A SURVEY OF ERDŐS-SZEKERES PRODUCTS

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Abstract. Let $\{s_j\}_{j=1}^n$ be positive integers. In 1959, Erdős and Szekeres posed a number of problems about the size of polynomials of the form

$$\prod_{j=1}^{n}\left(1-z^{s_j}\right),$$

where $\{s_j\}_{j=1}^n$ are positive integers. We survey results on these problems and closely related questions.

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1. The Original Paper

A celebrated short 1959 paper of Erdős and Szekeres [22] posed a number of problems about the growth or decay of "pure power products"

$$P_n(z)=\prod_{j=1}^{n}\left(1-z^{s_j}\right) \tag{1.1}$$

and their norms $\|P_n\|_{L_\infty(|z|=1)}$. Here $\{s_j\}_{j=1}^n$ are positive integers. This is a generating function for partitioning integers in a restricted way, and much of the motivation arises from this connection.

In this section, we shall list the problems stated in the original paper, and report on progress primarily in subsequent sections. The most well known is the following:

Problem 1
Let

$$M(s_1,s_2,...,s_n)=\left\|\prod_{j=1}^{n}\left(1-z^{s_j}\right)\right\|_{L_\infty(|z|=1)}$$

and

$$f(n):=\inf\left\{M(s_1,s_2,...,s_n):s_1,s_2,...,s_n\geq 1\right\}.$$

Determine the growth of $f(n)$ *as* $n\to\infty$.

Erdős and Szekeres proved that

$$\lim_{n\to\infty} f(n)^{1/n}=1. \tag{1.2}$$

They also claimed to prove that $f(n)\geq\sqrt{2n}$, though there are gaps in their proof, that can be repaired using other methods [13]. Erdős [21, p. 55] later conjectured that $f(n)$ should grow faster than any power of n.

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The special case where all $s_j = j$,

$$\prod_{j=1}^{n} (1 - z^j),$$

is often called a *Sudler product*, after C. Sudler's 1964 paper [42], where it was shown that

$$\lim_{n\to\infty} M\,(1, 2, ..., n)^{1/n} = 1.219... > 1.$$

This is quite a contrast to (1.2). In their paper, Erdős and Szekeres stated that "it is easy to show that $\lim_{n\to\infty} M\,(1, 2, ..., n)^{1/n}$ exists and is between 1 and 2." Of course, in the terminology of basic hypergeometric series, the Sudler product is $(z; z)_n$. We shall discuss this important special case below.

Erdős and Szekeres used elementary estimates from the theory of diophantine approximation to show that for a.e. α,

$$\liminf_{n\to\infty} \left| \prod_{j=1}^{n} (1 - e^{2\pi i j\alpha}) \right| = 0. \tag{1.3}$$

Hardy and Littlewood earlier obtained an nth root asymptotic (see Section 3). Erdős and Szekeres stated that perhaps this holds for all α :

Problem 2
Show that for all real α, (1.3) holds.

We shall discuss S. Grepstad, L. Kaltenböck, and M. Neumüller's surprising disproof of this [23] below. As a contrast to (1.3), Erdos and Szekeres noted that for a.e. α,

$$\limsup_{n\to\infty} \left| \prod_{j=1}^{n} (1 - e^{2\pi i j\alpha}) \right| = \infty, \tag{1.4}$$

and indeed one can choose a sequence of integers $n = q_k - 1$, where q_k is a denominator in the continued fraction expansion of α. Of course the lim sup is 0 for rational α. They suggested that one might deterrmine the rates in the lim sup and lim inf, for almost all α :

Problem 3
How fast can $\left| \prod_{j=1}^{n} (1 - e^{2\pi i j\alpha}) \right|$ *tend to* 0 *or* ∞ *for almost all* α?

Without being aware of this problem, the author investigated this issue in [33], see below. Replacing 1 by z, they also asked:

Problem 4
Is it true that for all α,

$$\limsup_{n\to\infty} \left\| \prod_{j=1}^{n} (z - e^{2\pi i j\alpha}) \right\|_{L_\infty(|z|=1)} = \infty? \tag{1.5}$$

Of course this is now a problem about polynomials in z, which is quite different in nature. For rational α, (1.5) follows by regarding α as a root of unity. This problem is related to an older more general one of Erdős, namely that if $\{z_j\}_{j=1}^{\infty}$ is any sequence on the unit circle,

$$\limsup_{n\to\infty}\left\|\prod_{j=1}^{n}(z-z_j)\right\|_{L_\infty(|z|=1)}=\infty.$$

This was established by Wagner in 1980 [45], see below. For the corresponding lim inf, the authors noted that for rational α, the lim inf (and hence the limit) is ∞. They conjectured:

Problem 5
Is it true that for all irrational α,

$$\liminf_{n\to\infty}\left\|\prod_{j=1}^{n}(z-e^{2\pi ij\alpha})\right\|_{L_\infty(|z|=1)}<\infty?$$

This was established by Avila, Jitomirskaya and Marx [7].

