THE EFFECT OF TWO EXTERIOR MASSPOINTS ON BOUNDS FOR ORTHOGONAL POLYNOMIALS

D. S. LUBINSKY

Abstract. Let ν be a positive measure supported on $[-1,1]$, with infinitely many points in its support. Let $\{p_n(\nu,x)\}_{n\geq 0}$ be its sequence of orthonormal polynomials. Suppose we add masspoints at r,s with $r\geq 1, |s|\geq 1$,and $r\neq s$, giving a new measure $\mu=\nu+M\delta_r+N\delta_s$. How much larger can $|p_n(\mu,0)|$ be than $|p_n(\nu,0)|$? We study this question for symmetric measures ν, and obtain bounds that are uniform in n,M,N,r,s. In an earlier paper, we studied the case where $r=1, s=-1$, and investigated asymptotics.

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1. Results

Let μ be a finite positive Borel measure on the real line with infinitely many points in its support, and all finite moments

$$\int t^j d\mu(t), \, j=0,1,2,\dots .$$

Then we may define orthonormal polynomials

$$p_n(\mu,x)=\gamma_n(\mu)x^n+\dots,\gamma_n(\mu)>0,$$

$n=0,1,2,\dots$ satisfying the orthonormality conditions

$$\int p_n(\mu,x)\,p_m(\mu,x)\,d\mu(x)=\delta_{mn}.$$

The nth reproducing kernel for μ is

$$K_n(\mu,x,t)=\sum_{j=0}^{n-1}p_j(\mu,x)\,p_j(\mu,t)=\frac{\gamma_{n-1}}{\gamma_n}(\mu)\frac{p_n(\mu,x)\,p_{n-1}(\mu,t)-p_{n-1}(\mu,x)\,p_{n-1}(\mu,t)}{x-t}.$$

The three term recurrence relation has the form

$$(x-b_n(\mu))\,p_n(\mu,x)=a_{n+1}(\mu)\,p_{n+1}(\mu,x)+a_n(\mu)\,p_{n-1}(\mu,x)\,,$$

where

$$a_n(\mu)=\frac{\gamma_{n-1}}{\gamma_n}(\mu)\,.$$

Note that when μ is symmetric about 0, then $b_n(\mu)=0$.

A central problem in the theory of orthonormal polynomials is to establish bounds on $p_n(\mu,x)$. Indeed, one of the most celebrated questions in this subject is the Steklov Conjecture, posed in 1920 and solved by E. A. Rakhmanov in 1979

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[16], [17]. Steklov conjectured that if μ has support $[-1,1]$, and is absolutely continuous there, while μ' is bounded below, then $\{p_n(\mu,x)\}_{n\geq 0}$ is uniformly bounded in $[-1,1]$, that is

$$\sup_{n\geq 0}\sup_{x\in[-1,1]}|p_n(\mu,x)|<\infty.$$

Rakhmanov showed that Steklov's conjecture is false, and in fact, given $\varepsilon>0$, one can find a measure satisfying its hypotheses such that for infinitely many n,

$$|p_n(\mu,0)|\geq Cn^{1/2-\varepsilon}.$$

There have since been a host of developments, due to amongst others, Ambroladze, Aptekarev, Denissov, Rush, and Tulyakov. In particular, Aptekarev, Denisov and Tulyakov [3] showed that $|p_n(\mu,0)|$ may grow faster than $n^{1/2}/\varepsilon_n$, where $\{\varepsilon_n\}$ is any sequence with limit 0, and this is the best one can aim for. Ambroladze showed [1] that even if μ' is continuous and satisfies some weak smoothness condition, Steklov's conjecture may still be false.

What about positive results? Most uniform bounds on orthonormal polynomials are consequences of much deeper asymptotics for orthonormal polynomials. Perhaps the most general result for measures supported on $[-1,1]$ is that of Badkov [5]. He proved that if μ satisfies Szegő's condition

$$\int_{-1}^{1}\frac{\log\mu'(x)}{\sqrt{1-x^2}}dx>-\infty$$

and μ' satisfies a local Dini condition, that is in some interval $[a,b]\subset(-1,1)$, the modulus of continuity

$$\omega(\delta)=\sup\{|\mu'(s)-\mu'(t)|:s,t\in[a,b]\text{ and }|s-t|\leq\delta\},\delta>0$$

satisfies

$$\int_0^1\frac{\omega(t)}{t}dt<\infty,$$

then the corresponding orthonormal polynomials are uniformly bounded in compact subsets of (a,b), and admit local Szegő asymptotics. For non-Szegő weights, Rakhmanov [18] showed that orthogonal polynomials for Dini smooth weights on the unit circle admit uniform bounds. For exponential weights, supported on finite or infinite intervals, Eli Levin and the author established [12] global bounds, using extensions of a technique that goes back to Stan Bonan, a student of Paul Nevai. In many of these results, the bounds are then superseded by asymptotics. See [8] for further developments.

