eta_+\right\}$.
Necessity of (24) and its analogous limit at $-\infty$
For $p = 1$ and $p = \infty$, the necessity was established in the proof of Theorem 3.1 in [10]. Suppose now $1 < p < \infty$. Let $a > 0$ and

$$f(x) = \int_0^{\min\{x,a\}} W^{-q}, x \geq 0.$$

Then

$$\left\|f'W\right\|_{L_p[0,\infty)} = \left(\int_0^a W^{(1-q)p}\right)^{\frac{1}{p}} = \left\|W^{-1}\right\|_{L_q[0,a]}^{\frac{1}{p-1}},$$

so

$$\begin{aligned}
& \left\|f'W\right\|_{L_p[0,\infty)} \phi(a) \\
= & \left\|f'W\right\|_{L_p[0,\infty)} \left\|W^{-1}\right\|_{L_q[0,a]} \|W\|_{L_p[a,\infty)} \\
= & \left\|W^{-1}\right\|_{L_q[0,a]}^{\left(\frac{1}{p-1}+1\right)} \|W\|_{L_p[a,\infty)} \\
= & \left(\int_0^a W^{-q}\right) \|W\|_{L_p[a,\infty)} = \|fW\|_{L_p[a,\infty)}.
\end{aligned}$$

Our hypothesis gives

$$\eta(a) \geq \|fW\|_{L_p[a,\infty)} / \left\|f'W\right\|_{L_p[0,\infty)} = \phi(a).$$

So ϕ has limit 0 at ∞. Similarly, the analogous limit follows at -∞. ■

4. Weighted Approximation

We begin with two lemmas, which are similar to corresponding lemmas in [10]. We shall use notation specific to this section: we use integers $n \geq 4$ and $1 \leq m \leq \frac{n}{4}$, as well as parameters

$$1 < \lambda \leq \frac{1}{2} q_m,$$

where $\{q_n\}_{n=1}^{\infty}$ are as in Theorem 2.1. We let $\rho(m)$ denote an increasing function that depends on m and W, while $\sigma(\lambda)$ denotes a function increasing in λ. These functions change in different occurrences. The essential feature is that σ is independent of m, n, p and functions f, while ρ is independent of λ, p and functions f. At the end, we choose m to grow slowly enough as a function of n, and then $\lambda \to \infty$ sufficiently slowly. We let $\mathcal{P}_m$ denote the set of polynomials of degree $\leq m$ with real coefficients.

Lemma 4.1
Let $W : \mathbb{R} \to (0,\infty)$ be continuous and satisfy (6), with an analogous limit at $-\infty$.
(a) There exists an increasing function $\sigma : [0,\infty) \to [0,\infty)$ with the following properties: let $m, \lambda \geq 1$. For $1 \leq p \leq \infty$ and all absolutely continuous f with $f'W \in L_p(\mathbb{R})$, there exists $R_m \in \mathcal{P}_m$ such that

$$\|(f - R_m) W\|_{L_p[-2\lambda,2\lambda]} \leq \frac{\sigma(\lambda)}{m} \left\|f'W\right\|_{L_p(\mathbb{R})}.$$

(b) There is an increasing function $\rho : \mathbb{Z}_+ \to (0,\infty)$ depending only on W such that

$$\|R_m W\|_{L_p(\mathbb{R})} \leq \rho(m) \left(\|fW\|_{L_p(\mathbb{R})} + \left\|f'W\right\|_{L_p(\mathbb{R})}\right).$$

Proof

(a) By the classical Jackson's Theorem [3, (6.4), Theorem 6.2, p. 219], there exists $R_m \in \mathcal{P}_m$ such that

$$\|f - R_m\|_{L_p[-2\lambda,2\lambda]} \leq \frac{\pi\lambda}{m+1} \|f'\|_{L_p[-2\lambda,2\lambda]}.$$

Then

$$\|(f - R_m) W\|_{L_p[-2\lambda,2\lambda]} \leq \frac{\pi\lambda}{m} \|W\|_{L_\infty[-2\lambda,2\lambda]} \left\|W^{-1}\right\|_{L_\infty[-2\lambda,2\lambda]} \left\|f'W\right\|_{L_p(\mathbb{R})}.$$

