ON CONVERSE MARCINKIEWICZ-ZYGMUND INEQUALITIES IN $L_p, p > 1$

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Abstract. We obtain converse Marcinkiewicz-Zygmund inequalities such as

$$\| P\nu \|_{L_p[-1,1]} \leq C \left(\sum_{j=1}^{n} \mu_j \, |P(t_j)|^p \right)^{1/p}$$

for polynomials P of degree $\leq n-1$, under general conditions on the points $\{t_j\}_{j=1}^n$ and on the function ν. The weights $\{\mu_j\}_{j=1}^n$ are appropriately chosen. We illustrate the results by applying them to extended Lagrange interpolation for exponential weights on $[-1, 1]$.

1. Introduction and Results

Let

$$-1 \leq t_1 < t_2 < ... < t_n \leq 1. \tag{1.1}$$

Converse Marcinkiewicz-Zygmund Inequalities have the form

$$\| P\nu \|_{L_p[-1,1]} \leq C \left(\sum_{j=1}^{n} \mu_j \, |P(t_j)|^p \right)^{1/p}. \tag{1.2}$$

They are valid for all $P \in \mathcal{P}_{n-1}$ ($\mathcal{P}_m$ denotes the polynomials of degree $\leq m$), and for appropriate choices of the $\{\mu_j\}_{j=1}^n$, the function $\nu : [-1, 1] \to \mathbb{R}$ and the constant C. One of their main applications is to Lagrange interpolation: let $L_n[f]$ denote the Lagrange interpolation polynomial to a function f at $\{t_j\}_{j=1}^n$ so that $L_n[f]$ has degree $\leq n-1$ and

$$L_n[f](t_j) = f(t_j), 1 \leq j \leq n.$$

From (1.2) we deduce

$$\| L_n[f]\nu \|_{L_p[-1,1]} \leq C \| f \|_{L_\infty[-1,1]} \left(\sum_{j=1}^{n} \mu_j \right)^{1/p}.$$

In particular, if we vary n, and the constant C as well as the sum of the weights $\sum_{j=1}^n \mu_j$ are bounded above independently of n, we obtain that the operators L_n are a sequence of uniformly bounded operators between appropriate spaces. The projection property

$$L_n[P] = P, P \in \mathcal{P}_{n-1}$$

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and density of polynomials then allow us to deduce mean convergence.

The oldest method to derive such inequalities involves mean convergence of orthogonal expansions [24], [25], [26]. Its scope is restricted by the requirement that the points $\{t_k\}$ be zeros of an orthogonal polynomial. See [10], [11] for the most recent and general results on this method and see [6] for a survey of earlier results. A second method due to König [3], [4] effectively regards part of the Lagrange interpolation polynomials as a discrete Hilbert transform and uses boundedness of the Hilbert transform. It is this method that we modify in this paper. We note that when one requires that (1.2) only holds for polynomials P of degree $\leq \varepsilon n$ with some $\varepsilon > 0$, and not of degree $\leq n-1$, then far more general results are possible, the most impressive being due to Mastroianni and Totik [12]. However in this situation, the application to Lagrange interpolation is lost.

In stating our results, we need the representation

$$L_n[f](t) = \sum_{j=1}^{n} f(t_j)\ell_j(t) \tag{1.3}$$

where $\{\ell_j\}$ are the fundamental polynomials of Lagrange interpolation at the $\{t_j\}$. We also set

$$t_j := 1, j > n; t_j := -1, j \leq 0. \tag{1.4}$$

For a function $\nu : [-1,1] \to \mathbb{R}$, we set $\nu := 0$ outside $[-1,1]$. The simplest case of our results is:

Theorem 1.1
Let $n \geq 1$ and let $\{t_j\}_{j=1}^n$ satisfy (1.1). Let $\nu : [-1,1] \to [0,\infty)$ be measurable. Let $\pi_n(t)$ be a polynomial of degree n whose zeros are $\{t_j\}_{j=1}^n$, normalized by the condition

$$|\pi_n \nu| \leq 1 \text{ in } [-1,1]. \tag{1.5}$$

Let $L \geq 1$ and

$$\delta_j := t_{j+L} - t_{j-L}, 1 \leq j \leq n. \tag{1.6}$$

Assume there exists $\alpha > 0$ such that for $1 \leq j,k \leq n$ with $|j-k| \geq L$,

$$|t_j - t_k| \geq \alpha \, |j-k|^{1/3} \left[1 + \log|j-k|\right]^{2/3} \delta_j. \tag{1.7}$$

Let $\varepsilon > 0, 1 < p < \infty$. Then for $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \sum_{j=1}^{n} |P(t_j)|^p \left\{ \int_{t_j-\varepsilon\delta_j}^{t_j+\varepsilon\delta_j} |\ell_j \nu|^p + \frac{\delta_j}{\left[\delta_j \left|\pi_n'(t_j)\right|\right]^p} \right\}. \tag{1.8}$$

The constant C depends on $L, \alpha, \varepsilon, p$ but is independent of $\nu, \{t_j\}_{j=1}^n, n, P$.

Remarks
(a) We note that while (1.5) is a normalization condition, our main hypothesis is (1.7). In it one could replace $|j-k|^{1/3} \left[1 + \log|j-k|\right]^{2/3}$ by $\psi\left(|j-k|\right)$ where $\psi : [1,\infty) \to (0,\infty)$ is any function satisfying

$$\sum_{i=1}^{\infty} \psi(i)^{-3} < \infty.$$

(b) Some insight into the spacing condition (1.7) is provided by the Chebyshev points

$$t_j := \cos((n-j+\frac{1}{2})\frac{\pi}{n}), 1 \le j \le n.$$

It is easily seen that for $|j-k| \ge 1$,

$$|t_j - t_k| \ge C\,|k-j|\,|t_{j+1} - t_{j-1}| \tag{1.9}$$

where C is independent of j, k, n so (1.7) holds with $L = 1$ in a much stronger form. More generally, given points

$$t_j = \cos\theta_j, 0 \le j \le n+1,$$

where $\theta_0 = \pi, \theta_{n+1} = 0$ and

$$\frac{C_1}{n} \le \theta_j - \theta_{j+1} \le \frac{C_2}{n}, 0 \le j \le n \tag{1.10}$$

then (1.9) holds with some C depending only on C_1, C_2. Indeed orthonormal polynomials corresponding to Jacobi weights and their myriad of generalisations satisfy this with C_1, C_2 independent of j, n [13].

However, there are orthonormal polynomials for weights on $(-1, 1)$ whose zeros do not satisfy (1.10). For the class of weights

$$W(x) := \exp(-\underbrace{\exp(\exp(\ldots\exp}_{k \text{ times}}(1-x^2)^{-\beta}))) \tag{1.11}$$

where $k \ge 0$ and $\beta > 0$, it follows from the results of [5] that if $\{t_j\}_{j=1}^n$ are the zeros of the nth orthonormal polynomial for the weight W^2, then

$$|t_j - t_k| \ge C\,|k-j|^{2/3}\,|t_{j+1} - t_{j-1}|$$

and $2/3$ is the largest exponent independent of j, k, n. We shall establish this in Lemma 6.1 below.

(c) Obviously any set of distinct points $\{t_j\}_{j=1}^n$ satisfies the hypotheses of the theorem with *some* $\alpha > 0$. However the theorem is useful only when applied to varying n, and in such a situation one would require α to be independent of n. Likewise the fact that the constant C in (1.8) is independent of ν is essential in applications where one needs to vary ν.

(d) The estimate (1.7) is required only for t_j, t_k with $|j-k| \ge L$. Thus, if $L = 2$, we could allow even and odd order points t_{2j} and t_{2j+1} to be close, but of course this would be reflected in the size of $\pi_n'(t_{2j})$ or $\pi_n'(t_{2j+1})$.

In some applications, it is unfortunate that the weight ν in the normalising condition (1.5) is the same as that in the right-hand side of (1.8). We can insert other weights $\omega : \mathbb{R} \to [0, \infty)$ provided their pth power satisfies the A_p condition. Let $1 < p < \infty$. Recall that ω^p satisfies the A_p condition if for all intervals $I \subset \mathbb{R}$ with length $|I|$,

$$\left[\frac{1}{|I|}\int_I \omega^p\right]\left[\frac{1}{|I|}\int_I \omega^{-\frac{p}{p-1}}\right]^{p-1} \le \beta. \tag{1.12}$$

The smallest such β is called the A_p bound of ω^p. The main feature we use of A_p weights is that they admit a weighted bound on the Hilbert transform. Recall that if $g \in L_p(\mathbb{R})$ its Hilbert transform

$$H[g](x) := \lim_{\varepsilon \to 0+} \int_{|t-x| \geq \varepsilon} \frac{g(t)}{x-t} dt \tag{1.13}$$

is defined for a.e. x. For an A_p weight ω^p, there exists $\gamma > 0$ such that for all $g \in L_p(\mathbb{R})$,

$$\| H[g]\omega \|_{L_p(\mathbb{R})} \leq \gamma \| g\omega \|_{L_p(\mathbb{R})} . \tag{1.14}$$

The proofs of (1.14) indicate that the size of γ depends only on the size of the A_p bound β of ω^p. However we could not find a reference where this is formally stated, so shall avoid using it. Another feature of A_p weights, is that they are *doubling weights*: there exists $\Delta > 0$ such that for all $a \in \mathbb{R}, \delta > 0$,

$$\int_{a-2\delta}^{a+2\delta} \omega^p \leq \Delta \int_{a-\delta}^{a+\delta} \omega^p. \tag{1.15}$$

The smallest such Δ is called the doubling constant; we may take $\Delta = 2^p \beta$. This is an easy consequence of (1.12), see [20, p.196].

Theorem 1.1 is the case $\omega = 1$ of

Theorem 1.2

Assume the hypotheses of Theorem 1.1, except that instead of (1.5), we assume

$$|\pi_n \nu| \leq \omega \text{ in } [-1, 1] \tag{1.16}$$

where $\omega : \mathbb{R} \to [0, \infty)$ *and for some* $1 < p < \infty$, ω^p *is an* A_p *weight. Assume moreover, that there exists* $\kappa > 0$ *such that for* $1 \leq j \leq n$,

$$\int_{\{t \in [-2,2] : |t-t_j| \geq \delta_j\}} \frac{\omega^p(t)}{|t-t_j|^3} dt \leq \kappa \frac{\int_{t_j-\delta_j}^{t_j+\delta_j} \omega^p}{\delta_j^3}. \tag{1.17}$$

(*In particular, there exists such a* κ *if the doubling constant* Δ *in (1.15) satisfies* $\Delta < 8$.) *Then there exists* $C > 0$ *and an integer* K *such that for* $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \sum_{j=1}^{n} |P(t_j)|^p \left\{ \int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p + \frac{\int_{t_j-\varepsilon\delta_j}^{t_j+\varepsilon\delta_j} \omega^p}{[\delta_j |\pi_n'(t_j)|]^p} \right\}. \tag{1.18}$$

The integer K *depends only on* L, α, *and the constant* C *depends on* $L, \alpha, \gamma, \Delta, \kappa, \varepsilon, p$ *but is independent of* $\nu, \omega, \{t_j\}_{j=1}^n, n, P$.

We note that one may replace $[t_{j-K}, t_{j+K+1}]$ by $[t_j - \varepsilon\delta_j, t_j + \varepsilon\delta_j]$ in the first integral in the right-hand side of (1.18) if $p \geq 3$ or if we assume that the ratio

$$(t_{j+L} - t_{j-L}) / (t_{j+1+L} - t_{j+1-L})$$

is bounded above and below by suitable positive constants - in applications, this is usually the case, with constants independent of n.