In a recent paper, we investigated how adding masspoints at ± 1 can increase the size of the orthonormal polynomial at the origin. Consider a fixed positive measure ν supported on $[-1,1]$ with infinitely many points in its support, and that is symmetric about 0, so that $\nu([-b,-a])=\nu([a,b])$ for all $[a,b]\subset[-1,1]$. Let $\mathcal{M}(\nu)$ denote the class of all measures

$$\mu=\nu+M\delta_1+N\delta_{-1}$$

where $M,N\geq 0$. Also let

$$\begin{aligned}U_n &= K_n(\nu,1,1)-K_n(\nu,-1,1);\\ V_n &= K_n(\nu,1,1)+K_n(\nu,-1,1).\end{aligned}$$

We proved:

Theorem A
Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support. Assume also that ν is symmetric, so that $\nu([-b,-a]) = \nu([a,b])$ for all subintervals $[a,b]$ of $[-1,1]$. Let $n \geq 2$ be even. Then

$$\sup_{\mu\in\mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \max\left\{1, \frac{U_n^2}{V_n V_{n+1}}\right\}.$$

We deduced:

Corollary B
Assume in addition to the hypotheses of Theorem 1.1, that ν lies in the Nevai class, so that the recurrence coefficients satisfy

$$\lim_{n\to\infty} a_n(\nu) = \frac{1}{2}; \lim_{n\to\infty} b_n(\nu) = 0. \tag{1.1}$$

Then

$$\lim_{n\to\infty}\left(\sup_{\mu\in\mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2\right) = 1.$$

Further more detailed results were established in [13], with precise asymptotics for ultraspherical weights.

In this paper, we consider what happens when we add masspoints at two arbitrary points exterior to $(-1,1)$. We prove:

Theorem 1
Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support. Assume also that ν is symmetric, so that $\nu([-b,-a]) = \nu([a,b])$ for all subintervals $[a,b]$ of $[-1,1]$. Let $n \geq 2$ be even. Let $M, N \geq 0$, $r \geq 1, s \leq -1$, and

$$\mu = \nu + M\delta_r + N\delta_s.$$

Then

$$\left|\frac{p_n(\mu,0)}{p_n(\nu,0)}\right| \leq 1 + \sqrt{202}\frac{\gamma_{n-1}}{\gamma_n}(\nu) / \frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu).$$

For the case $r, s \geq 1$, we prove:

Theorem 2
Assume the hypotheses of Theorem 1, except that we let $r, s \geq 1$. Then

$$\left|\frac{p_n(\mu,0)}{p_n(\nu,0)}\right| \leq 1 + \sqrt{12}\frac{\gamma_{n-1}}{\gamma_n}(\nu) / \frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu).$$

Remark
If μ belongs to the Nevai class, then (1.1) holds and Theorem 1 and gives bounds that are independent of n, M, N, r, s. However, much less is needed. Since the support is $[-1,1]$, $\frac{\gamma_{n-1}}{\gamma_n}(\nu) \leq 1$ so we need $\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)$ to be bounded below by a positive constant. This is true if ?

Thus we have an explicit bound that does not depend on the size of the masspoints, nor their location. This raises the interesting question of bounds for more exterior masspoints. We prove the theorems in Section 2.

In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurences.

2. The Basic Identity

Throughout this section, we let $M, N \geq 0$ and $r \geq 1$, $|s| > 1, s \neq r$, and

$$\mu = \nu + M\delta_r + N\delta_s. \tag{2.1}$$

We assume that ν is a symmetric measure supported on $[-1, 1]$. We fix even $n \geq 2$. We let

$$X = \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}; Y = \frac{\gamma_{n-1}}{\gamma_n}(\nu) \begin{bmatrix} \frac{p_{n-1}(\nu,s)}{s} \\ \frac{p_{n-1}(\nu,r)}{r} \end{bmatrix}; \tag{2.2}$$

$$L = \begin{bmatrix} K_n(\nu, r, r) & -K_n(\nu, r, s) \\ -K_n(\nu, r, s) & K_n(\nu, s, s) \end{bmatrix}; \tag{2.3}$$

$$B = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} + MNL; \tag{2.4}$$

$$W = \det L = K_n(\nu, s, s) K_n(\nu, r, r) - K_n(\nu, s, r)^2; \tag{2.5}$$

$$d = 1 + MK_n(\nu, r, r) + NK_n(\nu, s, s) + MNW; \tag{2.6}$$

and

$$C = \frac{1}{d}B. \tag{2.7}$$

We prove the following identity. For the case $r = 1, s = -1$, it appeared in [13]. Various identities of this type have been established for adding the effect of mass-points and we are not sure it is new. Nevertheless, we provide a derivation for the form we need.

Theorem 2.1

$$\left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 = \frac{\{1 - X^T CY\}^2}{\{1 + X^T CX\}}. \tag{2.8}$$

We begin the proof with the following

Lemma 2.2
(a) Let

$$\pi_{n-1}(y) = p_n(\mu, y) - \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, y); \tag{2.9}$$

$$A = \begin{bmatrix} 1 + MK_n(\nu, r, r) & -MK_n(\nu, r, s) \\ -NK_n(\nu, r, s) & 1 + NK_n(\nu, s, s) \end{bmatrix}. \tag{2.10}$$

Then

$$p_n(\mu, y) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \left\{ p_n(\nu, y) + \frac{1}{d} \begin{bmatrix} -NK_n(\nu, y, s) \\ -MK_n(\nu, y, r) \end{bmatrix}^T AX \right\}. \tag{2.11}$$

(b)

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)} \right)^2 \left\{ 1 + \frac{1}{d} X^T A^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} X \right\} = 1. \tag{2.12}$$

Proof

(a) Using orthogonality, we see that

$$\begin{aligned} \pi_{n-1}(y) &= \int_{-1}^{1} K_n(\nu, y, t)\, \pi_{n-1}(t)\, d\nu(t) \\ &= \int_{-1}^{1} K_n(\nu, y, t)\, p_n(\mu, t)\, d\nu(t) \\ &= -MK_n(\nu, y, r)\, p_n(\mu, r) - NK_n(\nu, y, s)\, p_n(\mu, s). \end{aligned}$$