So we may take

$$\sigma(\lambda) = \pi\lambda \|W\|_{L_\infty[-2\lambda,2\lambda]} \left\|W^{-1}\right\|_{L_\infty[-2\lambda,2\lambda]}.$$

(b) From our restricted range inequalities, and continuity of W,

$$\|R_m W\|_{L_p(\mathbb{R})} \leq C \|R_m\|_{L_p[-q_m,q_m]} \|W\|_{L_\infty[-q_m,q_m]}.$$

Moreover, from the proof of (a),

$$\begin{aligned} & \|R_m\|_{L_p[-2\lambda,2\lambda]} \\ \leq \ & \|f\|_{L_p[-2\lambda,2\lambda]} + \frac{\pi\lambda}{m} \|f'\|_{L_p[-2\lambda,2\lambda]} \\ \leq \ & \left\|W^{-1}\right\|_{L_\infty[-2\lambda,2\lambda]} \left[\|fW\|_{L_p[-2\lambda,2\lambda]} + \pi\lambda \left\|f'W\right\|_{L_p[-2\lambda,2\lambda]}\right]. \end{aligned}$$

We shall show that

$$\|R_m\|_{L_p[-q_m,q_m]} \leq C m^{2/p} \left(\frac{q_m}{\lambda}\right)^{m+\frac{1}{p}} \|R_m\|_{L_p[-2\lambda,2\lambda]}, \tag{26}$$

where C is independent of $m, \lambda, q_m, \{R_m\}$. (Recall that $2\lambda \leq q_m$).Then, on combining the above inequalities, we obtain

$$\|R_m W\|_{L_p(\mathbb{R})} \leq \rho(m) \left[\|fW\|_{L_p[-2\lambda,2\lambda]} + \left\|f'W\right\|_{L_p[-2\lambda,2\lambda]}\right]$$

where

$$\rho(m) = C m^{2/p} q_m^{m+1/p} \|W\|_{L_\infty[-q_m,q_m]} \left\|W^{-1}\right\|_{L_\infty[-q_m,q_m]} (1 + \pi q_m).$$

Now we proceed to establish (26). Recall the Chebyshev inequality [3, Proposition 2.3, p. 101], valid for polynomials P of degree $\leq m$:

$$|P(x)| \leq |T_m(x)| \, \|P\|_{L_\infty[-1,1]}, |x| > 1.$$

Here T_m is the classical Chebyshev polynomial of the first kind. By dilating this, and using the bound

$$|T_m(x)| \leq (2|x|)^m, |x| > 1,$$

we obtain

$$\|R_m\|_{L_\infty[-q_m,q_m]} \leq \left(\frac{q_m}{\lambda}\right)^m \|R_m\|_{L_\infty[-2\lambda,2\lambda]}.$$

Using Nikolskii inequalities [3, Theorem 2.6, p. 102], we continue this as

$$\begin{aligned}\|R_m\|_{L_p[-q_m,q_m]} &\leq (2q_m)^{1/p}\|R_m\|_{L_\infty[-q_m,q_m]}\\ &\leq (2q_m)^{1/p}\left(\frac{q_m}{\lambda}\right)^m\left(\frac{(p+1)\,m^2}{2\lambda}\right)^{1/p}\|R_m\|_{L_p[-2\lambda,2\lambda]},\end{aligned}$$

and then we have (26). ■

Lemma 4.2
There exists $C>0$ such that for large enough n, and for $1\leq\lambda\leq\frac{1}{2}q_n$, there are nonnegative polynomials V_n of degree $\leq 3n/4$ such that

$$|1-V_n(x)|\leq C\frac{q_n}{n\lambda}, x\in[-\lambda,\lambda]; \tag{27}$$

$$0\leq V_n(x)\leq C, |x|\in[\lambda,2\lambda]\,; \tag{28}$$

$$0\leq V_n(x)\leq C\left(\frac{q_n}{n\lambda}\right)^2, |x|\in[2\lambda,q_n]\,. \tag{29}$$

Here C is independent of n,λ and x.
Proof
See Lemma 4.2 in [10]. ■

Proof of the sufficiency part of Theorem 1.2
This is quite similar to that of Theorem 1.1 in [10], but there is an important difference: there we introduced estimates for R_mW in the uniform norm, while here we need to restrict ourselves to a given L_p norm. So we include all the details.