In some situations instead of a sup norm condition on $\pi_n \nu$ one may only have an L_r condition with $r < \infty$. This is the focus of the following result:

Theorem 1.3

Assume the hypotheses of Theorem 1.1, except that instead of (1.5), we assume that for some $1 < r < \infty$,

$$\| \pi_n \nu / \omega \|_{L_r[-1,1]} \leq 1 \tag{1.19}$$

where $\omega : \mathbb{R} \to [0, \infty)$ *and for some* $1 < p < r$, $\omega^{rp/(r-p)}$ *is an* $A_{rp/(r-p)}$ *weight. Assume moreover that (1.17) holds with* p *replaced by* $pr/(r-p)$. *Then for* $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \left\{ \sum_{j=1}^{n} |P(t_j)|^p \int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p \right\} + C \left\{ \sum_{j=1}^{n} \left(\int_{t_j - \varepsilon\delta_j}^{t_j + \varepsilon\delta_j} \omega^{rp/(r-p)} \right) \left[\frac{|P(t_j)|}{\delta_j \, |\pi_n'(t_j)|} \right]^{\frac{rp}{r-p}} \right\}^{\frac{r-p}{r}}. \tag{1.20}$$

The integer K *depends only on* L, α, *and the constant* C *depends on* $L, \alpha, \gamma, \Delta, \kappa, \varepsilon, p$ *but is independent of* $\nu, \omega, \{t_j\}_{j=1}^n, n, P$.

It is obviously of interest to have conditions on π_n that allow us to replace the two terms in the right-hand side of (1.8) or (1.18) by a single one. This is the focus of the next two corollaries:

Corollary 1.4

Assume the hypotheses of Theorem 1.2 and in addition that for some $\varepsilon, \rho > 0$ *and* $1 \leq j \leq n$,

$$\int_{t_{j-L}}^{t_{j+L}} |\pi_n \nu|^p \geq \rho \int_{t_j - \varepsilon\delta_j}^{t_j + \varepsilon\delta_j} \omega^p. \tag{1.21}$$

Then for $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \sum_{j=1}^{n} |P(t_j)|^p \int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p . \tag{1.22}$$

The integer K *depends only on* L, α, *and the constant* C *depends on* $L, \alpha, \gamma, \Delta, \kappa, \varepsilon, \rho, p$ *but is independent of* $\nu, \omega, \{t_j\}_{j=1}^n, n, P$.

Corollary 1.5

Assume the hypotheses of Theorem 1.2 and in addition that one of the following holds:

(I) ν' *exists a.e. and for some* $\rho > 0$ *and* $1 \leq j \leq n$,

$$\int_{t_{j-K}}^{t_{j+K+1}} \left| (\pi_n \nu)' \right|^p \leq \rho \delta_j^{-p} \int_{t_j - \varepsilon\delta_j}^{t_j + \varepsilon\delta_j} \omega^p. \tag{1.23}$$

(II) For some $\rho, \rho_1, \rho_2 > 0$ *and* $1 \leq j \leq n$,

$$\nu(t_j)^p \int_{t_{j-K}}^{t_{j+K+1}} |\pi_n'|^p \leq \rho \delta_j^{-p} \int_{t_j - \varepsilon\delta_j}^{t_j + \varepsilon\delta_j} \omega^p \tag{1.24}$$

and

$$\rho_1 \leq \nu(t)/\nu(t_j) \leq \rho_2, t \in [t_{j-1}, t_{j+1}] \cap [-1, 1]. \tag{1.25}$$

Then for $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \sum_{j=1}^{n} |P(t_j)|^p \frac{\int_{t_j-\varepsilon\delta_j}^{t_j+\varepsilon\delta_j} \omega^p}{[\delta_j |\pi_n'(t_j)|]^p}. \tag{1.26}$$

The constant C *depends on* $L, \alpha, \gamma, \Delta, \kappa, \rho, \rho_1, \rho_2, \varepsilon, p$ *but is independent of* $\nu, \omega, \{t_j\}_{j=1}^n, n, P$.

Following is a general example of how Theorem 1.2 can be applied.

Example 1.6
Let us assume the hypotheses of Theorem 1.2 and let $\mu : [-1,1] \to [0,\infty)$ be measurable. Then for functions $f : [-1,1] \to \mathbb{R}$, we have

$$\| L_n[f]\nu \|_{L_p[-1,1]} \leq C\Omega \| f\mu \|_{L_\infty[-1,1]} \tag{1.27}$$

where

$$\Omega := \left[\sum_{j=1}^{n} \left\{ \frac{\int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p}{\mu(t_j)^p} + \frac{\int_{t_j-\varepsilon\delta_j}^{t_j+\varepsilon\delta_j} \omega^p}{[\delta_j |\pi_n'(t_j)\mu(t_j)|]^p} \right\} \right]^{1/p} \tag{1.28}$$

and C, K are independent of $\nu, \mu, n, f, \{t_j\}$, but depend on the constants $L, \alpha, \gamma, \Delta, \kappa, \varepsilon, p$. In particular, if we vary n, and possibly also ν, μ but the constants $L, \alpha, \gamma, \Delta, \kappa$ may be chosen independent of n, while Ω is bounded above independently of n, and (1.17) holds uniformly in n, then we obtain uniform boundedness of $\{L_n\}$ in appropriately weighted spaces.

The point of this is that in many of the classical situations where mean convergence of Lagrange interpolation has been studied, notably at the zeros of orthogonal polynomials for the many generalisations of Jacobi weights [10], [11], [13], [16], [18] or the exponential weights on $[-1,1]$ [7] or the Freud or Erdös weights on $\mathbb{R}$ [1], [8] all the information that is needed to estimate Ω is readily available: namely lower bounds on $\pi_n'(t_j)$, upper and lower bounds on δ_j, and upper bounds on π_n and ℓ_j. For those weights, one may scale the interpolation points to $[-1,1]$ and may choose $L = 1$ and (1.7) will be satisfied so all one needs to check is whether Ω is bounded independently of n and whether (1.17) is satisfied with appropriate κ. This should in principle allow one to unify a wide range of results.

In most of the applications to interpolation at zeros of orthogonal polynomials, it suffices to consider ω^p of a very special form. This is the subject of the following theorem:

Theorem 1.7
Let $b \in \left[\frac{1}{2}, 1\right]$, $c \in [0, \frac{1}{2}]$ *and*

$$-\frac{1}{p} < \sigma < 1 - \frac{1}{p}. \tag{1.29}$$

Let

$$\omega(t) := \left(\left| 1 - \left| \frac{t}{b} \right| \right| + c \right)^{\sigma}. \tag{1.30}$$

Assume that $n \geq 1$ and that $\{t_j\}_{j=1}^n$ satisfy (1.7). Assume moreover, that for some $\tau > 0$, and $1 \leq j \leq n$,

$$\left|1 - \left|\frac{t_j}{b}\right|\right| + c \geq \tau \delta_j. \tag{1.31}$$

Let $\nu : [-1,1] \to [0,\infty)$ be measurable and let the polynomial $\pi_n(t)$, whose zeros are $\{t_j\}_{j=1}^n$, satisfy (1.16). Then for $P \in \mathcal{P}_{n-1}$,

$$\int_{-1}^{1} |P\nu|^p \leq C \sum_{j=1}^{n} |P(t_j)|^p \left\{ \int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p + \frac{\delta_j \omega\left(t_j\right)^p}{\left[\delta_j \left|\pi_n'(t_j)\right|\right]^p} \right\}. \tag{1.32}$$

The integer K depends only on L, α, and the constant C depends on $L, \alpha, \sigma, \tau, p$ but is independent of $\nu, \omega, \{t_j\}_{j=1}^n, b, c, n, P$.

We believe that several of the results on mean convergence of Lagrange interpolation associated with generalized Jacobi weights [13], [16], [18], with Freud weights[8], with exponential weights on $[-1,1]$ [7] or Erdös weights [1] may be deduced from Theorem 1.7.

As an illustration of Theorem 1.7, we deduce a new result on extended Lagrange interpolation associated with exponential weights on $[-1,1]$. In defining a suitable class of weights and in the sequel, we use $\sim$ in the following sense: given a real interval I and functions $f, g : I \to \mathbb{R}$, we write

$$f \sim g \text{ in } I$$

if there exist positive constants C_1, C_2 independent of t such that

$$C_1 \leq f\left(t\right)/g\left(t\right) \leq C_2, t \in I.$$

Similar notation is used for sequences and sequences of functions. Moreover, $C, C_1, C_2, \ldots$ denote positive constants independent of n, t, x and $P \in P_n$. The same symbol need not denote the same constant in different occurrences. To indicate that C does or does not depend on a parameter τ we write respectively $C = C\left(\tau\right)$ or $C \neq C\left(\tau\right)$.

Definition 1.8
Let $W = e^{-Q}$, where $Q : (-1,1) \to [0,\infty)$ is even, twice continuously differentiable and $Q' \geq 0, Q'' \geq 0$ in $(-1,1)$. Assume moreover that the function

$$T(t) := 1 + tQ''(t)/Q'(t), t \in (-1,1)\backslash\{0\} \tag{1.33}$$

is increasing in $(0,1)$ with

$$\lim_{t \to 0+} T(t) > 1 \tag{1.34}$$

and for t close enough to 1,

$$T(t) \sim Q'(t)/Q(t)$$

while for some $A > 2$ and t close enough to 1,

$$T(t) \geq \frac{A}{1-t^2}. \tag{1.35}$$

Then we write $W \in \mathcal{W}$.

The archetypal examples of $W \in \mathcal{W}$ are given by (1.11). See [5], [7] for further orientation on these weights.

Associated with the weight W^2 (note the square), we can define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., n \geq 0,$$

satisfying

$$\int_{-1}^{1} p_n p_m W^2 = \delta_{mn}.$$

We order the zeros of p_n as

$$-1 < x_{nn} < x_{n-1,n} < ... < x_{1n} < 1.$$

Mean convergence of Lagrange interpolation at the zeros of p_n for $W \in \mathcal{W}$ was studied in [7]. One of the unfortunate features is that in working in L_p norms with $p > 4$, severe weighting factors are required. This comes from the fact that $p_n(x)W(x)$ behaves essentially like $\left|1 - \frac{|x|}{a_n}\right|^{-1/4}$, where a_n is the nth Mhaskar-Rahmanov-Saff number (we shall define this in Section 6 below). The latter also has an impact on Lebesgue functions of Lagrange interpolation.

It was J. Szabados who came up with the idea of reducing the Lebesgue function of Lagrange interpolation by adding two extra interpolation points. Let ξ_n be the positive point where $p_n W$ attains its maximum in $[-1, 1]$:

$$|p_n W|(\pm\xi_n) = \|p_n W\|_{L_\infty[-1,1]}. \tag{1.36}$$

Szabados [21] considered for Freud weights the interpolation points

$$\{t_{jn}\}_{j=1}^{n} := \{x_{j,n-2}\}_{j=1}^{n-2} \cup \{-\xi_{n-2}, \xi_{n-2}\} \tag{1.37}$$

and showed how the Lebesgue constant reduced dramatically from $O\left(n^{1/6}\right)$ to $O(\log n)$. This was subsequently extended to exponential weights on $[-1, 1]$ [1].