(2.13)

Taking $y = s$ and $y = r$, gives

$$p_n(\mu, s) - \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, s) = -MK_n(\nu, s, r)\, p_n(\mu, r) - NK_n(\nu, s, s)\, p_n(\mu, s);$$

$$p_n(\mu, r) - \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, r) = -MK_n(\nu, r, r)\, p_n(\mu, r) - NK_n(\nu, r, s)\, p_n(\mu, s);$$

and gathering the terms involving $p_n(\mu, r)$, $p_n(\mu, s)$ gives the matrix equation

(2.14)

$$\begin{bmatrix} 1 + NK_n(\nu, s, s) & MK_n(\nu, s, r) \\ NK_n(\nu, s, r) & 1 + MK_n(\nu, r, r) \end{bmatrix} \begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix} = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}.$$

The determinant d of the matrix is

$$\begin{aligned} d &= (1 + NK_n(\nu, s, s))(1 + MK_n(\nu, r, r)) - MNK_n(\nu, s, r)^2 \\ &= 1 + MK_n(\nu, r, r) + NK_n(\nu, s, s) + MNW. \end{aligned}$$

Solving the matrix equation (2.14) gives

$$\begin{aligned} &\begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \frac{1}{d} \begin{bmatrix} 1 + MK_n(\nu, r, r) & -MK_n(\nu, r, s) \\ -NK_n(\nu, r, s) & 1 + NK_n(\nu, s, s) \end{bmatrix} \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \frac{1}{d} A \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}. \end{aligned}$$

From (2.13) and this last identity,

$$\begin{aligned} \pi_{n-1}(y) &= p_n(\mu, y) - \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, y) \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \frac{1}{d} \begin{bmatrix} -NK_n(\nu, y, s) \\ -MK_n(\nu, y, r) \end{bmatrix}^T A \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}. \end{aligned}$$

Then (2.11) follows.
(b) We obtain equations for $\frac{\gamma_n(\mu)}{\gamma_n(\nu)}$ in two ways: from the definition (2.9) of π_n and orthogonality,

$$\begin{aligned}
&\int_{-1}^{1} \pi_{n-1}^2(y)\, d\nu(y) \\
&= \int_{-1}^{1} p_n^2(\mu, y)^2\, d\nu(y) - 2\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 + \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\
&= 1 - M p_n(\mu, r)^2 - N p_n(\mu, s)^2 - \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2.
\end{aligned}$$

Also, from (2.13),

$$\begin{aligned}
&\int_{-1}^{1} \pi_{n-1}^2(y)\, d\nu(y) \\
&= \int_{-1}^{1} \left(-N K_n(\nu, y, s)\, p_n(\mu, s) - M K_n(\nu, y, r)\, p_n(\mu, r)\right)^2 d\nu(y) \\
&= N^2 p_n^2(\mu, s)\, K_n(\nu, s, s) + M^2 p_n^2(\mu, r)\, K_n(\nu, r, r) + 2MN p_n(\mu, s)\, p_n(\mu, r)\, K_n(\nu, r, s).
\end{aligned}$$

Then using the last two equations and solving for $1 - \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2$,

$$\begin{aligned}
&1 - \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\
&= p_n^2(\mu, s)\left\{N + N^2 K_n(\nu, s, s)\right\} + p_n^2(\mu, r)\left\{M + M^2 K_n(\nu, r, r)\right\} \\
&\quad + 2MN p_n(\mu, s)\, p_n(\mu, r)\, K_n(\nu, r, s) \\
&= \begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix}^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 1 + N K_n(\nu, s, s) & M K_n(\nu, s, r) \\ N K_n(\nu, s, r) & 1 + M K_n(\nu, r, r) \end{bmatrix} \begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix} \\
&= d \begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix}^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} A^{-1} \begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix}
\end{aligned}$$

Using

$$\begin{bmatrix} p_n(\mu, s) \\ p_n(\mu, r) \end{bmatrix} = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \frac{1}{d} A \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix},$$

gives

$$\begin{aligned}
&1 - \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\
&= \frac{1}{d}\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \left(A \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}\right)^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}
\end{aligned}$$

so

$$1 = \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \left\{1 + \frac{1}{d} \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}^T A^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix}\right\}.$$

■

Proof of Theorem 2.1
Setting $y = 0$, and using $p_{n-1}(\nu, 0) = 0$, the Christoffel-Darboux formula gives

$$K_n(\nu, 0, y) = -\frac{\gamma_{n-1}(\nu)}{\gamma_n(\nu)} p_n(\nu, 0) \frac{p_{n-1}(\nu, y)}{y}.$$

Then from (2.11)

$$\begin{aligned} p_n(\mu, 0) &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \left\{ p_n(\nu, 0) + \frac{1}{d} \begin{bmatrix} -NK_n(\nu, 0, s) \\ -MK_n(\nu, 0, r) \end{bmatrix}^T A \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix} \right\} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, 0) \left\{ 1 + \frac{\gamma_{n-1}(\nu)}{\gamma_n(\nu)} \frac{1}{d} \begin{bmatrix} N \frac{p_{n-1}(\nu, s)}{s} \\ M \frac{p_{n-1}(\nu, r)}{r} \end{bmatrix} A \begin{bmatrix} p_n(\nu, s) \\ p_n(\nu, r) \end{bmatrix} \right\} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, 0) \left\{ 1 + \frac{1}{d} Y^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} AX \right\}, \end{aligned}$$

so from (2.12),

$$\begin{aligned} &\left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)} \right)^2 \left\{ 1 + \frac{1}{d} X^T A^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} X \right\} \\ &= \left\{ 1 + \frac{1}{d} Y^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} AX \right\}^2 \\ &= \left\{ 1 + \frac{1}{d} X^T A^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} Y \right\}^2. \end{aligned}$$