We may assume that $f(0)=0$. (If not, replace f by $f-f(0)$ and absorb the constant $f(0)$ into the approximating polynomial). We choose $n\geq 1$ and $1\leq m\leq n/4$, and let λ satisfy $1\leq\lambda\leq\frac{1}{2}q_m$. Let R_m and V_n denote the polynomials of Lemma 4.1 and 4.2 respectively, and let

$$P_n=R_mV_n.$$

Then P_n is a polynomial of degree $\leq n$, and

$$\begin{aligned}&\inf_{\deg(P)\leq n}\|(f-P)\,W\|_{L_p(\mathbb{R})}\\ \leq\ &\|(f-P_n)\,W\|_{L_p(\mathbb{R})}\\ \leq\ &\|(f-P_n)\,W\|_{L_p[-q_n,q_n]}+\|fW\|_{L_p(\mathbb{R}\setminus[-q_n,q_n])}+\|P_nW\|_{L_p(\mathbb{R}\setminus[-q_n,q_n])}\\ \leq\ &\|(f-P_n)\,W\|_{L_p[-q_n,q_n]}+\|fW\|_{L_p(\mathbb{R}\setminus[-\lambda,\lambda])}+C4^{-n}\|P_nW\|_{L_p[-q_n,q_n]},\end{aligned} \tag{30}$$

by Theorem 2.1 and as $q_n > \lambda$. Here,

$$
\begin{aligned}
& \| (f - P_n) W \|_{L_p[-q_n, q_n]} \\
\leq \; & \| (f - P_n) W \|_{L_p[-\lambda,\lambda]} + \| fW \|_{L_p(\mathbb{R} \setminus [-\lambda,\lambda])} + \| P_n W \|_{L_p([-q_n,q_n] \setminus [-\lambda,\lambda])} \\
= \; & : T_1 + T_2 + T_3. \qquad (31)
\end{aligned}
$$

Firstly

$$
\begin{aligned}
T_1 \leq \; & \| (f - R_m) W \|_{L_p[-\lambda,\lambda]} + \| R_m (1 - V_n) W \|_{L_p[-\lambda,\lambda]} \\
\leq \; & \| (f - R_m) W \|_{L_p[-\lambda,\lambda]} + \| R_m W \|_{L_p[-\lambda,\lambda]} \| 1 - V_n \|_{L_\infty[-\lambda,\lambda]} \\
\leq \; & \frac{\sigma(\lambda)}{m} \| f' W \|_{L_p(\mathbb{R})} + \rho(m) \left(\| fW \|_{L_p(\mathbb{R})} + \| f' W \|_{L_p(\mathbb{R})} \right) \| 1 - V_n \|_{L_\infty[-\lambda,\lambda]} \\
\leq \; & \frac{\sigma(\lambda)}{m} \| f' W \|_{L_p(\mathbb{R})} + \rho(m) \frac{q_n}{n} \| f' W \|_{L_p(\mathbb{R})}, \qquad (32)
\end{aligned}
$$

by Lemmas 4.1, 4.2 and Theorem 3.1. Note that since $f(0) = 0$, the latter gives

$$\| fW \|_{L_p(\mathbb{R})} \leq \eta(0) \| f' W \|_{L_p(\mathbb{R})}.$$

The crucial thing in (32) is that σ and ρ are independent of f, n, p. Next, Theorem 3.1 gives,

$$T_2 \leq \eta(\lambda) \| f' W \|_{L_p(\mathbb{R})}. \qquad (33)$$

Of course this estimate also applies to the middle term in the right-hand side of (30). Next,

$$
\begin{aligned}
T_3 \leq \; & \| P_n W \|_{L_p(\lambda \leq |x| \leq 2\lambda)} + \| P_n W \|_{L_p(2\lambda \leq |x| \leq q_n)} \\
= \; & : T_{31} + T_{32}.
\end{aligned}
$$