The addition of the two points also leads to more elegant results in mean convergence of Lagrange interpolation. For Freud weights, this was explored in [9]. Here we use Theorem 1.7 to treat exponential weights on $[-1, 1]$:

Theorem 1.9
Let $W \in \mathcal{W}$, let $1 < p < \infty$ and $d > -\frac{1}{p}$. Let $f : [-1, 1] \to \mathbb{R}$ be such that fW is bounded and Riemann integrable in $[-1, 1]$. Then if $L_n[f]$ denotes the Lagrange interpolation polynomial to f at $\{t_{jn}\}_{j=1}^{n}$, we have

$$\lim_{n\to\infty} \|(f - L_n[f])(t) W(t) \left(1 - t^2\right)^d \|_{L_p[-1,1]} = 0. \tag{1.38}$$

In particular, this is true for $d = 0$.

One may compare this to the results in [7], where without the additional two points, more complicated results arise for $p > 4$.

This paper is organised as follows: we prove Theorem 1.2 in Section 2, and then immediately deduce Theorem 1.1. We prove Theorem 1.3 in Section 3 by indicating the modifications required to the proofs of the previous section. In Section 4, we prove Corollaries 1.4 and 1.5 and in Section 5, we prove Theorem 1.7. In Section 6, we prove Theorem 1.9.

2. The Proof of Theorem 1.2

Throughout this section, we assume the hypotheses and notation of Theorem 1.2. In particular (1.1) holds and $t_j = 1, j > n, t_j = -1, j \leq 0$. We let

$$I_j := [t_j - \delta_j, t_j + \delta_j], 1 \leq j \leq n,$$

where

$$\delta_j = t_{j+L} - t_{j-L},$$

and

$$\mathcal{J}_j := [t_j, t_{j+1}], 0 \leq j \leq n. \tag{2.1}$$

Note that

$$\mathcal{J}_j \cup \mathcal{J}_{j-1} \subset I_j.$$

We let χ_j denote the characteristic function of I_j and $|I_j| = 2\delta_j$ denote the length of I_j. As a consequence of our hypothesis (1.7), if $K \geq L$ satisfies

$$\alpha K^{1/3} (1 + \log K)^{2/3} \geq 4 \tag{2.2}$$

then

$$|j - k| \geq K \Rightarrow |t_j - t_k| \geq 4 \max\{\delta_j, \delta_k\} \Rightarrow I_j \cap I_k = \emptyset. \tag{2.3}$$

Of course, the size of K depends only on L and α. We fix a polynomial P of degree $\leq n-1$ and set

$$y_j := P(t_j)/\pi_n'(t_j), 1 \leq j \leq n. \tag{2.4}$$

We may then write

$$P(t) = L_n[P](t) = \pi_n(t) \sum_{j=1}^{n} \frac{y_j}{t - t_j}$$

$$= \left(\pi_n(t) H \left[\sum_{j=1}^{n} y_j \frac{\chi_j}{|I_j|} \right] \right) + \left(\pi_n(t) \sum_{j=1}^{n} y_j \left\{ \frac{1}{t - t_j} - H \left[\frac{\chi_j}{|I_j|} \right](t) \right\} \right)$$

$$=: J_1(t) + J_2(t). \tag{2.5}$$

Thus,

$$\| P\nu \|_{L_p[-1,1]} \leq \| J_1 \nu \|_{L_p[-1,1]} + \| J_2 \nu \|_{L_p[-1,1]} . \tag{2.6}$$

We begin with the estimation of the first term on the last right-hand side:

Lemma 2.1

$$\| J_1 \nu \|^p_{L_p[-1,1]} \leq \gamma^p (2K)^{p+1} \sum_{j=1}^{n} \left(\frac{|y_j|}{\delta_j} \right)^p \int_{I_j} \omega^p. \tag{2.7}$$

Proof
Now by our hypothesis (1.16)

$$|J_1 \nu| \leq \omega \left| H \left[\sum_{j=1}^{n} y_j \frac{\chi_j}{|I_j|} \right] \right| .$$

The boundedness of the Hilbert transform (1.14) then gives

$$\begin{aligned}\| J_1\nu \|^p_{L_p[-1,1]} &\leq \gamma^p \| \omega \sum_{j=1}^{n} y_j \frac{\chi_j}{|I_j|} \|^p_{L_p(\mathbb{R})} \\ &\leq \gamma^p \sum_{k=1}^{n} \int_{I_k} \left| \omega \sum_{j=k-K+1}^{k+K-1} y_j \frac{\chi_j}{|I_j|} \right|^p .\end{aligned}$$

Here we have used (2.3) which shows that if $|j-k| \geq K$ then $I_k \cap I_j$ is empty. We also set $y_j := 0$ if j lies outside the range $\{1, 2, ...n\}$. Using the inequality

$$(a_1 + a_2 + ...a_s)^p \leq s^p (a_1^p + a_2^p + ...a_s^p) \text{ if all } a_j \geq 0$$

we continue this as

$$\begin{aligned}\| J_1\nu \|^p_{L_p[-1,1]} &\leq \gamma^p (2K)^p \sum_{k=1}^{n} \sum_{j=k-K+1}^{k+K-1} \int_{I_k} \left| \omega y_j \frac{\chi_j}{|I_j|} \right|^p \\ &\leq \gamma^p (2K)^{p+1} \sum_{j=1}^{n} \left(\frac{|y_j|}{|I_j|} \right)^p \int_{I_j} \omega^p .\end{aligned}$$

Here we have used the fact that χ_j vanishes outside I_j. Then (2.7) follows.□

We next turn to the estimation of J_2. First we estimate a term in the sum defining J_2. It is here that the symmetry of the interval I_j about t_j is essential, and it is here that we obtain a substantial improvement over the method of König.

Lemma 2.2
Let

(2.8) $$g_j(t) := \frac{1}{t-t_j} - H\left[\frac{\chi_j}{|I_j|}\right](t).$$

Let $\beta > 0$. *Then for* $|t-t_j| \geq (1+\beta)\delta_j$, *we have*

(2.9) $$|g_j(t)| \leq \frac{1+\beta}{\beta} \frac{\delta_j^2}{|t-t_j|^3}.$$

Moreover, for $s \in I_j$,

(2.10) $$\frac{\beta}{1+\beta} \leq \frac{|t-s|}{|t-t_j|} \leq \frac{2+\beta}{1+\beta}.$$

Proof
Now as $t \notin I_j$, we have ordinary Riemann and not principal value integrals in the Hilbert transform:

$$\begin{aligned}g_j(t) &= \frac{1}{|I_j|} \int_{I_j} \left[\frac{1}{t-t_j} - \frac{1}{t-s} \right] ds \\ &= \frac{1}{|I_j|} \int_{I_j} \frac{t_j - s}{(t-t_j)(t-s)} ds \\ &= \frac{1}{|I_j|} \int_{I_j} \left\{ \frac{t_j - s}{(t-t_j)(t-s)} - \frac{t_j - s}{(t-t_j)^2} \right\} ds\end{aligned}$$

as t_j is the midpoint of I_j. We continue this as

$$g_j(t) = -\frac{1}{|I_j|\,(t-t_j)^2}\int_{I_j}\frac{(s-t_j)^2}{t-s}ds.$$

Next for $s \in I_j$,

$$|t-s| \geq |t-t_j| - \delta_j \geq |t-t_j|\left(1-\frac{1}{1+\beta}\right)$$

and we deduce the left inequality in (2.10), the right inequality in (2.10) follows similarly. Applying the lower bound in (2.10) to the last identity for g_j gives (2.9).□

We can now proceed with the estimation of J_2 of (2.5). We note that up to this step we have not used the full power of (1.7), all we needed was (2.3). The same is true of the following lemma: (note that $K \geq 1$).

Lemma 2.3

$$\| J_2\nu \|^p_{L_p[-1,1]} \leq 2^{6p}K^{p+1}\left(\sum_{k=1}^{n}|P(t_k)|^p\int_{t_{k-K}}^{t_{k+K+1}}|\ell_k\nu|^p + \gamma^p\sum_{k=1}^{n}\left(\frac{|y_k|}{|I_k|}\right)^p\int_{I_k}\omega^p\right)$$

$$+2^{3p}\sum_{j=1}^{n}\left(\int_{I_j}\omega^p\right)\left(\sum_{k:|k-j|\geq K}\frac{|y_k|\,\delta_k^2}{|t_j-t_k|^3}\right)^p. \tag{2.11}$$

Proof
Recall from (2.1) that the endpoints of $\{\mathcal{J}_j\}_{j=0}^n$ form a partition of $[-1,1]$ and that $\mathcal{J}_j \cup \mathcal{J}_{j-1} \subset I_j$. We have, with the notation (2.8),

$$\| J_2\nu \|^p_{L_p[-1,1]} = \sum_{j=0}^{n}\int_{\mathcal{J}_j}\left|\pi_n\nu\sum_{k=1}^{n}y_kg_k\right|^p$$

$$\leq 2^p\sum_{j=0}^{n}\int_{\mathcal{J}_j}\left|\pi_n\nu\sum_{k:|k-j|\leq K}y_kg_k\right|^p + 2^p\sum_{j=1}^{n}\int_{I_j}\left|\pi_n\nu\sum_{k:|k-j|\geq K}y_kg_k\right|^p$$

$$=: S_1 + S_2. \tag{2.12}$$

(Note that the inclusion of $\int_{\mathcal{J}_0}$ into $\int_{I_1}$ prevents strict inequality in the index of summation in each sum). Here

$$S_1 \leq 2^p(2K+1)^p\sum_{j=0}^{n}\sum_{k:|k-j|\leq K}\int_{\mathcal{J}_j}|\pi_n\nu y_kg_k|^p. \tag{2.13}$$

By the form (2.8) of g_k, and our choice (2.4) of y_k

$$\int_{\mathcal{J}_j}|\pi_n\nu y_kg_k|^p$$

$$\leq 2^p\left[|P(t_k)|^p\int_{\mathcal{J}_j}|\nu\ell_k|^p + \left(\frac{|y_k|}{|I_k|}\right)^p\int_{\mathcal{J}_j}|\pi_n\nu H[\chi_k]|^p\right]$$

and by first (1.16) and then (1.14),

$$\int_{\mathcal{J}_j} |\pi_n \nu H[\chi_k]|^p \leq \int_{\mathcal{J}_j} |\omega H[\chi_k]|^p$$
$$\leq \gamma^p \int_{\mathbb{R}} |\omega \chi_k|^p = \gamma^p \int_{I_k} \omega^p.$$

Thus,

$$S_1 \leq 2^{4p} K^p \sum_{j=0}^{n} \sum_{k:|k-j|\leq K} |P(t_k)|^p \int_{\mathcal{J}_j} |\nu \ell_k|^p + 2^{4p} K^p \gamma^p \sum_{j=0}^{n} \sum_{k:|k-j|\leq K} \left(\frac{|y_k|}{|I_k|}\right)^p \int_{I_k} \omega^p$$

$$\leq 2^{4p} K^p \sum_{k=1}^{n} |P(t_k)|^p \int_{t_{k-K}}^{t_{k+K+1}} |\nu \ell_k|^p + 2^{4p+2} K^{p+1} \gamma^p \sum_{k=1}^{n} \left(\frac{|y_k|}{|I_k|}\right)^p \int_{I_k} \omega^p. \tag{2.14}$$

Next, we consider S_2. If $|j-k| \geq K$ and $t \in I_j$, we have by (2.3),

$$|t - t_k| \geq |t_j - t_k| - \delta_j \geq |t_j - t_k| \left(1 - \frac{1}{4}\right),$$

so

$$t \in I_j \Rightarrow |t - t_k| \geq \frac{3}{4} |t_j - t_k| \geq 3\delta_k. \tag{2.15}$$

Then Lemma 2.2 gives

$$\int_{I_j} \left| \pi_n \nu \sum_{k:|k-j|\geq K} y_k g_k \right|^p$$
$$\leq \int_{I_j} \omega(t)^p \left(\frac{3}{2} \sum_{k:|k-j|\geq K} |y_k| \frac{\delta_k^2}{|t-t_k|^3} \right)^p dt$$
$$\leq \left(\int_{I_j} \omega^p \right) \left(4 \sum_{k:|k-j|\geq K} |y_k| \frac{\delta_k^2}{|t_j - t_k|^3} \right)^p.$$

Thus,

$$S_2 \leq 2^{3p} \sum_{j=1}^{n} \left(\int_{I_j} \omega^p \right) \left(\sum_{k:|k-j|\geq K} |y_k| \frac{\delta_k^2}{|t_j - t_k|^3} \right)^p. \tag{2.16}$$

Together with (2.14), this gives the result.□

We turn to the estimation of the third term in the right-hand side of (2.11). It is only here that we need the full power of (1.7). It is also only here that we need (1.17). We shall attend to the case where ω^p has a doubling constant < 8 in the proof of Theorem 1.2 itself.