Here

$$\begin{aligned} &A^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \\ &= \begin{bmatrix} 1 + MK_n(\nu, r, r) & -NK_n(\nu, r, s) \\ -MK_n(\nu, r, s) & 1 + NK_n(\nu, s, s) \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \\ &= \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} + MNL = B. \end{aligned}$$

So

$$\left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)} \right)^2 \left\{ 1 + \frac{1}{d} X^T BX \right\} = \left\{ 1 - \frac{1}{d} X^T BY \right\}^2$$

and

$$\left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)} \right)^2 = \frac{\left\{ 1 - X^T CY \right\}^2}{\left\{ 1 + X^T CX \right\}}.$$

■

3. Proof of Theorems 1 and 2

Let us go back and look at

$$C = \frac{1}{d} B = \frac{\begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} + MNL}{1 + MK_n(\nu, r, r) + NK_n(\nu, s, s) + MNW}. \tag{3.1}$$

We begin with a result that applies to both the situations of Theorem 1 and 2. Throughout we assume the hypotheses of Theorem1 on ν and μ.

Lemma 3.1
For $r \geq 1, |s| \geq 1$,
(a)

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 \leq \left(1+\left(Y^T CY\right)^{1/2}\right)^2. \tag{3.2}$$

(b) Moreover,

$$Y^T CY \leq \left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\right)^2 \left\{\frac{1}{s^2}+\frac{1}{r^2}\right\} + \frac{Y^T LY}{W}. \tag{3.3}$$

Proof
(a) Note that C is positive definite, so that

$$\left|X^T CY\right| \leq \left(\left(X^T CX\right)\left(Y^T CY\right)\right)^{1/2}.$$

Indeed, this follows from the Cauchy-Schwarz inequality as $< X, Y >= X^T CY$ defines an inner product. Then from (2.8),

$$\begin{aligned}
\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 &= \frac{\left\{1-X^T CY\right\}^2}{\left\{1+X^T CX\right\}} \\
&= \frac{1-2X^T CY+\left(X^T CY\right)^2}{1+X^T CX} \\
&\leq 1+\frac{2\left(\left(X^T CX\right)\left(Y^T CY\right)\right)^{1/2}+\left(X^T CX\right)\left(Y^T CY\right)}{1+X^T CX} \\
&\leq 1+2\left(\frac{Y^T CY}{1+X^T CX}\right)^{1/2}+Y^T CY \\
&\leq 1+2\left(Y^T CY\right)^{1/2}+Y^T CY=\left(1+\left(Y^T CY\right)^{1/2}\right)^2.
\end{aligned}$$

(b) Here

$$Y^T CY = \frac{Y^T\left(\begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}+MNL\right)Y}{1+MK_n(\nu,r,r)+NK_n(\nu,s,s)+MNW}$$

$$\begin{aligned}
&= \frac{N\left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\frac{p_{n-1}(\nu,s)}{s}\right)^2+M\left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\frac{p_{n-1}(\nu,r)}{r}\right)^2}{MK_n(\nu,r,r)+NK_n(\nu,s,s)}+\frac{Y^T LY}{W} \\
&\leq \left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\right)^2\left\{\frac{p_{n-1}^2(\nu,s)}{s^2K_n(\nu,s,s)}+\frac{p_{n-1}^2(\nu,r)}{r^2K_n(\nu,r,r)}\right\}+\frac{Y^T LY}{W} \\
&\leq \left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\right)^2\left\{\frac{1}{s^2}+\frac{1}{r^2}\right\}+\frac{Y^T LY}{W}.
\end{aligned}$$

■

Lemma 3.2
Let $r \geq 1$ and $s \leq -1$. Then
(a)

$$W \geq \frac{1}{25}\left(K_n(\nu,r,r)K_{n-1}(\nu,s,s)+K_n(\nu,s,s)K_{n-1}(\nu,r,r)\right).$$

(b) Hence,

$$Y^T CY \leq 202 \left\{ \frac{\gamma_{n-1}}{\gamma_n} (\nu) / \frac{\gamma_{n-2}}{\gamma_{n-1}} (\nu) \right\}^2 .$$

Proof

(a) We use the error formula for the Cauchy-Schwartz inequality:

$$\begin{aligned} & K_n(\nu, r, r) K_n(\nu, s, s) - K_n(\nu, r, s)^2 \\ = & \frac{1}{2} \sum_{j,k=0}^{n-1} (p_j(\nu, r) p_k(\nu, s) - p_j(\nu, s) p_k(\nu, r))^2 \end{aligned} \tag{3.4}$$

so using that p_j is odd or even according as j is and that $s < 0$,

$$\begin{aligned} & K_n(\nu, r, r) K_n(\nu, s, s) - K_n(\nu, r, s)^2 \\ \geq & \frac{1}{2} \sum_{j \text{ odd},\ k \text{even}} (p_j(\nu, r) p_k(\nu, |s|) + p_j(\nu, |s|) p_k(\nu, r))^2 \\ & + \frac{1}{2} \sum_{j \text{ even},\ k \text{ odd}} (p_j(\nu, r) p_k(\nu, |s|) + p_j(\nu, |s|) p_k(\nu, r))^2 \end{aligned}$$