Here

$$
\begin{aligned}
T_{31} \leq \; & \| R_m W \|_{L_p(\lambda \leq |x| \leq 2\lambda)} \| V_n \|_{L_\infty(\lambda \leq |x| \leq 2\lambda)} \\
\leq \; & C \left(\| (R_m - f) W \|_{L_p(\lambda \leq |x| \leq 2\lambda)} + \| fW \|_{L_p(\lambda \leq |x| \leq 2\lambda)} \right) \\
\leq \; & C \left(\frac{\sigma(\lambda)}{m} \| f' W \|_{L_p(\mathbb{R})} + \eta(\lambda) \| f' W \|_{L_p(\mathbb{R})} \right), \qquad (34)
\end{aligned}
$$

by Lemmas 4.1, 4.2 and Theorem 3.1. Also,

$$
\begin{aligned}
T_{32} \leq \; & \| R_m W \|_{L_p(2\lambda \leq |x| \leq q_n)} \| V_n \|_{L_\infty(2\lambda \leq |x| \leq q_n)} \\
\leq \; & \rho(m) \| f' W \|_{L_p(\mathbb{R})} C_1 \left(\frac{q_n}{n} \right)^2,
\end{aligned}
$$

by Lemmas 4.1, 4.2 and another application of Theorem 3.1. Combining this and the estimates in (31) to (34) gives

$$
\begin{aligned}
& \| (f - P_n) W \|_{L_p[-q_n, q_n]} \\
\leq \; & \| f' W \|_{L_p(\mathbb{R})} C \left\{ \frac{\sigma(\lambda)}{m} + \rho(m) \frac{q_n}{n} + \eta(\lambda) \right\}. \qquad (35)
\end{aligned}
$$

Then using this estimate and Theorem 3.1, we deduce that

$$\|P_nW\|_{L_p[-q_n,q_n]} \leq \|f'W\|_{L_p(\mathbb{R})} C\left\{\frac{\sigma(\lambda)}{m} + \rho(m)\frac{q_n}{n} + 1\right\}.$$

Combining this estimate, (30) and (35) gives

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})} \leq \|f'W\|_{L_p(\mathbb{R})} C\left\{\frac{\sigma(\lambda)}{m} + \rho(m)\frac{q_n}{n} + \eta(\lambda) + 4^{-n}\right\},$$

with C independent of $n, m, \lambda, \rho, \sigma$. The functions σ and ρ obey the conventions listed at the beginning of this section, and are independent of f, n, m, p, as is the constant C. For a given large enough $n \geq 1$, we choose $m = m(n)$ to be the largest integer $\leq n/2$ such that

$$\rho(m)\frac{q_n}{n} \leq \left(\frac{q_n}{n}\right)^{1/2}.$$

Since (by Theorem 2.1) $q_n/n \to 0$ as $n \to \infty$, while ρ is increasing and finite valued, necessarily $m = m(n)$ approaches ∞ as $n \to \infty$. Next, for the given $m = m(n)$, we choose the largest $\lambda = \lambda(n) \leq m$ such that

$$\sigma(\lambda) \leq \sqrt{m}$$

As σ is finite valued, necessarily $\lambda(n) \to \infty$, so $\eta(\lambda(n)) \to 0, n \to \infty$. Then for some sequence $\{\eta_n\}_{n=1}^{\infty}$ with limit 0, and which is independent of f,

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})}.$$

For the remaining finitely many n, we can set $\eta_n = \eta(0)$, and use

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbb{R})} \leq \|fW\|_{L_p(\mathbb{R})} \leq \eta(0)\|f'W\|_{L_p(\mathbb{R})}.$$