Lemma 2.4

(2.17)
$$\sum_{j=1}^{n} \left(\int_{I_j} \omega^p \right) \left(\sum_{k:|k-j|\geq K} \frac{|y_k| \delta_k^2}{|t_j - t_k|^3} \right)^p \leq \left[\max\{ \frac{2}{\alpha^3 \log(K-1)}, 4\kappa K\} \right]^p \sum_{k=1}^{n} \left(\frac{|y_k|}{|I_k|} \right)^p \int_{I_k} \omega^p.$$

Proof
Define an $n\times n$ matrix $B=(b_{jk})_{j,k=1}^{n}$ as follows:

$$b_{jk}:=\begin{cases} 0, & |j-k|<K \\ \left(\frac{\int_{I_j}\omega^p}{\int_{I_k}\omega^p}\right)^{1/p}\frac{\delta_k^3}{|t_j-t_k|^3}, & |j-k|\geq K \end{cases}.$$

Moreover, define the n-vector $Z:=(z_k)_{k=1}^{n}$ where

$$z_k:=\frac{|y_k|}{\delta_k}\left(\int_{I_k}\omega^p\right)^{1/p}, 1\leq k\leq n.$$

We then see that the left-hand side of (2.17) is exactly

$$\| BZ \|_{\ell_p^n}^p$$

where (in this proof only) $\|\cdot\|_{\ell_p^n}$ denotes the usual norm on ℓ_p^n so that for example

$$\| Z \|_{\ell_p^n}=\left(\sum_{k=1}^{n}|z_k|^p\right)^{1/p}.$$

Thus the left-hand side of (2.17) is bounded above by

$$\left(\| B \|_{\ell_p^n\to\ell_p^n}\| Z \|_{\ell_p^n}\right)^p=\| B \|_{\ell_p^n\to\ell_p^n}^p\left[\sum_{k=1}^{n}\left(\int_{I_k}\omega^p\right)\left(\frac{|y_k|}{\delta_k}\right)^p\right]. \tag{2.18}$$

Here the norm on B is the usual operator norm on ℓ_p^n. It thus remains to estimate the norm of the matrix B. To do this, we use the following proposition, involving the notation

$$\| g \|_{L_p(d\mu)}:=\left(\int_\Omega |g|^p\,d\mu\right)^{\frac{1}{p}},$$

for μ measurable functions g on a measure space (Ω,μ). See [2,p.745] for example, for a proof of this.

Proposition 2.5
Let $q:=\frac{p}{p-1}$. Let (Ω,μ) be a measure space, $s,r:\Omega^2\longrightarrow\mathbb{R}$ and,

$$T[f](u):=\int_\Omega s(u,v)\,f(v)\,d\mu(v)$$

for μ measurable $f:\Omega\to\mathbb{R}$. Assume that

$$\sup_u\int_\Omega |s(u,v)|\,|r(u,v)|^q\,d\mu(v)\leq N.$$

$$\sup_v\int_\Omega |s(u,v)|\,|r(u,v)|^{-p}\,d\mu(u)\leq N.$$

Then T is a bounded operator from $L_p(d\mu)$to $L_p(d\mu)$. More precisely,

$$\| T \|_{L_p(d\mu)\longrightarrow L_p(d\mu)}\leq N.$$

To apply this to the matrix B, we use the discrete space $\Omega=\{1,2,3,...,n\}$ and the measure $\mu(\{j\}) = 1, 1 \leq j \leq n$, and let $s(j,k) := b_{jk}$, for all j,k. We must then find $\{r_{jk}\}$ such that for an appropriate N,

$$\sup_j \sum_{k=1}^{n} b_{jk} r_{jk}^{q} \leq N; \tag{2.19}$$

$$\sup_k \sum_{j=1}^{n} b_{jk} r_{jk}^{-p} \leq N; \tag{2.20}$$

We set

$$r_{jk} := \left(\frac{\int_{I_k} \omega^p}{\int_{I_j} \omega^p} \right)^{\frac{1}{pq}}$$

and then (2.19-2.20) become

$$\sup_j \sum_{k:|k-j|\geq K} \frac{\delta_k^3}{|t_j - t_k|^3} \leq N; \tag{2.21}$$

$$\sup_k \frac{\delta_k^3}{\int_{I_k} \omega^p} \sum_{j:|k-j|\geq K} \frac{\int_{I_j} \omega^p}{|t_j - t_k|^3} \leq N; \tag{2.22}$$

Our hypothesis (1.7) gives

$$\sum_{k:|k-j|\geq K} \frac{\delta_k^3}{|t_j - t_k|^3} \leq \frac{2}{\alpha^3} \sum_{i=K}^{\infty} \frac{1}{i (\log i)^2} \leq \frac{2}{\alpha^3 \log(K-1)}.$$

Next, (2.3) and (2.10) of Lemma 2.2 (with $\beta = 3$) show that for $|j-k| \geq K$ and $s \in I_j$, we have

$$\frac{|t_k - s|}{|t_k - t_j|} \leq \frac{5}{4}$$

so

$$\begin{aligned} \sum_{j:|k-j|\geq K} \frac{\int_{I_j} \omega^p}{|t_j - t_k|^3} &\leq 2 \sum_{j:|k-j|\geq K} \int_{I_j} \frac{\omega^p(s)}{|s - t_k|^3} ds \\ &\leq 4K \int_{\{s\in[-2,2]:|s-t_k|\geq 2\delta_k\}} \frac{\omega^p(s)}{|s - t_k|^3} ds \\ &\leq 4\kappa K \frac{\int_{I_k} \omega^p}{\delta_k^3}. \end{aligned}$$

In the second last line we used (2.3), which shows that each $s \in [-2,2]$ can belong to at most $2k-1$ of $\{I_j\}$, and in the last line we used (1.17). It follows that we have

$$\| B \|_{\ell_p^n \to \ell_p^n} \leq \max\{\frac{2}{\alpha^3 \log(K-1)}, 4\kappa K\}$$

and the result follows from (2.18).□

We can now turn to

The proof of Theorem 1.2
From (2.6) and Lemmas 2.1, 2.3, 2.4,

$$\| P\nu \|^p_{L_p[-1,1]} \leq C \left[\sum_{k=1}^{n} |P(t_k)|^p \int_{t_{k-K}}^{t_{k+K+1}} |\ell_k \nu|^p + \sum_{k=1}^{n} \left(\int_{I_k} \omega^p \right) \left(\frac{|y_k|}{\delta_k} \right)^p \right] \tag{2.23}$$

where C depends only on $\alpha, \gamma, \kappa, L, K, p$. Next, let m be the smallest integer such that

$$2^m \varepsilon \geq 1.$$

We see that if Δ is the doubling constant of ω^p,

$$\int_{I_k} \omega^p \leq \Delta^m \int_{t_k - \varepsilon\delta_k}^{t_k + \varepsilon\delta_k} \omega^p$$

and then recalling the definition (2.4) of y_k, we obtain the conclusion of Theorem 1.2. It remains to show that (1.17) holds if the doubling constant $\Delta < 8$. Now

$$\int_{\{t\in[-2,2]:|t-t_j|\geq\delta_j\}} \frac{\omega^p(t)}{|t-t_j|^3} dt \leq \sum_{k=1}^{\infty} \frac{1}{(2^{k-1}\delta_j)^3} \int_{\{t:2^{k-1}\delta_j\leq|t-t_j|\leq 2^k\delta_j\}} \omega^p(t)dt$$

$$\leq \frac{8 \int_{t_j-\delta_j}^{t_j+\delta_j} \omega^p}{\delta_j^3} \sum_{k=1}^{\infty} \left(\frac{\Delta}{8} \right)^k = \frac{\Delta}{1-\frac{\Delta}{8}} \frac{\int_{t_j-\delta_j}^{t_j+\delta_j} \omega^p}{\delta_j^3}$$

So (1.17) holds.□

We turn to

The Proof of Theorem 1.1
We apply Theorem 1.2 with $\omega = 1$. Firstly it is easily seen that (1.17) holds with $\kappa = \frac{1}{2}$. Next, our hypothesis (1.5) gives

$$\int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p \leq \int_{t_j-\varepsilon\delta_j}^{t_j+\varepsilon\delta_j} |\ell_j \nu|^p + \int_{t:|t-t_j|\geq\varepsilon\delta_j} \frac{dt}{|\pi_n'(t_j)|^p |t-t_j|^p}$$

$$= \int_{t_j-\varepsilon\delta_j}^{t_j+\varepsilon\delta_j} |\ell_j \nu|^p + \frac{2\delta_j}{(|\pi_n'(t_j)|\,\delta_j)^p} \frac{\varepsilon^{1-p}}{p-1}.$$

Then (1.8) follows from (1.18).□

3. The Proof of Theorem 1.3

In this section, we briefly indicate the modifications required to the proofs in the previous section. We continue to use the decomposition (2.5). Instead of Lemma 2.1, we have:

Lemma 3.1

$$\| J_1 \nu \|^p_{L_p[-1,1]} \leq C_1 \left(\sum_{j=1}^{n} \left(\frac{|y_j|}{\delta_j} \right)^{\frac{rp}{r-p}} \int_{I_j} \omega^{\frac{rp}{r-p}} \right)^{\frac{r-p}{r}}. \tag{3.1}$$

Here C_1 depends only on γ and K.

Proof

We have by Hölder's inequality, with parameters $\sigma = \frac{r}{p}, \tau = \frac{r}{r-p}$ so that $\frac{1}{\sigma} + \frac{1}{\tau} = 1$,

$$\| J_1 \nu \|_{L_p[-1,1]} = \| \pi_n \nu H \left[\sum_{j=1}^{n} y_j \frac{\chi_j}{|I_j|} \right] \|_{L_p[-1,1]}$$

$$\leq \| \pi_n \nu / \omega \|_{L_r[-1,1]} \| \omega H \left[\sum_{j=1}^{n} y_j \frac{\chi_j}{|I_j|} \right] \|_{L_{\frac{rp}{r-p}}[-1,1]} .$$

The rest of the proof follows as before.□

The analogue of Lemma 2.3 and 2.4 is:

Lemma 3.2

(3.2)

$$\| J_2 \nu \|^p_{L_p[-1,1]} \leq C_2 \left(\sum_{k=1}^{n} |P(t_k)|^p \int_{t_{k-K}}^{t_{k+K+1}} |\ell_k \nu|^p + \left(\sum_{k=1}^{n} \left(\frac{|y_k|}{|I_k|} \right)^{\frac{pr}{r-p}} \int_{I_k} \omega^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}} \right)$$

Here C_2 depends only on α, γ, κ and K.