$$\begin{aligned} \geq & \frac{1}{2} \left\{ \begin{array}{c} \sum_{j \text{ odd}} p_j^2(\nu, r) \sum_{k \text{even}} p_k^2(\nu, |s|) + \sum_{k \text{even}} p_k^2(\nu, r) \sum_{j \text{ odd}} p_j^2(\nu, |s|) \\ + \sum_{j \text{ odd}} p_j^2(\nu, |s|) \sum_{k \text{even}} p_k^2(\nu, r) + \sum_{k \text{even}} p_k^2(\nu, |s|) \sum_{j \text{ odd}} p_j^2(\nu, r) \end{array} \right\} \\ \geq & K_n^o(r, r) K_n^e(|s|, |s|) + K_n^e(r, r) K_n^o(|s|, |s|), \end{aligned}$$

(3.5)

where

$$K_n^e(x, y) = \sum_{0 \leq j \leq n-1, j \text{ even}} p_j(x) p_j(y)$$

$$K_n^o(x, y) = \sum_{0 \leq j \leq n-1, j \text{ odd}} p_j(x) p_j(y) .$$

Next, from the recurrence relation,

$$x p_j(\nu, x) = \frac{\gamma_j}{\gamma_{j+1}} (\nu) p_{j+1}(\nu, x) + \frac{\gamma_{j-1}}{\gamma_j} (\nu) p_{j-1}(\nu, x)$$

so as $\frac{\gamma_j}{\gamma_{j+1}} (\nu) \leq 1$, we have

$$\begin{aligned} |p_j(\nu, x)| \leq & \frac{|p_{j+1}(\nu, x) + p_{j-1}(\nu, x)|}{|x|} \\ & \Rightarrow p_j(\nu, x)^2 \leq 2 \left\{ \left(\frac{p_{j+1}(\nu, x)}{x} \right)^2 + \left(\frac{p_{j-1}(\nu, x)}{x} \right)^2 \right\} \end{aligned}$$

Then (recall n is even) for $|x| \geq 1$,

$$\begin{aligned} K_n(\nu, x, x) &= K_n^o(x, x) + K_n^e(x, x) \\ &\leq K_n^o(x, x) + \frac{2}{x^2} \{ K_n^o(x, x) + K_n^o(x, x) \} \leq 5 K_n^o(x, x) . \end{aligned}$$

Similarly,

$$K_{n-1}(\nu, x, x) \leq 5 K_n^e(x, x) .$$

Hence from (3.5),

$$W \geq \frac{1}{25}\left(K_n\left(\nu,r,r\right)K_{n-1}\left(\nu,s,s\right)+K_n\left(\nu,s,s\right)K_{n-1}\left(\nu,r,r\right)\right).$$

(b) Next,

$$\begin{aligned} & Y^TLY \\ = & \left(\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)\right)^2\left\{K_n\left(\nu,r,r\right)\left(\frac{p_{n-1}\left(\nu,s\right)}{s}\right)^2+K_n\left(\nu,s,s\right)\left(\frac{p_{n-1}\left(\nu,r\right)}{r}\right)^2-2K_n\left(\nu,r,s\right)\frac{p_{n-1}\left(\nu,s\right)}{s}\frac{p_{n-1}\left(\nu,\right.}{r} \right. \\ \leq & \left(\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)\right)^2 K_n\left(\nu,r,r\right)K_n\left(\nu,s,s\right)\left(\frac{1}{|s|}+\frac{1}{r}\right)^2, \end{aligned}$$

by Cauchy-Schwarz, so that from (3.3),

$$\begin{aligned} & Y^TCY \\ \leq & \left(\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)\right)^2\left[\begin{array}{c} \frac{1}{s^2}+\frac{1}{r^2} \\ +\frac{25K_n(\nu,s,s)K_n(\nu,r,r)}{K_n(\nu,r,r)K_{n-1}(\nu,s,s)+K_n(\nu,s,s)K_{n-1}(\nu,r,r)}\left(\frac{1}{|s|}+\frac{1}{r}\right)^2 \end{array}\right] \\ \leq & \left(\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)\right)^2\left[2+\frac{50K_n\left(\nu,s,s\right)}{K_{n-1}\left(\nu,s,s\right)s^2}+\frac{50K_n\left(\nu,r,r\right)}{K_{n-1}\left(\nu,r,r\right)r^2}\right]. \end{aligned}$$

We continue this as

$$\leq\left(\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)\right)^2\left[2+\frac{50}{s^2}+\frac{50}{r^2}+50\left(\frac{p_{n-1}\left(\nu,s\right)^2}{K_{n-1}\left(\nu,s,s\right)s^2}+\frac{p_{n-1}\left(\nu,r\right)^2}{K_{n-1}\left(\nu,r,r\right)r^2}\right)\right].$$

Here

$$\begin{aligned} \left|sp_{n-2}\left(\nu,s\right)\right| &= \left|\frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)p_{n-1}\left(\nu,s\right)+\frac{\gamma_{n-3}}{\gamma_{n-2}}\left(\nu\right)p_{n-3}\left(\nu,s\right)\right| \\ &\geq \frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)\left|p_{n-1}\left(\nu,s\right)\right| \qquad (3.6) \end{aligned}$$

as p_{n-1}, p_{n-3} have the same sign at s, so

$$\frac{p_{n-1}\left(\nu,s\right)^2}{K_{n-1}\left(\nu,s,s\right)s^2}\leq\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)\right)^{-2}.$$