■

Proof of the necessity part of Theorem 1.2

We assume that (5) is true for every absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, where $p = 1$ or $p = \infty$. In particular, if we choose f to be 0 outside $[-1,1]$, and not a.e. a polynomial in $[-1,1]$, we obtain for some sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials with degrees tending to ∞,

$$\|P_nW\|_{L_p(|x|\geq 1)} \to 0, n \to \infty.$$

As P_n behaves for large $|x|$ like its leading term, this forces

$$\|x^nW(x)\|_{L_p(\mathbb{R})} < \infty,$$

for each $n \geq 0$. Then the hypothesis (21) of Lemma 2.4 is fulfilled, and consequently there exist $\{\xi_n\}_{n=1}^{\infty}$ such that (22) holds for all polynomials P_n of degree $\leq n$. Let us consider an absolutely continuous f with $f(0) = 0$ and $\|f'W\|_{L_p(\mathbb{R})}$ finite. Our hypothesis asserts that there are for large n polynomials $\{P_n\}_{n=1}^{\infty}$ of degree $\leq n$ with

$$\|(f-P_n)W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})}$$

$$\Rightarrow \|fW\|_{L_p(|x|\geq\xi_n)} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})} + \|P_nW\|_{L_p(|x|\geq\xi_n)}.$$

By Lemma 2.4, and then our hypothesis on $\{P_n\}_{n=1}^{\infty}$,

$$\begin{aligned}\|P_nW\|_{L_p(|x|\geq\xi_n)} &\leq C2^{-n}\|P_nW\|_{L_p[-1,1]} \\ &\leq C2^{-n}\left(\|fW\|_{L_p[-1,1]} + \eta_n\|f'W\|_{L_p(\mathbb{R})}\right).\end{aligned}$$

Here

$$\begin{aligned}\|fW\|_{L_p[0,1]} &\leq \|W\|_{L_\infty[0,1]} \left\|\int_0^x f'(t)\,dt\right\|_{L_p[0,1]} \\ &\leq \|W\|_{L_\infty[0,1]} \left\|f'\right\|_{L_p[0,1]} \\ &\leq \|W\|_{L_\infty[0,1]}\|W^{-1}\|_{L_\infty[0,1]} \left\|f'W\right\|_{L_p[0,1]}.\end{aligned}$$

A similar inequality holds over $[-1,0]$ and hence

$$\|fW\|_{L_p[-1,1]} \leq 2\|W\|_{L_\infty[-1,1]}\|W^{-1}\|_{L_\infty[-1,1]}\|f'W\|_{L_p[-1,1]}.$$

The case $p=\infty$ is easier. Combining all the above inequalities gives

$$\|fW\|_{L_p(|x|\geq\xi_n)} \leq \eta_n^*\|f'W\|_{L_p(\mathbb{R})},$$

where $\{\eta_n^*\}_{n=1}^{\infty}$ has limit 0 and is independent of f. The same inequality then holds for the L_p norm of fW over $|x|\geq\lambda$, where $\lambda\in\left[\xi_n,\xi_{n+1}\right]$. It follows that there is a positive decreasing function η with limit 0 at ∞ such that (23) holds for absolutely continuous f with $f(0)=0$ and $\|f'W\|_{L_p(\mathbb{R})}$ finite. Then Theorem 3.1 gives the limit (6). ■

5. Proof of Theorem 1.3

In this section, we let

$$W_2(x) = \exp\left(-x^2\right), x\in\mathbb{R},$$

denote the Hermite weight. Moreover, we determine q, s by the equations

$$\frac{1}{r}+\frac{1}{s}=1 \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$

The construction is more complicated than that in [10], but the general idea is the same. We choose intervals

$$[j-\alpha_j, j+\alpha_j], j\geq 3$$

where $\alpha_j\leq\frac{1}{2j}$, $j\geq 3$. We set

$$W(x) = W_2(x), \quad x\in\mathbb{R}\backslash\bigcup_{j=3}^{\infty}(j-\alpha_j, j+\alpha_j). \tag{36}$$

(I) For the case where $p<r$, we set

$$W(j) = W_2(j)/[j\log j], \; j\geq 3, \tag{37}$$

choose

$$\beta\in(s,q) \tag{38}$$

and

$$\alpha_j = \frac{1}{2j\left(\log j\right)^{\beta}},\ j \geq 3. \tag{39}$$

(II) For the case where $p > r$, we set

$$W\left(j\right) = W_2\left(j\right)\left[j \log j\right],\ j \geq 3, \tag{40}$$