Returning to general choices of $\{s_j\}$, Erdős and Szekeres pose their final problems:

Problem 6
Let $\{s_j\}_{j=1}^{\infty}$ be a strictly increasing sequence of positive integers. Is it true that for almost all α,

$$\limsup_{n\to\infty}\left|\prod_{j=1}^{n}(1-e^{2\pi is_j\alpha})\right|=\infty$$

but

$$\liminf_{n\to\infty}\left|\prod_{j=1}^{n}(1-e^{2\pi is_j\alpha})\right|=0?$$

There has been little progress on this.

In Section 2, we discuss progress primarily related to the growth of the sup norm of the Erdős-Szekeres polynomials $\prod_{j=1}^{n}(1-z^{s_j})$, namely Problem 1. In Section 3, we discuss Problems 4 and 5. In Section 4, we discuss Sudler products, Problems 2 and 3. In Section 5, we discuss Problem 6 on pointwise growth of the general products.

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2. The Erdős-Szekeres polynomials: Problem 1

Let's start by discussing progress in the original paper. Erdős and Szekeres proved (1.2) using some results from diophantine approximation and some special polynomials. Assume that $m \geq 2$ and

$$m^2 \leq n < (m+1)^2.$$

They set $s_j = 1$ for $j \leq n - m^2$ and the remaining $\{s_j\}$ are the integers $2^k \ell$ for $1 \leq k \leq m, 1 \leq \ell \leq m$. So their choice is

$$P_n(z) = (1-z)^{n-m^2} \prod_{k=1}^m \prod_{\ell=1}^m \left(1 - z^{2^k \ell}\right).$$

They note that for $|z| \leq 1$,

$$\left|(1-z)^{n-m^2}\right| \leq 2^m \leq 2^{2\sqrt{n}}.$$

To estimate the rest of the terms in P_n, they prove a preliminary result on diophantine approximation. Then they show that for each $\varepsilon > 0$,

$$\|P_n\|_{L_\infty(|z|=1)} \leq C(\varepsilon)(1+\varepsilon)^n,$$

giving (1.2).

They also claim the lower bound

$$f(n) := \inf\{M(s_1, s_2, ..., s_n) : s_1, s_2, ..., s_n \geq 1\} \geq \sqrt{2n}.$$

However, their proof has a gap. They assume that for some increasing integers, $\{a_j\}$ and another distinct set of increasing integers $\{b_j\}$, (so that there is no intersection between the $\{a_j\}, \{b_j\}$)

$$\sum_j z^{a_j} - \sum_j z^{b_j} = \prod_{j=1}^n (1 - z^{s_j}) = P_n(z). \tag{2.1}$$

In particular all coefficients of powers of z are ± 1. Then as the right-hand side has a zero of multiplicity n at 1, we can differentiate the left-hand side k times and set $z = 1$ to deduce (after some manipulation) that

$$\sum_j a_j^k = \sum_j b_j^k, \; k = 0, 1, ..., n-1.$$

Yes, for those familiar with the topic, this is the Prouhet-Tarry-Escott problem [17], [35], [40]. They then conclude that at least n of the $\{a_j\}$ and at least n of the $\{b_j\}$ are non-zero, so that at least $2n$ Maclaurin series coefficients of $P_n(z)$ are non-zero. Then Parseval's inequality gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P_n\left(e^{it}\right)\right|^2 dt = \sum_j |a_j|^2 + \sum_j |b_j|^2 \geq 2n,$$

giving $f(n) \geq \sqrt{2n}$.

However, the identity (2.1) is simply not true for all n, as very often there are coefficients other than ± 1. For example,

$$\prod_{j=1}^4 \left(1 - z^j\right) = 1 - z - z^2 + 2z^5 - z^8 - z^9 + z^{10}.$$

This gap was repaired in [13]. Since P_n has a zero of order n at 1, and is not identically 0, one can show it has at least n non-zero coefficients. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P_n\left(e^{it}\right)\right|^2 dt = \sum_{k=0}^N |a_k|^2 \geq n.$$

Since all zeros are on the unit circle, a result of O'Hara and Rodriguez [38, Corollary 1, p. 333] shows that

$$\|P_n\|^2_{L_\infty(|z|=1)} \geq 2\sum_{k=0}^{N} |a_k|^2 \geq 2n$$

so that indeed

$$f(n) \geq \sqrt{2n}.$$

If the original Erdős-Szekeres proof could be fixed, the O'Hara Rodriguez bound would give $f(n) \geq 2\sqrt{n}$.