Proof

Firstly (2.12) persists, and the estimate (2.13) for S_1 persists. Instead of proceeding as after (2.13), we use Hölder's inequality:

$$\int_{\mathcal{J}_j} |\pi_n \nu y_k g_k|^p$$

$$\leq 2^p \left[|P(t_k)|^p \int_{\mathcal{J}_j} |\nu \ell_k|^p + \left(\frac{|y_k|}{|I_k|} \right)^p \left(\int_{\mathcal{J}_j} |\pi_n \nu / \omega|^r \right)^{\frac{p}{r}} \left(\int_{\mathcal{J}_j} |\omega H[\chi_k]|^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}} \right]$$

$$\leq 2^p \left[|P(t_k)|^p \int_{\mathcal{J}_j} |\nu \ell_k|^p + \gamma^p \left(\frac{|y_k|}{|I_k|} \right)^p \left(\int_{\mathcal{J}_j} |\pi_n \nu / \omega|^r \right)^{\frac{p}{r}} \left(\int_{I_k} \omega^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}} \right]$$

by (1.14). Here of course γ is the constant in (1.14) but with p replaced by $pr/(r-p)$ and then (2.13) and another application of Hölder's inequality gives

$$S_1 \leq C \sum_{k=1}^{n} |P(t_k)|^p \int_{t_{k-K}}^{t_{k+K+1}} |\nu \ell_k|^p$$

$$+ C \sum_{k=1}^{n} \left(\frac{|y_k|}{|I_k|} \right)^p \left(\int_{\mathcal{J}_k} |\pi_n \nu / \omega|^r \right)^{\frac{p}{r}} \left(\int_{I_k} \omega^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}}$$

$$\leq C \sum_{k=1}^{n} |P(t_k)|^p \int_{t_{k-K}}^{t_{k+K+1}} |\nu \ell_k|^p$$

$$+ C \left(\int_{-1}^{1} |\pi_n \nu / \omega|^r \right)^{\frac{p}{r}} \left(\sum_{k=1}^{n} \left(\frac{|y_k|}{|I_k|} \right)^{\frac{pr}{r-p}} \int_{I_k} \omega^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}} .$$

Next, instead of (2.16), we obtain via an application of Hölder's inequality first to an integral and then to a sum,

$$S_2 \leq C \sum_{j=1}^{n} \left(\int_{I_j} |\pi_n \nu|^p \right) \left(\sum_{k:|k-j|\geq K} |y_k| \frac{\delta_k^2}{|t_j - t_k|^3} \right)^p$$

$$\leq C \left(\int_{-1}^{1} |\pi_n \nu / \omega|^r \right)^{\frac{p}{r}} \left(\sum_{j=1}^{n} \left(\int_{I_j} \omega^{\frac{pr}{r-p}} \right) \left(\sum_{k:|k-j|\geq K} |y_k| \frac{\delta_k^2}{|t_j - t_k|^3} \right)^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}}.$$

By proceeding exactly as in Lemma 2.4 with p replaced by $pr/(r-p)$, we obtain

$$S_2 \leq C \left(\sum_{k=1}^{n} \left(\frac{|y_k|}{|I_k|} \right)^{\frac{pr}{r-p}} \int_{I_k} \omega^{\frac{pr}{r-p}} \right)^{\frac{r-p}{r}}.$$

Then (3.2) follows.□

The Proof of Theorem 1.3

This is the same as that of Theorem 1.2, except that we use Lemmas 3.1 and 3.2.□

4. The Proof of Cor. 1.4 and 1.5

We begin with

The Proof of Corollary 1.4

By our hypothesis (1.21) (recall that $|t - t_j| \leq \delta_j$ for $t \in [t_{j-L}, t_{j+L}] \subset [t_{j-K}, t_{j+K+1}]$)

$$\int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p \geq \left(\frac{1}{|\pi_n'(t_j)|\, \delta_j} \right)^p \int_{t_{j-L}}^{t_{j+L}} |\pi_n \nu|^p \geq \frac{\rho}{(|\pi_n'(t_j)|\, \delta_j)^p} \int_{t_j - \varepsilon\delta_j}^{t_j + \varepsilon\delta_j} \omega^p$$

so the result follows from (1.18).□

We turn to

The Proof of Corollary 1.5

We assume (I), the proof under (II) is similar. Let $M[g](x)$ denote the usual maximal function of a function $g \in L_1(\mathbb{R})$, so that

$$M[g](x) = \sup_{\delta > 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |g(y)|\, dy.$$

We see that if χ_j^* denotes the characteristic function of (t_{j-K}, t_{j+K+1}), then for t in this interval,

$$\left| \frac{(\pi_n \nu)(t)}{t - t_j} \right| = \left| \frac{1}{t - t_j} \int_{t_j}^{t} (\pi_n \nu)'(s) ds \right| \leq M[(\pi_n \nu)' \chi_j^*](t)$$

and then boundedness of the maximal function operator from L_p to L_p gives

$$\int_{t_{j-K}}^{t_{j+K+1}} |\ell_j \nu|^p \leq \frac{1}{|\pi_n'(t_j)|^p} \int_{t_{j-K}}^{t_{j+K+1}} \left| M[(\pi_n \nu)' \chi_j^*] \right|^p$$

$$\leq \frac{C^p}{|\pi_n'(t_j)|^p} \int_{t_{j-K}}^{t_{j+K+1}} |(\pi_n \nu)'|^p \leq \frac{C^p \rho}{|\pi_n'(t_j)\delta_j|^p} \int_{t_j - \varepsilon\delta_j}^{t_j + \varepsilon\delta_j} \omega^p$$

by (1.23). The constant C is of course just the L_p norm of the maximal function operator.$\square$

5. The Proof of Theorem 1.7

We begin with a technical lemma:

Lemma 5.1
(a) For $t \in I_j$,

$$C_1 \leq \omega(t) / \omega(t_j) \leq C_2, \tag{5.1}$$

where C_1, C_2 depend only on τ, L.
(b) If j, k are such that $|j-k| \geq L$ and

$$|t_j - t_k| \leq 2\left(\left|1 - \frac{|t_j|}{b}\right| + c\right), \tag{5.2}$$

then

$$\delta_k \leq C\left(\left|1 - \frac{|t_j|}{b}\right| + c\right), \tag{5.3}$$

where $C = C(\alpha, L)$ only.
(c) ω is an A_p weight, and the constant γ in (1.14) may be taken to be independent of b, c (but depending on σ, p).
Proof
(a) Now for $k = j \pm 1$,

$$\left| \frac{\left|1 - \frac{|t_k|}{b}\right| + c}{\left|1 - \frac{|t_j|}{b}\right| + c} - 1 \right| \leq \frac{\frac{1}{b}|t_k - t_j|}{\left|1 - \frac{|t_j|}{b}\right| + c} \leq \frac{\frac{1}{b}\delta_j}{\left|1 - \frac{|t_j|}{b}\right| + c} \leq \frac{1}{b\tau} \leq \frac{2}{\tau}.$$

Here we have used (1.31). It follows that

$$\left|1 - \frac{|t_k|}{b}\right| + c \sim \left|1 - \frac{|t_{k+1}|}{b}\right| + c$$

uniformly in k, with constants in $\sim$ depending only on τ. Then also uniformly in k,

$$\omega(t_k) \sim \omega(t_{k+1}), \tag{5.4}$$

with constants in $\sim$ depending only on τ, σ. Since (2.3) shows that

$$t_{j-K} \leq t_j - \delta_j \leq t_j + \delta_j \leq t_{j+K},$$

we obtain from the local monotonicity of ω that (5.1) holds.
(b) We use (1.7):

$$\begin{aligned} \delta_k &\leq |t_j - t_k| \alpha^{-1} |j-k|^{-1/3} [1 + \log|j-k|]^{-2/3} \\ &\leq |t_j - t_k| \alpha^{-1} L^{-1/3} [1 + \log L]^{-2/3}. \end{aligned}$$

Now use our hypothesis on $|t_j - t_k|$.
(c) This essentially goes back to Muckenhoupt [15]. A detailed proof of the fact that γ in (1.14) may be taken independent of n can be found for example in [7,Lemma

3.1]. $\square$

We turn to

The Proof of Theorem 1.7
We assume that K is as in (2.2). Moreover, we write, as at (2.5),

$$P(t) = L_n[P](t) =: J_1(t) + J_2(t).$$

The conclusions of Lemmas 2.1 to 2.3 remain valid, since they do not use (1.17). We merely have to replace Lemma 2.4 by a suitable analogue. We define the matrix B as in the proof of Lemma 2.4. As at (2.17-2.18), there is the estimate

$$\sum_{j=1}^{n}\left(\int_{I_j}\omega^p\right)\left(\sum_{k:|k-j|\geq K}\frac{|y_k|\,\delta_k^2}{|t_j-t_k|^3}\right)^p \leq \| B \|^p_{\ell_p^n\to\ell_p^n}\sum_{k=1}^{n}\left(\frac{|y_k|}{|I_k|}\right)^p\int_{I_k}\omega^p.$$

Now

$$\int_{I_k}\omega^p \sim \delta_k\omega(t_k)^p, \tag{5.5}$$

by (5.1), with constants in $\sim$ depending only on τ and L. Then if we can show that

$$\| B \|_{\ell_p^n\to\ell_p^n} \leq C, \tag{5.6}$$

where C depends only on σ, τ, p, L, the desired result follows from Lemmas 2.1 and 2.3.

To prove (5.6), we use Proposition 2.5, but with a different choice of r_{jk} from that used for Theorem 1.2. We set

$$\rho := \begin{cases} -q, & \sigma \geq 0, \\ p, & \sigma < 0 \end{cases}$$

and

$$r_{jk} := \left(\frac{\delta_k}{\delta_j}\right)^{\frac{1}{pq}}\left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\frac{\rho}{pq}}. \tag{5.7}$$

Let us set

$$\beta_1 := -\left(1-\frac{\rho}{p}\right); \beta_2 := 1+\frac{\rho}{q}.$$

Then (2.19) becomes (instead of (2.21)), and because of (5.5),

$$\sup_j \sum_{k:|k-j|\geq K}\frac{\delta_k^3}{|t_j-t_k|^3}\left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\beta_1} \leq N; \tag{5.8}$$

After swopping the roles of j,k in (2.20), it becomes (instead of (2.22))

$$\sup_j \sum_{k:|k-j|\geq K}\frac{\delta_k\delta_j^2}{|t_j-t_k|^3}\left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\beta_2} \leq N. \tag{5.9}$$