Similarly,

$$\frac{p_{n-1}\left(\nu,r\right)^2}{K_{n-1}\left(\nu,r,r\right)r^2}\leq\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)\right)^{-2}.$$

Then as $r,|s|\geq 1$, and $\frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)\leq 1$,

$$\begin{aligned} Y^TCY &\leq \left(\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)\right)^2\left\{102+100\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)\right)^{-2}\right\}. \\ &\leq 202\left\{\frac{\gamma_{n-1}}{\gamma_n}\left(\nu\right)/\frac{\gamma_{n-2}}{\gamma_{n-1}}\left(\nu\right)\right\}^2. \end{aligned}$$

■

Proof of Theorem 1
From Lemma 3.1 and Lemma 3.2,

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 \le \left(1+\sqrt{202}\left\{\frac{\gamma_{n-1}}{\gamma_n}(\nu)\Big/\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right\}\right)^2 .$$

■

For the case where both $r, s \ge 1$, we first prove:

Lemma 3.3
Assume that $r, s \ge 1$. Then

$$Y^T C Y \le 12\left\{\frac{\gamma_{n-1}}{\gamma_n}(\nu)\Big/\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right\}^2 .$$

Proof
(a)

$$\begin{aligned}
& Y^T L Y \Big/ \left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\right)^2 \\
= \; & K_n(\nu,r,r)\left(\frac{p_{n-1}(\nu,s)}{s}\right)^2 + K_n(\nu,s,s)\left(\frac{p_{n-1}(\nu,r)}{r}\right)^2 - 2K_n(\nu,r,s)\frac{p_{n-1}(\nu,s)}{s}\frac{p_{n-1}(\nu,r)}{r} \\
= \; & \sum_{j=0}^{n-1}\left\{p_j(\nu,r)\frac{p_{n-1}(\nu,s)}{s} - p_j(\nu,s)\frac{p_{n-1}(\nu,r)}{r}\right\}^2 \\
= \; & \frac{1}{(rs)^2}\sum_{j=0}^{n-2}\left\{r p_j(\nu,r)\,p_{n-1}(\nu,s) - s p_j(\nu,s)\,p_{n-1}(\nu,r)\right\}^2 + \frac{(r-s)^2}{(rs)^2}\left(p_{n-1}(r)\,p_{n-1}(\nu,s)\right)^2 .
\end{aligned}$$

Using the recurrence relation and $\frac{\gamma_j}{\gamma_{j+1}}(\nu) \le 1$, we continue this as

$$\begin{aligned}
= \; & \frac{1}{(rs)^2}\sum_{j=0}^{n-2}\left\{\begin{array}{c} \left[\frac{\gamma_j}{\gamma_{j+1}}(\nu)\,p_{j+1}(\nu,r) + \frac{\gamma_{j-1}}{\gamma_j}(\nu)\,p_{j-1}(\nu,r)\right]p_{n-1}(\nu,s) \\ -\left[\frac{\gamma_j}{\gamma_{j+1}}(\nu)\,p_{j+1}(\nu,s) + \frac{\gamma_{j-1}}{\gamma_j}(\nu)\,p_{j-1}(\nu,s)\right]p_{n-1}(\nu,r)\end{array}\right\}^2 + \frac{(r-s)^2}{(rs)^2}\left(p_{n-1}(r)\,p_{n-1}(\nu,s)\right)^2 \\
\le \; & \frac{2}{(rs)^2}\sum_{j=0}^{n-2}\left(\frac{\gamma_j}{\gamma_{j+1}}(\nu)\right)^2\left\{\begin{array}{c}\left\{p_{j+1}(\nu,r)\,p_{n-1}(\nu,s) - p_{j+1}(\nu,s)\,p_{n-1}(\nu,r)\right\}^2 \\ +\left\{p_{j-1}(\nu,r)\,p_{n-1}(\nu,s) - p_{j-1}(\nu,s)\,p_{n-1}(\nu,r)\right\}^2\end{array}\right\} + \frac{(r-s)^2}{(rs)^2}\left(p_{n-1}(\nu,r)\,p_{n-\right. \\
\le \; & \frac{2}{(rs)^2}\sum_{j=0}^{n-2}\left\{\begin{array}{c}\left\{p_{j+1}(\nu,r)\,p_{n-1}(\nu,s) - p_{j+1}(\nu,s)\,p_{n-1}(\nu,r)\right\}^2 \\ +\left\{p_{j-1}(\nu,r)\,p_{n-1}(\nu,s) - p_{j-1}(\nu,s)\,p_{n-1}(\nu,r)\right\}\end{array}\right\}^2 + \frac{(r-s)^2}{(rs)^2}\left(p_{n-1}(\nu,r)\,p_{n-1}(\nu,s)\right)^2 \\
\le \; & \frac{4}{(rs)^2}\sum_{j,k=0}^{n-1}\left(p_j(r)\,p_k(s) - p_j(s)\,p_k(r)\right)^2 + \frac{(r-s)^2}{(rs)^2}\left(p_{n-1}(\nu,r)\,p_{n-1}(\nu,s)\right)^2 \\
= \; & \frac{8W}{(rs)^2} + \frac{(r-s)^2}{(rs)^2}\left(p_{n-1}(\nu,r)\,p_{n-1}(\nu,s)\right)^2 .
\end{aligned}$$