As noted above, Erdős and Szekeres remarked that one can prove that

$$\lim_{n\to\infty} \left\| \prod\nolimits_{j=1}^{n} \left(1 - z^j\right) \right\|^{1/n}_{L_\infty(|z|=1)}$$

exists and assumes a value in $(1, 2)$. This was made precise in a 1964 paper of Cuthbert Sudler, who seemed unaware of [22]. He proved that

$$\begin{aligned} \log M(1, 2, ...n) &= \log \left\| \prod\nolimits_{j=1}^{n} \left(1 - z^j\right) \right\|_{L_\infty(|z|=1)} \\ &= Kn + O(\log n), \end{aligned} \tag{2.2}$$

where

$$K = \max\left\{ \frac{1}{w}\int_0^w \log|2\sin \pi t|\, dt : w \in \left(\frac{1}{2}, 1\right) \right\} = 0.19861... .$$

Sudler also considered the coefficients in the Maclaurin series

$$\prod\nolimits_{j=1}^{n} \left(1 - z^j\right) = \sum_j a_{n,j} z^j$$

and formed

$$A_n = \max_j |a_{n,j}|.$$

He noted that (2.2) also holds with A_n replacing $M(1, 2, ..., n)$, referring to earlier work of Nicol, Vandiver, and Motzkin. He conjectured the ratio asymptotic

$$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = K.$$

E.M. Wright proved in 1964 [46] not only this ratio asymptotic, but the stronger result

$$A_n = \frac{B}{n} e^{Kn} (1 + o(1)),$$

where

$$B = 2e^K \left(1 - \frac{1}{4}e^{2K}\right)^{-1/4} = 2.7424...$$

In a 2013 paper, Bell gave precise asymptotics for the L_p norm of $\prod_{j=1}^{n} \left(1 - z^j\right)$ [9].

The first improved estimate for $f(n)$ was given by Atkinson in 1961 [6]. In a short elegant paper, he showed that

$$\log f(n) \leq n^{1/2} \left(\frac{1}{2}\log n + 4\log 2\right),$$

by relating the Erdős-Szekeres problem to estimates of Fourier series, using

$$\log\left|1-e^{i\theta}\right| = -\sum_{j=1}^{\infty}\frac{\cos j\theta}{j}.$$

Some twenty years later, in 1982, Odlyzko [37] proved that

$$f(n) = \exp\left(O\left(n^{1/3}(\log n)^{4/3}\right)\right),$$

using random trigonometric polynomials, and estimates of cosine polynomials with nonnegative coefficients. Subsequently, Kolountzakis [31] also used random trigonometric polynomials, and improved Odlyzko's bound to

$$f(n) = \exp\left(O\left(n^{1/3}(\log n)\right)\right).$$

Whereas Odlyzko's $\{s_j\}$ need not be distinct, Kolountzakis ensured that

$$1 \le s_1 < s_2 < s_3 < ... < s_n < 2n + O\left(\sqrt{n}\right).$$

A major breakthrough came with the 1996 estimate of Belov and Konyagin [12]

$$f(n) = \exp\left(O\left((\log n)^4\right)\right),$$

a consequence of their work on nonnegative trigonometric polynomials. They considered, as did Atkinson and Odlyzko earlier, trigonometric polynomials $\sum_{k=0}^{\infty}\alpha_k\cos(kx)$ that are nonnegative on the real line, and for which $\alpha_1, \alpha_2, ...$ are nonnegative integers $\{\alpha_k\}$ such that

$$\sum_{k=1}^{\infty}\alpha_k = n.$$

Let $K_Z(n)$ denote the infimum of the constant coefficient α_0 of such trigonometric polynomials. In a powerful, long, technical paper, they show that

$$K_Z(n) = O\left((\log n)^3\right)$$

and then apply Odlyzko's observation that

$$\ln f(n) < (1+\log n)\,K_Z(n).$$

There has not been any improvement on this result.

Nor has there been an improvement on the lower bound $f(n) \ge \sqrt{2n}$ in general. Maltby [34], [35] obtained some improvements for specific values of n, focusing on the ℓ_1 norm of the coefficients, and using an algorithm to search for optimal $\{s_j\}$. Thus if

$$P_n(z) = \sum_j a_{n,j}z^j,$$

he studied the size of

$$\|P_n\|_1 = \sum_j |a_{n,j}|,$$

and also listed

$$\|P_n\|_2 = \sqrt{\sum_j |a_{n,j}|^2}.$$

For example for $n = 13$, it is known [17, p. 104] that the smallest $\|P_n\|_1$ is 44, and is attained for

$$\{s_j\} = \{1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23\}.$$

Optimal choices are known for all n up to 13. A table is given in a beautiful book of the late great Peter Borwein [17, p. 104]. Maltby listed the minimal known (in other words not yet proven minimal) for n up to 70 [34, pp. 240-241].

There have also been several results that treat $\{s_j\}$ with additional restrictions. For example, Peter Borwein [16] showed that if none of the $\{s_j\}$ are divisible by a given prime p,

$$M\left(s_1, s_2, ..., s_n\right) \geq \exp\left(\frac{\log p}{p-1} n\right).$$

Moreover, when $p = 2, 3, 5, 7, 11, 13$, the minimum of such products with no s_j divisible by p, achieve this rate in a precise form.