Here we want N to depend only on $\sigma, p, \tau, \alpha, K$. Note that

$$\beta_1\sigma = \begin{cases} -q\sigma, & \sigma \geq 0, \\ 0, & \sigma < 0 \end{cases}; \beta_2\sigma = \begin{cases} 0, & \sigma \geq 0, \\ p\sigma, & \sigma < 0 \end{cases}$$

and then from our restriction (1.29) on σ, we deduce that

$$-1 < \beta_\ell \sigma \leq 0, \ell = 1, 2. \tag{5.10}$$

Let

$$\mathcal{S}(j) := \left\{ j : |k - j| \geq K \text{ and } \left|1 - \frac{|t_k|}{b}\right| + c \geq \frac{1}{100}\left(\left|1 - \frac{|t_j|}{b}\right| + c\right)\right\}$$

and let

$$\mathcal{T}(j) := \left\{ j : |k - j| \geq K \text{ and } \left|1 - \frac{|t_k|}{b}\right| + c < \frac{1}{100}\left(\left|1 - \frac{|t_j|}{b}\right| + c\right)\right\}$$

denote the complementary range. For $\ell = 1, 2$,

$$k \in \mathcal{S}(j) \Rightarrow \left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\beta_\ell} \leq 100^{\beta_\ell \sigma},$$

as $\beta_\ell \sigma \leq 0$. It follows that

$$\sum_{k \in \mathcal{S}(j)} \frac{\delta_k^3}{|t_j - t_k|^3} \left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\beta_1}$$

$$\leq 100^{\beta_1 \sigma} \sum_{k:|k-j| \geq K} \frac{\delta_k^3}{|t_j - t_k|^3} \leq C, \tag{5.11}$$

with C depending only on $L, \sigma, \alpha, p, \tau$. We have used our spacing condition (1.7) here. Similarly,

$$\sum_{k \in \mathcal{S}(j)} \frac{\delta_j^2 \delta_k}{|t_j - t_k|^3} \left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\beta_2}$$

$$\leq 100^{\beta_2 \sigma} \delta_j^2 \sum_{k \in \mathcal{S}(j)} \frac{\delta_k}{|t_j - t_k|^3}$$

$$\leq C \delta_j^2 \int_{\{t:|t-t_j| \geq \delta_j\}} \frac{dt}{|t - t_j|^3} \leq C, \tag{5.12}$$

where C depends only on $L, \sigma, \alpha, p, \tau$. We have used (2.3) to estimate the sum by an integral. Next, let

$$\lambda := \left|1 - \frac{|t_j|}{b}\right| + c.$$

For $k \in \mathcal{T}(j)$,

$$\begin{aligned} |t_j - t_k| &\geq b\left[\left(\left|1 - \frac{|t_j|}{b}\right| + c\right) - \left(\left|1 - \frac{|t_k|}{b}\right| + c\right)\right] \\ &\geq \frac{99b}{100}\left(\left|1 - \frac{|t_j|}{b}\right| + c\right) = \frac{99b}{100}\lambda \geq \frac{99}{200}\lambda. \end{aligned}$$

Moreover,

$$|t_j - t_k| \geq C\delta_k,$$

with $C \neq C(j, k)$. Then

$$\sum_{k \in \mathcal{T}(j)} \frac{\delta_k^3}{|t_j - t_k|^3} \left(\frac{\omega(t_k)}{\omega(t_j)}\right)^{\beta_1}$$

$$\leq C\lambda^{-1}\omega\left(t_j\right)^{-\beta_1}\sum_{k\in\mathcal{T}(j)}\delta_k\omega\left(t_k\right)^{\beta_1}$$

$$\leq C\lambda^{-1}\omega\left(t_j\right)^{-\beta_1}\sum_{k\in\mathcal{T}(j)}\int_{I_k}\omega\left(t\right)^{\beta_1}dt. \tag{5.13}$$

Now $k\in\mathcal{T}(j)$ and $t\in I_k$ implies that

$$\left|1-\frac{|t|}{b}\right|+c\leq\left|1-\frac{|t_k|}{b}\right|+c+\frac{\delta_k}{b}$$

$$\leq\left(\left|1-\frac{|t_k|}{b}\right|+c\right)\left(1+\frac{2}{\tau}\right)\leq\frac{\lambda}{100}\left(1+\frac{2}{\tau}\right)\leq\frac{\lambda}{25\tau},$$

by (1.31), and assuming, as we may, that $\tau<1$. Thus we may continue (5.13) as

$$\leq C\lambda^{-1}\omega\left(t_j\right)^{-\beta_1}\int_{\left\{t:\left|1-\frac{|t|}{b}\right|+c<\frac{\lambda}{25\tau}\right\}}\omega\left(t\right)^{\beta_1}dt$$

$$\leq 2C\lambda^{-1-\beta_1\sigma}\int_{\left\{t:\left|1-\frac{t}{b}\right|+c<\frac{\lambda}{25\tau}\right\}}\left(\left|1-\frac{t}{b}\right|+c\right)^{\beta_1\sigma}dt.$$

We now make the substitution $1-\frac{t}{b}=\lambda s$ and then continue this as

$$\leq 2C\int_{\left\{s:|s|+\frac{c}{\lambda}<\frac{1}{25\tau}\right\}}\left(|s|+\frac{c}{\lambda}\right)^{\beta_1\sigma}ds\leq C_1, \tag{5.14}$$

with C_1 depending on L,σ,p,α,τ: note that $c/\lambda\in[0,1]$ and $\beta_1\sigma>-1$. Similarly,

$$\sum_{k\in\mathcal{T}(j)}\frac{\delta_j^2\delta_k}{\left|t_j-t_k\right|^3}\left(\frac{\omega\left(t_k\right)}{\omega\left(t_j\right)}\right)^{\beta_2}$$

$$\leq C\lambda^{-1}\omega\left(t_j\right)^{-\beta_2}\sum_{k\in\mathcal{T}(j)}\delta_k\omega\left(t_k\right)^{\beta_2}\leq C_1, \tag{5.15}$$

as above. Combining (5.11), (5.12), (5.14) and (5.15) gives the required bounds (5.8) and (5.9), with N depending only on L,σ,p,α,τ.$\square$

6. The Proof of Theorem 1.9

In the analysis of exponential weights $W=e^{-Q}$, the Mhaskar-Rahmanov-Saff number a_n plays an important role: it is the positive root of the equation

$$n=\frac{2}{\pi}\int_0^1 a_n t Q'\left(a_n t\right)\frac{dt}{\sqrt{1-t^2}}. \tag{6.1}$$

One of its features is the Mhaskar-Saff identity [14], [19]

$$\| PW \|_{L_\infty[-1,1]}=\| PW \|_{L_\infty[-a_n,a_n]}, P\in\mathcal{P}_n.$$

We also set

$$\eta_n:=\left(nT\left(a_n\right)\right)^{-2/3}, n\geq 1; \tag{6.2}$$

$$\psi_n\left(t\right):=\left|1-\frac{|t|}{a_n}\right|+\eta_n; \tag{6.3}$$

and

$$\phi_n(t) := \max\left\{\sqrt{\psi_n(t)}, \frac{1}{T(a_n)\sqrt{\psi_n(t)}}\right\}. \tag{6.4}$$

(In [5], [7], η_n was denoted by δ_n, but this would conflict with our use of δ_j in this paper). We let

$$d \in \left(-\frac{1}{p}, \frac{1}{4} - \frac{1}{p}\right) \text{ and } d \leq 0 \tag{6.5}$$

and

$$\sigma := \frac{3}{4} + d. \tag{6.6}$$

Note that then σ satisfies (1.29).

For a given n, we set $L = 1$ and

$$\delta_j := t_{j+1,n} - t_{j-1,n};$$

$$\omega(t) := \psi_n(t)^{\sigma} = \psi_n(t)^{\frac{3}{4}+d}; \tag{6.7}$$

$$\pi_n(t) := p_{n-2}(t)\left(1 - \left(\frac{t}{\xi_{n-2}}\right)^2\right); \tag{6.8}$$

$$\nu(t) := C_0 W(t)\, \psi_n(t)^d. \tag{6.9}$$

Here $C_0 \neq C_0(n,t)$ will be chosen small enough later. We begin by summarizing what we need to apply Theorem 1.7 and to prove Theorem 1.9: we assume throughout the hypotheses of the latter.

Lemma 6.1
(a) Uniformly in n and j,

$$t_{j+1,n} - t_{j-1,n} \sim t_{jn} - t_{j-1,n} \sim \frac{1}{n}\phi_n(t_{jn}). \tag{6.10}$$

Moreover, for some $C \neq C(j,n)$,

$$t_{jn} \leq a_n(1 + C\eta_n). \tag{6.11}$$

(b) For some $C \neq C(j,k,n)$,

$$\frac{|t_{jn} - t_{kn}|}{t_{j+1,n} - t_{j-1,n}} \geq C\,|j-k|^{2/3}. \tag{6.12}$$

(c) For some $\tau > 0$, with $\tau \neq \tau(j,k,n)$,

$$\psi_n(t_{jn}) \geq \tau\delta_j. \tag{6.13}$$

(d) Uniformly in j, n, t,

$$\psi_n(t_{jn}) \sim \psi_n(t) \sim \psi_n(t_{j+1,n}),\, t \in [t_{jn}, t_{j+1,n}]. \tag{6.14}$$

(e) If C_0 is appropriately chosen, then in $[-1,1]$,

$$|\pi_n \nu| \leq \omega.$$

(f) Let K be a fixed positive integer. There exists $C \neq C(j,n)$ such that

$$\int_{t_{j-K,n}}^{t_{j+K+1,n}} |\ell_j \nu|^p + \frac{\delta_j \omega\left(t_{jn}\right)^p}{\left[\delta_j \left|\pi_n'(t_{jn})\right|\right]^p} \leq C\delta_j W\left(t_{jn}\right)^p \psi_n\left(t_{jn}\right)^{dp}. \tag{6.15}$$

(g) There exists $C \neq C(n,P)$ such that for $P \in \mathcal{P}_n$

$$\| \left(PW\right)(t)\left(1-t^2\right)^d \|_{L_p[-1,1]} \leq C \| \left(PW\right)(t)\left(1-t^2\right)^d \|_{L_p[-a_{2n},a_{2n}]}. \tag{6.16}$$

We delay the proof of the lemma until after

The Proof of Theorem 1.9

We note first that we have verified all the hypotheses of Theorem 1.7 with the above choices of π_n, ν, ω. Indeed, (6.12) is a stronger form of the spacing condition (1.7), while (6.13) is a restatement of (1.31), and we have (1.16) from Lemma 6.1(e). Then setting $P := L_n[f]$ in (1.32), and taking account of (6.15) gives

$$\int_{-1}^{1} \left|L_n[f] W\psi_n^d\right|^p \leq C\sum_{j=1}^{n} \delta_j \left|fW\psi_n^d\right|^p (t_{jn}), \tag{6.17}$$

with $C \neq C(n,f)$. Let $\varepsilon > 0$ and $0 < \rho < \rho' < 1$ be chosen so that

$$\int_{\rho}^{1} (1-t)^{dp}\, dt < \varepsilon. \tag{6.18}$$

Then

$$\begin{aligned} &\sum_{|t_j|\geq\rho'} \delta_j \left|fW\psi_n^d\right|^p (t_{jn}) \\ &\leq C \| fW \|_{L_\infty[-1,1]}^p \sum_{|t_j|\geq\rho'} \left(t_{j+1,n}-t_{jn}\right)\psi_n^{dp}(t_{jn}) \\ &\leq C \| fW \|_{L_\infty[-1,1]}^p \int_{\rho}^{1} \psi_n^{dp}(t)\, dt. \end{aligned}$$

Here $C \neq C(f,n,\rho,\rho',\varepsilon)$ and we have used (6.10) and (6.14). Since as $n \to \infty$, $a_n \to 1-$ and $\eta_n \to 0+$, we may continue this for large n as

$$\leq 2C \| fW \|_{L_\infty[-1,1]}^p \int_{\rho}^{1} (1-t)^{dp}\, dt < 2C \| fW \|_{L_\infty[-1,1]}^p \varepsilon. \tag{6.19}$$