(b) Next, from (3.4), and the Christoffel-Darboux formula,

$$\begin{aligned} W &\geq \frac{1}{2}\left(p_{n-2}(\nu,r)\,p_{n-1}(\nu,s)-p_{n-2}(\nu,s)\,p_{n-1}(\nu,r)\right)^2 \\ &= \frac{1}{2}\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)^{-2}\left(K_{n-1}(\nu,r,s)(r-s)\right)^2 \\ &\geq \frac{1}{2}\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)^{-2}\left(p_{n-2}(\nu,r)\,p_{n-2}(\nu,s)(r-s)\right)^2. \end{aligned}$$

This last step uses that all $p_j(r), p_j(s) \geq 0$. Then

$$\begin{aligned} &\frac{(r-s)^2}{W(rs)^2}\left(p_{n-1}(\nu,r)\,p_{n-1}(\nu,s)\right)^2 \\ \leq\ & \frac{2}{(rs)^2}\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)^2\frac{\left(p_{n-1}(\nu,r)\,p_{n-1}(\nu,s)\right)^2}{\left(p_{n-2}(\nu,r)\,p_{n-2}(\nu,s)\right)^2}. \end{aligned}$$

Here from the recurrence relation,

$$\begin{aligned} rp_{n-2}(\nu,r) &= \frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\,p_{n-1}(\nu,r)+\frac{\gamma_{n-2}}{\gamma_{n-3}}(\nu)\,p_{n-3}(\nu,r) \\ &> \frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\,p_{n-1}(\nu,r), \end{aligned}$$

so

$$\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\,\frac{p_{n-1}(\nu,r)}{rp_{n-2}(\nu,r)}<1.$$

so that

$$\frac{(r-s)^2}{W(rs)^2}\left(p_{n-1}(\nu,r)\,p_{n-1}(\nu,s)\right)^2 \leq 2\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)^{-2}.$$

Thus

$$\frac{Y^TLY}{W}\left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\right)^{-2} \leq \frac{8}{(rs)^2}+2\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)^{-2} \leq 10\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)^{-2}$$

so from (3.3),

$$Y^TCY \leq 12\left\{\frac{\gamma_{n-1}}{\gamma_n}(\nu)\,/\,\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right\}^2.$$

■

Proof of Theorem 2

$$\begin{aligned} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 &\leq \left(1+\left(Y^TCY\right)^{1/2}\right)^2 \\ &\leq \left(1+\sqrt{12}\left(\frac{\gamma_{n-1}}{\gamma_n}(\nu)\,/\,\frac{\gamma_{n-2}}{\gamma_{n-1}}(\nu)\right)\right)^2. \end{aligned}$$

■

4. An alternative proof?

We know from elsewhere that

$$Y^T C Y = Y^T \left(D^{-1} + K\right)^{-1} Y,$$

where

$$D^{-1} = \left[\begin{array}{cc} M^{-1} & 0 \\ 0 & N^{-1} \end{array}\right];$$

$$K = \left[\begin{array}{cc} K_n(\nu, r, r) & -K_n(\nu, r, s) \\ -K_n(\nu, r, s) & K_n(\nu, s, s) \end{array}\right].$$

Is is true that

$$\left(D^{-1} + K\right)^{-1} \leq L^{-1},$$

that is

$$X^T \left(D^{-1} + K\right)^{-1} X \leq X^T K^{-1} X,$$

or

$$X^T \left[\left(D^{-1} + K\right)^{-1} - K^{-1}\right] X \leq 0?$$

Now

$$\begin{aligned} & \left(D^{-1} + K\right)^{-1} - K^{-1} \\ = \; & \frac{1}{\det\left(D^{-1} + K\right)} \left[\begin{array}{cc} N^{-1} + K_n(\nu, s, s) & -K_n(\nu, r, s) \\ -K_n(\nu, r, s) & M^{-1} + K_n(\nu, r, r) \end{array}\right] \\ & - \frac{1}{\det K} \left[\begin{array}{cc} K_n(\nu, s, s) & -K_n(\nu, r, s) \\ -K_n(\nu, r, s) & K_n(\nu, r, r) \end{array}\right] \\ = \; & \frac{1}{\det\left(D^{-1} + K\right)} \left[\begin{array}{cc} N^{-1} & 0 \\ 0 & M^{-1} \end{array}\right] \\ & + \left(\frac{1}{\det\left(D^{-1} + K\right)} - \frac{1}{\det K}\right) \left[\begin{array}{cc} K_n(\nu, s, s) & -K_n(\nu, r, s) \\ -K_n(\nu, r, s) & K_n(\nu, r, r) \end{array}\right]. \end{aligned}$$

Then

$$\begin{aligned} & \left\{\left(D^{-1} + K\right)^{-1} - K^{-1}\right\} \det\left(D^{-1} + K\right) \det(K) \\ = \; & \det(K) \left[\begin{array}{cc} N^{-1} & 0 \\ 0 & M^{-1} \end{array}\right] \\ & + \left(\det K - \det\left(D^{-1} + K\right)\right) \left[\begin{array}{cc} K_n(\nu, s, s) & -K_n(\nu, r, s) \\ -K_n(\nu, r, s) & K_n(\nu, r, r) \end{array}\right]. \end{aligned}$$