choose

$$\beta \in \left(r, p\right) \tag{41}$$

and

$$\alpha_j = \frac{1}{2j\left(\log j\right)^{\beta}},\ j \geq 3. \tag{42}$$

In both cases we then define W so that W/W_2 is linear in $[j - \alpha_j, j]$ and in $[j, j + \alpha_j]$. This ensures that W is continuous in $\mathbb{R}$. (Of couse we could ensure it is C^∞ by smoothing at j and $j \pm \alpha_j$). It also implies under (38) that,

$$1 \geq W\left(x\right)/W_2\left(x\right) \geq \frac{1}{1+x^2}, x \in \mathbb{R}, \tag{43}$$

and under (40),

$$1 \leq W\left(x\right)/W_2\left(x\right) \leq 1 + x^2, x \in \mathbb{R}. \tag{44}$$

(Since $\log x = o\left(x\right)$, these inequalities are clear for large $|x|$. However they are even true for "small" $|x|$, as shown by some simple calculations.) We shall make repeated use of the fact that uniformly in j and x,

$$W_2\left(x\right) \sim W_2\left(j\right), \ x \in \left[j - \alpha_j, j + \alpha_j\right],$$

as follows since $\alpha_j \leq \frac{1}{2j}$. We now show that W fulfils the asymptotic behavior required for Theorem 1.3.

Lemma 4.2
(a) Let $p < r$ and W satisfy (37), (38) and (39). Then

$$\limsup_{x\to\infty} \left\|W^{-1}\right\|_{L_q[0,x]} \left\|W\right\|_{L_p[x,\infty)} = \infty \tag{45}$$

but

$$\lim_{x\to\infty} \left\|W^{-1}\right\|_{L_s[0,x]} \left\|W\right\|_{L_r[x,\infty)} = 0. \tag{46}$$

(b) Let $p > r$ and W satisfy (40), (41) and (42). Then (45) and (46) are valid.

Proof
(a) Note that as $1 \leq p < r$, so $p, s < \infty$. Let $c > 0$. Some simple calculations show that for $1 \leq a \leq b$,

$$\int_a^b W_2^{-c} \sim W_2^{-c}\left(b\right) \min\left\{\frac{1}{b}, b - a\right\} \tag{47}$$

and if also $b \leq 2a$,

$$\int_a^b W_2^c \sim W_2^c(a) \min\left\{\frac{1}{b}, b-a\right\}. \tag{48}$$

Since $\alpha_j = O\left(\frac{1}{j}\right)$, we see that $W_2(j+\alpha_j) \sim W_2(j)$ and hence applying (48),

$$\int_j^\infty W^p \geq \int_{j+\alpha_j}^{j+1-\alpha_{j+1}} W_2^p \geq \frac{C}{j} W_2(j)^p.$$

Moreover, by (47), if $q < \infty$,

$$\int_0^j W^{-q} \geq C(j\log j)^q \int_{j-\frac{\alpha_j}{2}}^j W_2^{-q} \geq C(j\log j)^q \alpha_j W_2(j)^{-q}.$$

Then

$$\begin{aligned}\left\|W^{-1}\right\|_{L_q[0,j]} \|W\|_{L_p[j,\infty)} &\geq C[j\log j]\, \alpha_j^{1/q} j^{-1/p} \\ &= C(\log j)^{1-\beta/q} \to \infty,\end{aligned}$$

$j \to \infty$, by (38). We then have (45) for the case $1 < p, q < \infty$. If $q = \infty$, it is easy to see that (45) persists, by minor modifications of the above arguments.

The proof of (46) is a little more difficult because it involves a full limit. Let $x \geq 2$ and j_0 denote the least integer $\geq x$. We see that as $\alpha_j = O\left(\frac{1}{j}\right)$,

$$\begin{aligned}\int_0^x W^{-s} &\leq \int_{(0,x)\setminus \bigcup_{j=3}^{j_0}(j-\alpha_j, j+\alpha_j)} W_2^{-s} + \sum_{j=3}^{j_0-1} \int_{j-\alpha_j}^{j+\alpha_j} W^{-s} + \int_{[j_0-\alpha_{j_0}, x]} W^{-s} \\ &\leq \int_0^x W_2^{-s} + C\sum_{j=3}^{j_0-1} \alpha_j W_2^{-s}(j)(j\log j)^s + C\alpha_{j_0} W^{-s}(x)(j_0\log j_0)^s \\ &\leq CW_2(x)^{-s}/x + CW_2^{-s}(x)\, x^{s-1}(\log x)^{s-\beta},\end{aligned}$$

as for large enough j, and some $\theta < 1$ independent of j,

$$\frac{\alpha_j W_2^{-s}(j)(j\log j)^s}{\alpha_{j-1} W_2^{-s}(j-1)((j-1)\log(j-1))^s} < \theta.$$