Bell, Borwein, and Richmond [10] obtained both asymptotic upper and lower bounds. They modified Atkinson's method, and cleverly used Vandermonde determinants, showing that if $\{\ell_n\}$ is an increasing sequence of integers, and we take $\{s_1, s_2, ..., s_n\}$ to be the first n of

$$\{\ell_1 - \ell_0, \ell_2 - \ell_1, \ell_2 - \ell_0, \ell_3 - \ell_2, \ell_3 - \ell_1, \ell_3 - \ell_0, ...\},$$

then

$$M\left(s_1, s_2, ..., s_n\right) \leq (32n)^{\sqrt{n/8}}.$$

They also showed that if L is a positive integer,

$$\liminf_{n\to\infty} M\left(1, 2^L, 3^L, ..., n^L\right)^{1/n} > 1,$$

Moreover, if $f(x)$ is a quadratic polynomial such that $\{f(n)\}$ is an increasing sequence of positive integers, then

$$\liminf_{n\to\infty} M\left(f(1), f(2), f(3), ..., f(n)\right)^{1/n} > 1.$$

All these results were proved by careful estimation of the products at carefully chosen points.

Bourgain and Chang [19] showed that we can choose $\{s_1, s_2, ..., s_n\} \subset \{1, 2, ..., N\}$ with $n/N \asymp 1/2$ such that

$$M\left(s_1, s_2, ..., s_n\right) \leq \exp\left(O\left(\sqrt{n}\sqrt{\log n}\log\log n\right)\right)$$

but if $\tau > 0$ is small enough and $n > (1-\tau) N$, then for all $\{s_1, s_2, ..., s_n\} \subset \{1, 2, ..., N\}$,

$$M\left(s_1, s_2, ..., s_n\right) > \exp\left(\tau n\right).$$

They use probabilistic methods and careful estimation of trigonometric sums.

The author and research students obtained in 2021 [13] lower bounds, using the Poisson integral of $\log|P_n|$ that are useful when the $\{s_j\}$ do not grow too fast. Let $1 \leq s_1 \leq s_2 \leq ... \leq s_n$ and $1 \leq L \leq n$. Then

$$M\left(s_1, s_2, ..., s_n\right)_\infty \geq \exp\left(\frac{1}{2e}\frac{L}{\left(s_1 s_2 ... s_L\right)^{1/L}}\right). \tag{2.3}$$

For example, if for some $r \in (0,1)$, we have $s_j \leq \frac{j}{1+(\log j)^2}$, for $1 \leq j \leq [rn]$, then this gives

$$M\left(s_1, s_2, ..., s_n\right) \geq \exp\left(\frac{1}{2}\left(\log[rn]\right)^2\left(1 + o(1)\right)\right).$$

A second result in [13], proved using Kellogg's extension of the Hausdorff-Young inequalities for coefficients of Fourier series, works well for rapidly growing or separated $\{s_j\}$: let $I_k = \left\{2^{k-1}, 2^{k-1}+1, ..., 2^k - 1\right\}$ for $k \geq 1$. Let $1 \leq s_1 \leq s_2 \leq ... \leq s_n$. Assume that I_k contains $\ell_k \geq 0$ of the $\{s_j\}_{j=1}^n$ for $k \geq 1$. Let $1 < p \leq 2$ and $\varepsilon = \frac{2}{p}(p-1)$. Then for $n \geq 2$,

$$M(s_1, s_2, ..., s_n) \geq \exp\left(C\left\{\frac{\sum_{k=1}^{\infty} \ell_k^{\varepsilon}}{(n \log n)^{\varepsilon}}\right\}^{p/2}\right). \tag{2.4}$$

Here C depends on p but is independent of n and the $\{s_j\}$.

As an example, let $B > 2$ and

$$s_j = \left[j (\log j)^B\right], \ j \geq 1.$$

Then one can use (2.4) to show that for some $C, \delta > 0$,

$$M(s_1, s_2, ..., s_n) \geq \exp(C (\log n)^{1+\delta}).$$

In a 2022 paper [14], the author and research students explored the average and variance of L_p norms of Erdős-Szekeres polynomials. Here the average is taken over all $s_1, s_2, ..., s_n \in [1, M]$ and we then let $M \to \infty$. Unsurprisingly, the average behavior is geometric growth.

Clearly, there is a lot of scope for work on both upper and lower bounds. Problem 1 is very far from solved.