Again we emphasise that $C \neq C(\varepsilon)$. Next, for $n \geq n_0(\rho')$ and $t \in [-\rho', \rho']$, we have

$$\psi_n^{dp}(t) \leq 2\left|1-|t|\right|^{dp}$$

so (6.10) gives for such n,

$$\begin{aligned} &\sum_{t_j\in(-\rho',\rho')} \delta_j \left|fW\psi_n^d\right|^p (t_{jn}) \\ &\leq C \sum_{t_j\in(-\rho',\rho')} \left(t_{j+1,n}-t_{jn}\right)|fW|^p(t_{jn})\left|1-|t_{jn}|\right|^{dp} \\ &\to C\int_{-\rho'}^{\rho'} |fW|^p(t)\left|1-|t|\right|^{dp} dt, n \to \infty, \end{aligned}$$

recall that fW is bounded and Riemann integrable. Here $C \neq C(n, f, \rho, \rho', \varepsilon)$. Combined with (6.17) and (6.19) and letting $\varepsilon \to 0+$, gives

$$\limsup_{n\to\infty} \| L_n[f] W \psi_n^d \|_{L_p[-1,1]} \leq C \| (fW)(t) \left(1 - t^2\right)^d \|_{L_p[-1,1]},$$

for some $C \neq C(n, f)$. Next for $t \in [-a_n, a_n]$,

$$\psi_n(t) = 1 - \frac{|t|}{a_n} + \eta_n \leq 1 - |t| + \eta_n \leq C\left(1 - t^2\right),$$

as

$$1 - t^2 \geq a_{2n}^2 - a_n^2 \sim \frac{1}{T(a_n)} >> \eta_n,$$

see [5,pp.24-25]. Moreover, for $t \in [-a_{2n}, a_{2n}] \setminus [-a_n, a_n]$,

$$\begin{aligned}\psi_n(t) &\leq \left|1 - \frac{a_{2n}}{a_n}\right| + \eta_n \leq \frac{C}{T(a_n)} \\ &\leq C\left(a_{3n}^2 - a_n^2\right) \leq C\left(1 - t^2\right),\end{aligned}$$

again see [5,pp.24-25]. Thus,

$$\begin{aligned}&\psi_n(t) \leq C\left(1 - t^2\right), t \in [-a_{2n}, a_{2n}] \\ \Rightarrow\ &\psi_n(t)^d \geq C\left(1 - t^2\right)^d, t \in [-a_{2n}, a_{2n}].\end{aligned}$$

Then the restricted range inequality (6.16) gives

$$\begin{aligned}&\limsup_{n\to\infty} \| (L_n[f] W)(t) \left(1 - t^2\right)^d \|_{L_p[-1,1]} \\ &\leq C \limsup_{n\to\infty} \| (L_n[f] W)(t) \left(1 - t^2\right)^d \|_{L_p[-a_{2n}, a_{2n}]} \\ &\leq C \limsup_{n\to\infty} \| (L_n[f] W)(t) \psi_n(t)^d \|_{L_p[-1,1]} \\ &\leq C \| (fW)(t) \left(1 - t^2\right)^d \|_{L_p[-1,1]},\end{aligned}$$

where $C \neq C(n, f)$. For any fixed polynomial P, we then deduce that

$$\limsup_{n\to\infty} \| (f - L_n[f] W)(t) \left(1 - t^2\right)^d \|_{L_p[-1,1]} \leq C \| ((f - P) W)(t) \left(1 - t^2\right)^d \|_{L_p[-1,1]}.$$

Here as fW is bounded while $dp > -1$, we may choose P to make the last right-hand side arbitrarily small. Then (1.38) follows. Finally, if (1.38) holds for a given d, it holds for any larger d, so our assumption (6.5) is no real restriction.□

We turn to the proof of Lemma 6.1, but precede it with:

Lemma 6.2
(a)

(6.20) $$\left|1 - \frac{\xi_n}{a_n}\right| \leq C\eta_n, n \geq 1.$$

(b) For $1 \leq j \leq n$,

(6.21) $$|x_{jn} - \xi_n| \geq C\eta_n.$$

Proof

(a) It is known [5, Cor. 1.5, p.10] that

$$|p_nW|(x) \le C\left|1-\frac{|x|}{a_n}\right|^{-1/4}, x\in(-1,1) \tag{6.22}$$

and

$$|p_nW|(\xi_n) = \|p_nW\|_{L_\infty[-1,1]} \sim (nT(a_n))^{1/6} = \eta_n^{-1/4}.$$

Then (6.20) follows.

(b) It follows from (12.4) and (12.5) in [5,p.134] that for $1 \le j \le n$ and $x \in (-1,1)$,

$$|p_nW|(x) \le C|x-x_{jn}|\, n\phi_n^{-1}(x_{jn})\,\psi_n(x_{jn})^{-1/4}.$$

In particular for $x = \xi_n$, we obtain

$$\eta_n^{-1/4} \le C|\xi_n - x_{jn}|\, n\phi_n^{-1}(x_{jn})\,\psi_n(x_{jn})^{-1/4}. \tag{6.23}$$

Now if with C as in (a),

$$|1-x_{jn}/a_n| > 2C\eta_n,$$

then (a) immediately gives (b). In the contrary case $|1-x_{jn}/a_n| \le 2C\eta_n$, we see from the definitions (6.3) and (6.4) of ϕ_n, ψ_n that

$$\psi_n(x_{jn}) \sim \eta_n; \phi_n(x_{jn}) \sim \frac{1}{T(a_n)\sqrt{\eta_n}}$$

so (6.23) becomes

$$\eta_n^{-1/4} \le C|\xi_n - x_{jn}|\, nT(a_n)\sqrt{\eta_n}\eta_n^{-1/4}$$

and then again (b) follows.□

The Proof of Lemma 6.1(a), (d)

It is known that uniformly in j, n [5,p.9]

$$x_{jn} - x_{j-1,n} \sim \frac{1}{n}\phi_n(x_{jn}); \left|1-\frac{x_{1n}}{a_n}\right| \le C\eta_n$$

and uniformly in j, n [5,p.111,eqn. (10.12)]

$$\psi_n(x_{jn}) \sim \psi_n(x_{j-1,n}) \tag{6.24}$$

and hence also

$$\phi_n(x_{jn}) \sim \phi_n(x_{j-1,n}). \tag{6.25}$$

It is also follows easily from the fact that [5,p.24],

$$1-\frac{a_{n-1}}{a_n} \sim \frac{1}{nT(a_n)} = o(\eta_n), \tag{6.26}$$

that uniformly in n and $x \in (-1,1)$,

$$\psi_n(x) \sim \psi_{n-1}(x); \phi_n(x) \sim \phi_{n-1}(x). \tag{6.27}$$

Finally,

$$\psi_n(\xi_n) \sim \eta_n; \phi_n(\xi_n) \sim \frac{1}{T(a_n)\sqrt{\eta_n}}.$$

The above relations and Lemma 6.2 (a), (b), directly imply (6.10) and (6.14). Also (6.11) follows from Lemma 6.2 (a), and from the bound above for x_{1n}. □

The Proof of Lemma 6.1(c)
In view of (6.10), it suffices to show that for some $C \neq C(j,n)$,

$$\psi_n(t_{jn}) \geq \frac{C}{n} \max\left\{\sqrt{\psi_n(t_{jn})}, \frac{1}{T(a_n)\sqrt{\psi_n(t_{jn})}}\right\}.$$

This, in turn, is equivalent to both

$$\sqrt{\psi_n(t_{jn})} \geq \frac{C}{n};$$

and

$$\psi_n(t_{jn})^{3/2} \geq \frac{C}{nT(a_n)} = C\eta_n^{3/2}.$$

The last inequality immediately follows from the definition of ψ_n. Since [5,p.24], $T(a_n) = o\left(n^2\right)$,

$$\eta_n = (nT(a_n))^{-2/3} >> n^{-2},$$

and then the former inequality follows as

$$\sqrt{\psi_n(t_{jn})} \geq \sqrt{\eta_n} >> \frac{1}{n}.$$

□

Proof of Lemma 6.1(b)
From (6.10) and (6.14),

$$\int_{t_{jn}}^{t_{j+1,n}} \phi_n^{-1} \sim \frac{1}{n}$$

whence if $k \neq j$,

$$\left|\int_{t_{jn}}^{t_{kn}} \phi_n^{-1}\right| \sim \frac{|k-j|}{n}. \tag{6.28}$$

We consider 4 subcases, thereby treating all possibilities with $t_{jn}, t_{kn} \geq 0$; the remaining cases are similar:
(I) $0 \leq t_{jn}, t_{kn} \leq a_{3n/4}$.
Then for t between t_{jn} and t_{kn},

$$\phi_n(t) \sim \sqrt{\psi_n(t)} = \sqrt{1 - \frac{t}{a_n} + \eta_n}$$

whence (6.28) gives

$$\frac{|k-j|}{n} \sim \left|\sqrt{\psi_n(t_{jn})} - \sqrt{\psi_n(t_{kn})}\right| = \frac{a_n^{-1}|t_{jn} - t_{kn}|}{\sqrt{\psi_n(t_{jn})} + \sqrt{\psi_n(t_{kn})}}.$$

Then our spacing (6.10) gives

$$\frac{|t_{kn} - t_{jn}|}{t_{j+1,n} - t_{j-1,n}} \sim |k-j| \frac{\sqrt{\psi_n(t_{jn})} + \sqrt{\psi_n(t_{kn})}}{\sqrt{\psi_n(t_{jn})}} \geq |k-j|.$$

Thus we have (6.12) in a stronger form.
(II) $a_{n/2} \leq t_{jn}, t_{kn} < 1$.

Then for t between t_{jn} and t_{kn},

$$\phi_n(t) \sim \frac{1}{T(a_n)\sqrt{\psi_n(t)}} \sim \frac{1}{T(a_n)\sqrt{1-\frac{t}{a_n}+M\eta_n}},$$

where the large enough $M>0$ may be chosen independent of n,t,j,k and we are using the fact that all $t_{jn} \le a_n(1+C\eta_n)$. Then (6.28) gives

$$T(a_n)\left|\left(1-\frac{t_{jn}}{a_n}+M\eta_n\right)^{3/2}-\left(1-\frac{t_{kn}}{a_n}+M\eta_n\right)^{3/2}\right| \sim \frac{|k-j|}{n}. \tag{6.29}$$

Using

$$\left|v^3-u^3\right| \sim \left|v^2-u^2\right|(v+u)\,, u,v>0$$

gives

$$T(a_n)\left|t_{jn}-t_{kn}\right|\left|\left(1-\frac{t_{jn}}{a_n}+M\eta_n\right)^{1/2}+\left(1-\frac{t_{kn}}{a_n}+M\eta_n\right)^{1/2}\right| \sim \frac{|k-j|}{n}$$

and hence

$$\frac{|t_{kn}-t_{jn}|}{t_{j+1,n}-t_{j-1,n}} \sim |k-j|\,\frac{\sqrt{1-\frac{t_{jn}}{a_n}+M\eta_n}}{\sqrt{1-\frac{t_{jn}}{a_n}+M\eta_n}+\sqrt{1-\frac{t_{kn}}{a_n}+M\eta_n}}. \tag{6.30}$$

If

$$1-\frac{t_{jn}}{a_n}+M\eta_n \ge \frac{1}{4}\left(1-\frac{t_{kn}}{a_n}+M\eta_n\right),$$

then we obtain a stronger form of (6.12). Otherwise, (6.29) gives

$$|k-j| \sim nT(a_n)\left(1-\frac{t_{kn}}{a_n}+M\eta_n\right)^{3/2}$$

$$\Rightarrow |k-j|^{1/3} \sim (nT(a_n))^{1/3}\left(1-\frac{t_{kn}}{a_n}+M\eta_n\right)^{1/2} = \sqrt{\frac{1-\frac{t_{kn}}{a_n}+M\eta_n}{\eta_n}}$$

so (6.30) gives

$$\frac{|t_{kn}-t_{jn}|}{t_{j+1,n}-t_{j-1,n}} \sim |k-j|^{2/3}\sqrt{\frac{1-\frac{t_{kn}}{a_n}+M\eta_n}{\eta_n}}\,\frac{\sqrt{1-\frac{t_{jn}}{a_n}+M\eta_n}}{\sqrt{1-\frac{t_{kn}}{a_n}+M\eta_n}}$$

$$= |k-j|^{2/3}\sqrt{\frac{1-\frac{t_{jn}}{a_n}+M\eta_n}{\eta_n}} \ge \sqrt{M}\,|k-j|^{2/3}.$$