We also used (47). Then this and (43) give

$$\begin{aligned}\left\|W^{-1}\right\|_{L_s[0,x]} \|W\|_{L_r[x,\infty)} &\leq CW_2^{-1}(x)\, x^{1-1/s}(\log x)^{1-\beta/s} \|W_2\|_{L_r[x,\infty)} \\ &\leq CW_2^{-1}(x)\, x^{1-1/s}(\log x)^{1-\beta/s} W_2(x)\, x^{-1/r} \\ &= C(\log x)^{1-\beta/s} \to 0,\end{aligned}$$

$x \to \infty$ as $\beta > s$, recall (38).

(b) This is very similar to (a). Note that as $p > r \geq 1$, so $r, q < \infty$. By

(40), if $p<\infty$,

$$\int_j^\infty W^p \geq C \int_j^{j+\alpha_j/2} (j\log j)^p W_2^p \geq C\alpha_j j^p (\log j)^p W_2(j)^p .$$

Moreover,

$$\int_0^j W^{-q} \geq \int_{j-1+\alpha_{j-1}}^{j-\alpha_j} W_2^{-q} \geq Cj^{-1} W_2(j)^{-q},$$

by (47). Then

$$\begin{aligned} \left\|W^{-1}\right\|_{L_q[0,j]} \|W\|_{L_p[j,\infty)} &\geq Cj^{-1/q}\alpha_j^{1/p} j \log j \\ &= C(\log j)^{1-\beta/p} \to \infty, \end{aligned}$$

as $\beta<p$ (recall (41)). If $p=\infty$, this argument requires minor modifications. So we have (46). Next, if j_1 is the largest integer $\leq x$,

$$\begin{aligned} \int_x^\infty W^r &\leq \int_{(x,\infty)\setminus \bigcup\limits_{j=j_1}^{\infty}(j-\alpha_j,j+\alpha_j)} W_2^r + \sum_{j=j_1}^{\infty} \int_{j-\alpha_j}^{j+\alpha_j} W_2^r (j\log j)^r + \int_{\left[x,j_1+\alpha_{j_1}\right]} W_2^r (j_1 \log j_1)^r \\ &\leq \int_x^\infty W_2^r + C\sum_{j=j_1+1}^{\infty} \alpha_j (j\log j)^r W_2^r(j) + CW_2^r(x)\,\alpha_{j_1}(j_1\log j_1)^r \\ &\leq CW_2(x)^r/x + j_1^{r-1}(\log j_1)^{r-\beta} W_2^r(x) \\ &\leq Cx^{r-1}(\log x)^{r-\beta} W_2^r(x), \end{aligned}$$

by (48) and as again for large j and some $\theta<1$,

$$\frac{\alpha_j (j\log j)^r W_2^r(j)}{\alpha_{j-1}((j-1)\log(j-1))^r W_2^r(j-1)} < \theta.$$

Then (46) and (47) gives

$$\begin{aligned} \left\|W^{-1}\right\|_{L_s[0,x]} \|W\|_{L_r[x,\infty)} &\leq C\left\|W_2^{-1}\right\|_{L_s[0,x]} W_2(x)\, x^{1-1/r} (\log x)^{1-\beta/r} \\ &\leq CW_2^{-1}(x)\, x^{-1/s} W_2(x)\, x^{1-1/r} (\log x)^{1-\beta/r} \\ &= C(\log x)^{1-\beta/r} \to 0, \end{aligned}$$

$x\to\infty$, as $\beta>r$ (recall (41)). ■

Proof of Theorem 1.3

This follows directly from the limit conditions in Lemma 4.2 and from Theorem 1.2. ■

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