3. Generalized Products: Problems 4 and 5

As noted above, Problem 4 is a special case of an older problem of Erdős, on growth of sequences of polynomials with all zeros $\{z_j\}$ on the unit circle. Recall this asks if

$$\limsup_{n\to\infty} \left\| \prod_{j=1}^{n} (z - z_j) \right\|_{L_\infty(|z|=1)} = \infty.$$

G. Wagner [45] proved in 1980 that for some $\delta > 0$, independent of the sequence $\{z_j\}$, and infinitely many n,

$$\left\| \prod_{j=1}^{n} (z - z_j) \right\|_{L_\infty(|z|=1)} \geq (\log n)^{\delta}.$$

In a 1991 paper, J. Beck [8] proved that for some $c > 0$, $\left\| \prod_{j=1}^{n} (z - z_j) \right\|_{L_\infty(|z|=1)}$ grows at least as fast as n^c, for infinitely many n. In the other direction, Beck showed that there is a sequence $\{z_j\}$ such that the nth sup is bounded by $n+1$ for all n. While Beck's construction of these slowly growing polynomials uses roots of unity, it does not resolve Problem 1 of Erdős-Szekeres as the polynomials have a different form. An L_2 analogue of Wagner's lower bound was recently established by Steinerberger [41]: for infinitely many n,

$$\left\| \log \left| \prod_{j=1}^{n} (z - z_j) \right| \right\|_2 \geq C\sqrt{\log n}.$$

The solution to Problem 5 is much more recent: in a 2017 paper on almost Matthieu operators, Avila, Jitomirskaya, and Marx [7] showed that Problem 5 has a positive solution: for all irrational α,

$$\liminf_{n\to\infty}\left\|\prod_{j=1}^{n}(z-e^{2\pi ij\alpha})\right\|_{L_\infty(|z|=1)}<\infty. \tag{3.1}$$

Moreover, they specified the subsequence of integers for which (3.1) is true - one can choose $n=q_\ell-1$, where $\{q_\ell\}$ are the denominators in convergents to the continued fraction for α.

4. Sudler Products: Problems 2 and 3

The term Sudler product for

$$S_n(\alpha)=\prod_{j=1}^{n}(1-e^{2\pi ij\alpha})$$

now seems to be fairly common. I personally feel that it could just as well be called a Sudler-Wright product, or just the $q-$Pochhammer symbol $(q;q)_n$ with $q=e^{2\pi i\alpha}$. Whatever the name, it certainly has generated a lot of research, and has many connections. We shall use the notation $S_n(\alpha)$ throughout this section.

Possibly the earliest analytic estimate of these products appeared in a 1946 paper of Hardy and Littlewood [27], the twenty fourth (!) in a series of papers on series. Using the identity

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}=\exp\left(\sum_{n=1}^{\infty}\frac{z^n}{n(1-q^n)}\right), \tag{4.1}$$

and the Cauchy-Hadamard formula for the radius of convergence of a power series, they showed that

$$\liminf_{n\to\infty}|(q;q)_n|^{1/n}=\liminf_{n\to\infty}|1-q^n|^{1/n}.$$

Diophantine approximation then immediately yields that for

$$q=e^{2\pi i\alpha}$$

and a.e. $\alpha\in[0,1]$, we have

$$\liminf_{n\to\infty}|S_n(\alpha)|^{1/n}=\liminf_{n\to\infty}|(q;q)_n|^{1/n}=1.$$

By choosing suitable α, one may ensure that this last lim inf takes any values between 0 and 1. Related radii of convergence were explored, for example in [20], [32], [39]. In particular, Petruska proved in his 1992 paper that for suitable $A=e^{2\pi i\beta}$ and $q=e^{2\pi i\alpha}$,

$$\limsup_{n\to\infty}\left|\prod_{j=1}^{n}\left(A-q^j\right)\right|^{1/n}$$

may take any value in $[0,1]$.

The author began to look for finer estimates of $(q;q)_n$ as part of a project to resolve the Baker-Gammel-Wills Conjecture on Padé approximants. Without being really aware of Problem 3, he established a number of results in a 1999 paper [33]. Among the tools, were the continued fraction expansion,

$$\alpha = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + ..., \tag{4.2}$$

more specifically, the convergents

$$\frac{p_k}{q_k} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + ... \frac{1|}{|a_k}, k \geq 1,$$

the Ostrowski representation of positive integers n in terms of the denominators $\{q_k\}$ of the convergents, and careful estimation of trigonometric sums using methods that are quite common in discrepancy theory. In particular, he showed that given $\varepsilon > 0$, for a.e. α, we have

$$|\log |S_n(\alpha)|| = O\left((\log n)(\log \log n)^{1+\varepsilon}\right), n \to \infty,$$

but for a.e. α,

$$\limsup_{n\to\infty} \left(\log \frac{1}{|S_n(\alpha)|}\right) \frac{1}{(\log n)(\log \log n)} = \infty.$$

(A remarkable recent paper of Bence Borda [15] contains some improvements, see below.)

For irrational α, it was shown that

$$\limsup_{n\to\infty} \frac{\log |S_n(\alpha)|}{\log n} \geq 1.$$

Of course, this also gives a positive solution to Problem 4, with a rate, but only for irrational α. As noted above, Problem 4 is trivial for rational α.