(III) $t_{kn} \ge a_{3n/4}; 0 \le t_{jn} \le a_{n/2}$
Here (6.28) becomes

$$\int_{t_{jn}}^{a_{2n/3}} \frac{dt}{\sqrt{1-\frac{t}{a_n}+M\eta_n}} + \int_{a_{2n/3}}^{x_{kn}} T(a_n)\sqrt{1-\frac{t}{a_n}+M\eta_n}\,dt \sim \frac{k-j}{n},$$

Via some straightforward estimations, that are similar to those in (I), (II), and by using

$$\eta_n = o\left(\frac{1}{T(a_n)}\right) \text{ and } 1-\frac{a_{2n/3}}{a_n} \sim \frac{1}{T(a_n)},$$

we deduce that the first integral gives the dominant term:

$$\sqrt{1-\frac{t_{jn}}{a_n}} \sim \frac{k-j}{n}.$$

Next,

$$1-\frac{t_{kn}}{a_n} \leq 1-\frac{a_{3n/4}}{a_n} = 1-\frac{a_{n/2}}{a_n} - \left(\frac{a_{3n/4}-a_{n/2}}{a_n}\right)$$

and [5,p.25],

$$1-\frac{a_{n/2}}{a_n} \sim \frac{1}{T(a_n)} \sim \frac{a_{3n/4}-a_{n/2}}{a_n}$$

so for some $0 < C < 1$ with $C \neq C(j,k,n)$,

$$1-\frac{t_{kn}}{a_n} \leq \left(1-\frac{a_{n/2}}{a_n}\right)(1-C) \leq \left(1-\frac{t_{jn}}{a_n}\right)(1-C).$$

Then

$$t_{kn}-t_{jn} = a_n\left(\left(1-\frac{t_{jn}}{a_n}\right)-\left(1-\frac{t_{kn}}{a_n}\right)\right) \sim 1-\frac{t_{jn}}{a_n},$$

so

$$\frac{|t_{kn}-t_{jn}|}{t_{j+1,n}-t_{j-1,n}} \sim \frac{1-\frac{t_{jn}}{a_n}}{\frac{1}{n}\sqrt{1-\frac{t_{jn}}{a_n}}} \sim k-j.$$

Again, we have (6.12) in a stronger form.
(IV) $t_{jn} \geq a_{3n/4}; 0 \leq t_{kn} \leq a_{n/2}$
As in (III), we obtain

$$\sqrt{1-\frac{t_{kn}}{a_n}} \sim \frac{|k-j|}{n} \tag{6.31}$$

and also

$$t_{jn}-t_{kn} \sim 1-\frac{t_{kn}}{a_n}.$$

Then

$$\begin{aligned}\frac{|t_{kn}-t_{jn}|}{t_{j+1,n}-t_{j-1,n}} &\sim \frac{1-\frac{t_{kn}}{a_n}}{\frac{1}{nT(a_n)\sqrt{1-\frac{t_{jn}}{a_n}+M\eta_n}}} \\ &\geq CnT(a_n)\sqrt{\eta_n}\left(1-\frac{t_{kn}}{a_n}\right) \\ &= C\left[T(a_n)\left(1-\frac{t_{kn}}{a_n}\right)\right]^{2/3}\left[n^2\left(1-\frac{t_{kn}}{a_n}\right)\right]^{1/3} \\ &\geq C\left[T(a_n)\left(1-\frac{a_{n/2}}{a_n}\right)\right]^{2/3}\left[|j-k|^2\right]^{1/3} \geq C|j-k|^{2/3}.\end{aligned}$$

Here we have used (6.31) and [5,p.25],

$$1-\frac{a_{n/2}}{a_n} \sim \frac{1}{T(a_n)}.$$

□

Proof of Lemma 6.1(e)
Now from (6.8) and (6.9),

$$\left|\pi_n\nu\right|(t)=C_0\left|p_{n-2}W\right|(t)\left|1-\left(\frac{t}{\xi_{n-2}}\right)^2\right|\psi_n(t)^d.$$

Here from Lemma 6.2 (a) and (6.26),

$$\left|1-\left(\frac{t}{\xi_{n-2}}\right)^2\right|\leq C\left(\left|1-\frac{|t|}{a_n}\right|+\eta_n\right)=C\psi_n(t)$$

while from (6.22) and (6.27),

$$\left|p_{n-2}W\right|(t)\leq C\psi_n(t)^{-1/4}.$$

Thus

$$\left|\pi_n\nu\right|(t)\leq C_0C\psi_n(t)^{d+3/4}=C_0C\omega(t).$$

So, with $C_0:=1/C$, we obtain (1.16). $\square$

Proof of Lemma 6.1(f)
We have to show that for some $C\neq C(n,j)$,

$$\int_{t_{j-K,n}}^{t_{j+K+1,n}}\left|\ell_j\nu\right|^p\leq C\delta_jW(t_{jn})^p\,\psi_n(t_{jn})^{dp}\tag{6.32}$$

and

$$\frac{\delta_j\omega(t_{jn})^p}{\left[\delta_j\left|\pi_n'(t_{jn})\right|\right]^p}\leq C\delta_jW(t_{jn})^p\,\psi_n(t_{jn})^{dp}.\tag{6.33}$$

Let us set $m:=n-2$. Suppose first that $t_{jn}=x_{km}$, some $1\leq k\leq m$. Then

$$\ell_j(x)=\frac{p_m(x)}{p_m'(x_{km})(x-x_{km})}\frac{1-\left(\frac{x}{\xi_m}\right)^2}{1-\left(\frac{x_{km}}{\xi_m}\right)^2}.$$

Here by (12.5) in [5,p.134],

$$\left|\frac{p_m(x)W(x)W^{-1}(x_{km})}{p_m'(x_{km})(x-x_{km})}\right|\leq C_1,$$

where C_1 is independent of m,k,x. Also for $x\in[t_{j-k,n},t_{j+K+1,n}]$, Lemma 6.2(a), (b) and Lemma 6.1(a) give

$$\left|\frac{1-\left(\frac{x}{\xi_m}\right)^2}{1-\left(\frac{x_{km}}{\xi_m}\right)^2}\right|\leq C_2$$

whence

$$\left|\ell_j(x)\nu(x)\right|\leq C_1C_2W(x_{km})\,\psi_n(x)^d\sim W(t_{jn})\,\psi_n(t_{jn})^d,$$

recall (6.14). This and Lemma 6.1(a) gives (6.32). Next, suppose that $t_{jn}=\xi_m$. Then,

$$\ell_j(x)=\frac{p_m(x)}{p_m(\xi_m)}\frac{1+\frac{x}{\xi_m}}{2}$$

so

$$\left|\ell_j(x)W(x)\right| \leq C\left|\frac{(p_mW)(x)}{(p_mW)(\xi_m)}\right| W(\xi_m) \leq CW(\xi_m).$$

Then it is easily seen that (6.32) persists for such j.

We turn to the proof of (6.33). Now if $t_{jn} = x_{km}$, then

$$\pi_n'(t_{jn}) = p_m'(x_{km})\left(1 - \left(\frac{x_{km}}{\xi_m}\right)^2\right)$$

and by [5,p.11] and [5,p.9]

$$\left|(p_m'W)(x_{km})\right| \sim m\min\left\{\frac{1}{\psi_m(x_{km})}, T(a_m)\right\}\psi_m(x_{km})^{1/4}$$

$$\sim m\psi_m(x_{km})^{-1/4}\phi_m(x_{km})^{-1} \sim \psi_m(x_{km})^{-1/4}(x_{km} - x_{k+1,m}) \sim \psi_n(t_{jn})^{-1/4}\delta_j^{-1},$$

by Lemma 6.1(a) and (6.27). Since also by Lemma 6.1(a),

$$\left|1 - \left(\frac{x_{km}}{\xi_m}\right)^2\right| \sim \psi_n(x_{km}) = \psi_n(t_{jn})$$

we obtain

$$\left|\pi_n'(t_{jn})\right| \sim \psi_n(t_{jn})^{3/4}W^{-1}(t_{jn})\delta_j^{-1}.$$

Then

$$\frac{\delta_j\omega(t_{jn})^p}{[\delta_j\left|\pi_n'(t_{jn})\right|]^p} \sim \delta_jW^p(t_{jn})\left(\frac{\omega(t_{jn})}{\psi_n(t_{jn})^{3/4}}\right)^p = \delta_jW^p(t_{jn})\psi_n(t_{jn})^{dp}.$$

So we have (6.33). Finally, if $t_{jn} = \xi_m$,

$$\pi_n'(t_{jn}) = -\frac{2}{\xi_n}p_m(\xi_m),$$

then

$$\left|\pi_n'(t_{jn})\right| \sim \left|p_mW\right|(\xi_m)W^{-1}(\xi_m)$$

$$= \|p_mW\|_{L_\infty(I)}W^{-1}(\xi_m) \sim \eta_n^{-1/4}W^{-1}(t_{jn}).$$

Since also for such t_{jn}, $\delta_j \sim \eta_n \sim \psi_n(t_{jn})$, we obtain

$$\frac{\delta_j\omega(t_{jn})^p}{[\delta_j\left|\pi_n'(t_{jn})\right|]^p} \sim \delta_jW^p(t_{jn})\left(\frac{\omega(t_{jn})}{\eta_n^{3/4}}\right)^p \sim \delta_jW^p(t_{jn})\eta_n^{dp} \sim \delta_jW^p(t_{jn})\psi_n(t_{jn})^{dp}.$$

So we have (6.33) in all cases. $\square$

Finally, we give

The Proof of Lemma 6.1(g)

We pass over some of the details of this proof, as such proofs are standard. Now for $P \in \mathcal{P}_n$,

$$\|(PW)(t)\left(1-t^2\right)^d\|_{L_p(|t|\geq a_{2n})}$$

$$\leq \|PW\|_{L_\infty(|t|\geq a_{2n})}\|\left(1-t^2\right)^d\|_{L_p[-1,1]}$$

$$\leq \exp\left(-n^{C_1}\right)\|PW\|_{L_\infty[-1,1]}\|\left(1-t^2\right)^d\|_{L_p[-1,1]}.$$

Here $C_1 \neq C_1(n, P)$ −see [5,pp.61-62] for this last step. In turn, since the Christoffel functions for the weight W^2 decline no faster than a power of n [5,p.8,Cor.1.3], one may easily deduce that for some $C_2 \neq C_2(n, P)$,

$$\| PW \|_{L_\infty[-1,1]} \leq n^{C_2} \| PW \|_{L_p[-a_n,a_n]} .$$

Then we deduce that

$$\| (PW)(t)\left(1-t^2\right)^d \|_{L_p(|t|\geq a_{2n})}$$

$$\leq C \| PW \|_{L_p[-a_n,a_n]} \leq C \| (PW)(t)\left(1-t^2\right)^d \|_{L_p[-a_n,a_n]} .$$

Then (6.16) follows easily.□

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