There were also other results depending on the size of the continued fraction coefficients $\{a_j\}$: if

$$\sup_j a_j = \infty,$$

then

$$\liminf_{n\to\infty} |S_n(\alpha)| = 0. \tag{4.3}$$

Thus Problem 2 is resolved for the case of α having a continued fraction with unbounded entries. On the other hand, if the entries in the continued fraction are bounded, that is,

$$\sup_j a_j < \infty,$$

then it was shown that

$$|\log |S_n(\alpha)|| = O(\log n),$$

so that for some positive constants C_1, C_2,

$$n^{-C_1} \leq |S_n(\alpha)| \leq n^{C_2}. \tag{4.4}$$

It was also stated there that the proofs show that there is an integer K such that if

$$a_j \geq K \text{ for infinitely many } j,$$

then (4.3) is true, and it was stated that "we are certain that it is true in general". The question of the smallest C_1 in (4.4) was raised.

It is always dangerous to be certain about something that has not been proven. The first inklings that (4.3) might not always be true appear in a 2016 paper of

Verscheuren and Mestel [44]. That paper also contains an interesting review of connections to dynamical systems, including work on generalizations of the Sudler product, due for example to Knill and Lesieutre [30]. Let

$$\alpha = \frac{\sqrt{5}-1}{2} = \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + ...$$

("the golden rotation number" or "golden section ratio") and $\{F_n\}$ be the Fibonacci numbers. These are also the denominators in the continued fraction for α. Verscheuren and Mestel proved that

$$\lim_{n\to\infty} |S_{F_n}(\alpha)| = c = 2.407...$$

and

$$\lim_{n\to\infty} \frac{|S_{F_n-1}(\alpha)|}{F_n} = \frac{c\sqrt{5}}{2\pi}.$$

They conjectured that the first result can be extended to all quadratic irrationals α, provided we replace the Fibonacci numbers by the denominators $\{q_k\}$ of the continued fraction for α.

This conjecture was verified by Grepstad and Neumüller in a 2018 paper [24]. They consider α with periodic of period ℓ continued fraction entries $a_1, a_2, ...a_\ell$. As usual, $\{q_j\}$ are the denominators in the continued fraction for α. They prove that there exist positive constants $C_k, k = 0, 1, 2, ..., \ell - 1$, such that

$$\lim_{n\to\infty} \left|S_{q_{\ell n+k}}(\alpha)\right| = C_k.$$

They note that this offers substantial evidence that (4.3) might be false for quadratic irrationals.

Grepstad, Kaltenböck, and Neumüller turned this evidence into a dramatic breakthrough, resolving the 60 year old problem in a 2019 paper [23]:

$$\liminf_{n\to\infty} \left|S_n\left(\frac{\sqrt{5}-1}{2}\right)\right| > 0.$$

Their proof relies on the earlier result of Verscheuren and Mestel. (It was also independently established in the unpublished thesis of Verscheuren [43]). They use the Ostrowki representation in the special case of $q_n = F_n$, which is then called the Zeckendorf representation. They make a number of interesting conjectures, such as

$$\left|S_{F_{n-1}}\left(\frac{\sqrt{5}-1}{2}\right)\right| \le \left|S_m\left(\frac{\sqrt{5}-1}{2}\right)\right| \le \left|S_{F_n-1}\left(\frac{\sqrt{5}-1}{2}\right)\right|$$

if

$$F_{n-1} \le m \le F_n - 1.$$

In a subsequent paper [25], they used results of the author's 1999 paper to rigorously prove that there is a thereshhold K such that if infinitely many entries a_j in the continued fraction (4.2) satisfy $a_j \ge K$, then (4.3) is true. They also conjecture that if the continued fraction has period 1, so

$$\alpha = \frac{1|}{|a} + \frac{1|}{|a} + \frac{1|}{|a} + ... , \tag{4.5}$$

with $a \ge 6$, then (4.3) is true.

This was verified in another breakthrough, in the 2020 paper of Aistleitner, Technau, and Zafeiropoulos [5]: they consider quadratic irrationals of the form (4.5). They prove the remarkable result that if $a \leq 5$, then

$$\liminf_{n\to\infty} |S_n(\alpha)| > 0 \text{ and } \limsup_{n\to\infty} \frac{|S_n(\alpha)|}{n} < \infty; \tag{4.6}$$

On the other hand if $a \geq 6$, then

$$\liminf_{n\to\infty} |S_n(\alpha)| = 0 \text{ and } \limsup_{n\to\infty} \frac{|S_n(\alpha)|}{n} = \infty.$$

They have to separately consider the case $a = 6$. They noted that their proofs show more for $a \geq 6$, namely

$$\liminf_{n\to\infty} \frac{\log |S_n(\alpha)|}{\log n} < 0 \text{ and } \limsup_{n\to\infty} \frac{\log |S_n(\alpha)|}{\log n} > 1.$$

They pose a number of problems, such as when the Sudler product grows at most linearly; and in particular, under what conditions on distinct integers a, b, is the growth at most linear for the two periodic continued fraction

$$\alpha = \frac{1|}{|a} + \frac{1|}{|b} + \frac{1|}{|a} + \frac{1|}{|b} + \frac{1|}{|a} \dots .$$

In a 2022 paper, Hauke [28] determined the exact value of the lim inf and lim sup in (4.6) for $a = 1, 2, 3, 4, 5$. More precisely, Hauke proved that

$$\begin{aligned} \liminf_{n\to\infty} |S_n(\alpha)| &= \lim_{k\to\infty} |S_{q_k}(\alpha)| = C_a \\ \limsup_{n\to\infty} \frac{|S_n(\alpha)|}{n} &= \lim_{k\to\infty} \frac{|S_{q_k-1}(\alpha)|}{q_k - 1} = \frac{\sqrt{a^2+1}}{2\pi} C_a, \end{aligned}$$

where, with $\{\}$ denoting fractional part,

$$C_a = \frac{2\pi}{\sqrt{a^2+1}} \prod\nolimits_{n=1}^{\infty} \left(\left(1 - \frac{\{n\alpha\} - \frac{1}{2}}{n\sqrt{a^2+4}}\right)^2 - \frac{1}{\left(2n\sqrt{a^2+4}\right)^2} \right).$$

He also showed that $S_n(\alpha)$ is maximal if $n = q_k - 1$, and minimal when $n = 1$.

Several of the above papers use perturbed Sudler products, and the techniques and proofs are not for the faint hearted! A search for simplifying principles that might offer a general approach is welcome, and a recent paper of Aistleitner and Borda [1] offers some hope in this regard. They prove a reflection principle and a transfer principle. The former is the observation that if a, b are coprime positive integers, and $0 \leq N < b$, then

$$\log S_N(a/b) + \log S_{b-N-1}(a/b) = \log b.$$

The latter is the following estimate: if

$$\alpha = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \dots$$

has convergents $\left\{\frac{p_m}{q_m}\right\}$, then for $k \geq 1, 0 \leq N < q_k$,

$$\left| \log |S_N(\alpha)| - \log S_N\left(\frac{p_k}{q_k}\right) \right| \leq C \frac{\log(1 + \max_{1\leq \ell \leq k} a_\ell)}{a_{k+1}}.$$

The constant C is a universal constant. As a consequence they relate the upper and lower constants in (4.4). In particular for quadratic irrationals of period 1 with a as in (4.5), they show that

$$C_2 = C_1 + 1 = \frac{Vol\,(4_1)}{4\pi} \frac{a}{\log \frac{a+\sqrt{a^2+1}}{2}} + O\,(1)\,,$$

where

$$Vol\,(4_1) = 4\pi \int_0^{5/6} \log\,(2 \sin \pi x)\, dx = 2.02988...$$

is the hyperbolic volume of the complement of the figure eight knot(!). They also use their principle to show that when the continued fraction coefficients $\{a_j\}$ are bounded,

$$\liminf_{n\to\infty} |S_n\,(\alpha)| = 0 \text{ iff } \limsup_{n\to\infty} \frac{|S_n\,(\alpha)|}{n} = \infty$$

Can one extend the cutoff in the result of 6 in the result of Aistleitner, Technau, and Zafeiropoulos to general quadratic irrationals? A recent paper of Grepstad, Neumüller, and Zafeiropoulos [26] establishes that for quadratic irrationals whose continued fraction is periodic of length ℓ, so that

$$\alpha = \frac{1|}{|b_1} + \frac{1|}{|b_2} + ... \frac{1|}{|b_j} + \frac{1|}{|a_1} + \left(\overline{\frac{1|}{|a_2} + ... \frac{1|}{|a_\ell}} \right)$$

then as long as $\max_j a_j \geq 23$, we have

$$\liminf_{n\to\infty} |S_n\,(\alpha)| = 0 \text{ and } \limsup_{n\to\infty} \frac{|S_n\,(\alpha)|}{n} = \infty. \tag{4.7}$$

They again conjecture that $\max_j a_j \geq 6$ should suffice.

This was resolved as partly true and partly false in yet another recent breakthrough paper of Hauke [29]. He proved that if α is an arbitrary irrational with continued fraction entries $\{a_j\}$ satisfying

$$\limsup_{j\to\infty} a_j \geq 7,$$

then (4.7) holds. On the other hand, the conjecture is false for

$$\alpha = \overline{\frac{1|}{|6} + \frac{1|}{|5} + \frac{1|}{|5}...}$$

for which

$$\liminf_{n\to\infty} |S_n\,(\alpha)| > 0 \text{ and } \limsup_{n\to\infty} \frac{|S_n\,(\alpha)|}{n} < \infty.$$

Thus the case where the largest entry is 6 is still intriguing. Hauke notes that the length of the period in the continued fraction of the quadratic irrational is a determining factor in the cutoff of $\limsup_{j\to\infty} a_j$ for (4.7) to hold. When the length is 1 or 2, the cutoff is 6, but for length 3, the cutoff is 7. He conjectures that the cutoff is 7 for length $\ell \geq 4$, with an example being provided by

$$\alpha = \overline{\frac{1|}{|6} + \underbrace{\frac{1|}{|5} + \frac{1|}{|5} ... \frac{1|}{|5}}_{\ell-1 \text{ times}}}\,.$$

A recent paper of Neumüller [36] investigates subsequences of $\{S_n\,(\alpha)\}$. In particular, when the continued fraction coefficients $\{a_j\}$ are bounded, he shows that

$\{S_{q_n}(\alpha)\}_{n\geq 1}$ is bounded above and below by positive constants independent of n. When the coefficients are unbounded, a precise rate of decay is given for $S_{q_n}(\alpha)$, involving the digits in Ostrowski expansions. Aistleitner and Borda [2] give precise expressions for sums of powers of $|S_n(\alpha)|$ and $\max_{1\leq n\leq m}|S_n(a)|$.

It is fitting to end a discussion of the very difficult and deep results above by mentioning a remarkable central limit theorem, concentration result, and lower bound of Borda [15]: if α is given by (4.5), then for $m\geq 3$ and real t,

$$\frac{1}{m}\#\left\{1\leq n\leq m:\frac{\log S_n(\alpha)-\frac{1}{2}\log n}{\sigma(\alpha)\sqrt{\log n}}\leq t\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-x^2/2}dx+O\left(\frac{(\log\log m)^2}{(\log m)^{1/6}}\right).$$

Here $\sigma(\alpha)$ is an explicitly computable constant. Moreover, for irrational α with bounded continued fraction entries $\{a_j\}$, there is a concentration law of the form

$$\frac{1}{m}\#\left\{1\leq n\leq m:e^{-t\sqrt{\log 2n}}\leq S_n(\alpha)/\sqrt{n}\leq e^{t\sqrt{\log 2n}}\right\}=1-O\left(t^2\right).$$

In addition, for a.e. α,

$$\liminf_{m\to\infty}\frac{\max_{1\leq n\leq m}\log S_n(\alpha)}{(\log n)(\log\log n)}=\frac{12}{\pi^2}\int_0^{5/6}\log|2\sin\pi x|\,dx.$$

Note that this improves for a.e. α, the lower bounds from the author's 1999 paper.

Given the complexity of both the results and proofs above, I am tempted to chuck out the hope that by exploiting the original power series identity (4.1), one might obtain a simpler approach, for at least some basic results. What is clear is that researchers have discovered ever deeper and more profound results, especially in recent years.

5. Problem 6

As one might expect, when Sudler products are not fully understood, far less is known about

$$\left|\prod_{j=1}^{n}\left(1-e^{2\pi i s_j\alpha}\right)\right|.$$

In the case where $s_j=2^j$, there is a powerful result of Aistleitner, Hofer, and Larcher [3]. Let $\varepsilon>0$. Then for a.e. $\alpha\in(0,1)$, we have for large enough n,

$$\left|\prod_{j=1}^{n}\left(1-e^{2\pi i 2^j\alpha}\right)\right|\leq\exp\left((\pi+\varepsilon)\sqrt{n\log\log n}\right)$$

while for infinitely many n,

$$\left|\prod_{j=1}^{n}\left(1-e^{2\pi i 2^j\alpha}\right)\right|\geq\exp\left((\pi-\varepsilon)\sqrt{n\log\log n}\right).$$

The proofs are probabilistic in nature. In a 2018 paper, Aistleitner, Larcher, Pillichshammer, Saad Eddin, and Tichy [4] consider random products

$$\left|\prod_{j=1}^{n}\left(1-e^{2\pi i x_j}\right)\right|,$$

where the $\{x_j\}$ are uniformly distributed in $[0,1]$. A special case of their results has relevance to Problem 6: let the coefficients in the continued fraction expansion of α be bounded. Let $\{\xi_n\}$ be a sequence of i.i.d. $\{0,1\}$ valued random variables with mean $\frac{1}{2}$. This induces a random sequence $\{s_k\}$ as the sequence of numbers n

with $\xi_n = 1$, sorted in increasing order. Then for all $\varepsilon > 0$, one has almost surely, for all large enough n,

$$\left|\prod_{j=1}^{n}\left(1-e^{2\pi i s_j \alpha}\right)\right| \leq \exp\left(\left(\frac{\pi}{\sqrt{12}}+\varepsilon\right)\sqrt{n\log\log n}\right);$$

while infinitely often,

$$\left|\prod_{j=1}^{n}\left(1-e^{2\pi i s_j \alpha}\right)\right| \geq \exp\left(\left(\frac{\pi}{\sqrt{12}}-\varepsilon\right)\sqrt{n\log\log n}\right).$$

One can also interpret results on uniform distribution of $\{s_j\alpha\}$ as providing some limited insight into Problem 6.

6. Conclusion

Almost 65 years after its publication, the problems posed by Erdős and Szekeres continue to inspire arduous and deep research. Clearly a lot of work remains to be done, especially on Problems 1, 3 and 6.

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