Then

$$\begin{aligned} & \left\{X^T \left(\left(D^{-1} + K\right)^{-1} - K^{-1}\right) X\right\} \det\left(D^{-1} + K\right) \det(K) \\ = \; & x_1^2 N^{-1} \det(K) + x_2^2 M^{-1} \det(K) \\ & + x_1^2 \left(\det K - \det\left(D^{-1} + K\right)\right) K_n(\nu, s, s) \\ & + x_2^2 \left(\det K - \det\left(D^{-1} + K\right)\right) K_n(\nu, r, r) \\ & - 2 x_1 x_2 \left(\det K - \det\left(D^{-1} + K\right)\right) K_n(\nu, r, s). \end{aligned}$$

Now

$$\begin{aligned} & \det\left(D^{-1}+K\right) \\ = & \left(N^{-1}+K_n(\nu,s,s)\right)\left(M^{-1}+K_n(\nu,r,r)\right)-K_n(\nu,r,s)^2 \\ = & N^{-1}M^{-1}+M^{-1}K_n(\nu,s,s)+N^{-1}K_n(\nu,r,r)+\det(K) \end{aligned}$$

Then

$$\begin{aligned} & \left\{X^T\left(\left(D^{-1}+K\right)^{-1}-K^{-1}\right)X\right\}\det\left(D^{-1}+K\right)\det(K) \\ = & x_1^2N^{-1}\det(K)+x_2^2M^{-1}\det(K) \\ & -\left\{N^{-1}M^{-1}+M^{-1}K_n(\nu,s,s)+N^{-1}K_n(\nu,r,r)\right\} \\ & \times\left\{x_1^2K_n(\nu,s,s)+x_2^2K_n(\nu,r,r)-2x_1x_2K_n(\nu,r,s)\right\} \end{aligned}$$

$$\begin{aligned} = & x_1^2\left\{N^{-1}\det(K)-N^{-1}M^{-1}K_n(\nu,s,s)-M^{-1}K_n(\nu,s,s)^2-N^{-1}K_n(\nu,r,r)K_n(\nu,s,s)\right\} \\ & +x_2^2\left\{M^{-1}\det(K)-N^{-1}M^{-1}K_n(\nu,r,r)-M^{-1}K_n(\nu,s,s)K_n(\nu,r,r)-N^{-1}K_n(\nu,r,r)^2\right\} \\ & +2x_1x_2K_n(\nu,r,s)\left\{N^{-1}M^{-1}+M^{-1}K_n(\nu,s,s)+N^{-1}K_n(\nu,r,r)\right\} \\ = & x_1^2\left\{-N^{-1}K_n(\nu,r,s)^2-N^{-1}M^{-1}K_n(\nu,s,s)-M^{-1}K_n(\nu,s,s)^2\right\} \\ & +x_2^2\left\{-M^{-1}K_n(\nu,r,s)^2-N^{-1}M^{-1}K_n(\nu,r,r)-N^{-1}K_n(\nu,r,r)^2\right\} \\ & +2x_1x_2K_n(\nu,r,s)\left\{N^{-1}M^{-1}+M^{-1}K_n(\nu,s,s)+N^{-1}K_n(\nu,r,r)\right\} \\ = & N^{-1}M^{-1}\left\{-x_1^2K_n(\nu,s,s)-x_2^2K_n(\nu,r,r)+2x_1x_2K_n(\nu,r,s)\right\} \\ & +N^{-1}\left\{-x_1^2K_n(\nu,r,s)^2-x_2^2K_n(\nu,r,r)^2+2x_1x_2K_n(\nu,r,r)K_n(\nu,r,s)\right\} \\ & +M^{-1}\left\{-x_1^2K_n(\nu,s,s)^2-x_2^2K_n(\nu,r,s)^2+2x_1x_2K_n(\nu,s,s)K_n(\nu,r,s)\right\}. \end{aligned}$$

If x_1,x_2 have opposite sign, then as n is even and $K_n(\nu,r,s)>0$, all terms are negative. Now assume that x_1,x_2 have the same sign. We continue this as

$$\begin{aligned} = & N^{-1}M^{-1}\left\{\begin{array}{c} -\left(x_1\sqrt{K_n(\nu,s,s)}-x_2\sqrt{K_n(\nu,r,r)}\right)^2 \\ +2x_1x_2\left[K_n(\nu,r,s)-\sqrt{K_n(\nu,s,s)}\sqrt{K_n(\nu,r,r)}\right] \end{array}\right\} \\ & -N^{-1}\left\{-\left(x_1K_n(\nu,r,s)-x_2K_n(\nu,r,r)\right)^2\right\} \\ & -M^{-1}\left\{-\left(x_1K_n(\nu,s,s)-x_2K_n(\nu,r,s)\right)^2\right\}. \end{aligned}$$

This is indeed negative. Acutally note too that the expression above equals

$$\begin{aligned} = & N^{-1}M^{-1}\left\{\begin{array}{c} -\left(|x_1|\sqrt{K_n(\nu,s,s)}+|x_2|\sqrt{K_n(\nu,r,r)}\right)^2+2x_1x_2K_n(\nu,r,s) \\ -2|x_1||x_2|\sqrt{K_n(\nu,s,s)}\sqrt{K_n(\nu,r,r)} \end{array}\right\} \\ & +N^{-1}\left\{-x_1^2K_n(\nu,r,s)^2-x_2^2K_n(\nu,r,r)^2+2x_1x_2K_n(\nu,r,r)K_n(\nu,r,s)\right\} \\ & +M^{-1}\left\{-x_1^2K_n(\nu,s,s)^2-x_2^2K_n(\nu,r,s)^2+2x_1x_2K_n(\nu,s,s)K_n(\nu,r,s)\right\}. \end{aligned}$$

which for all x_1,x_2 (even if $K(\nu,r,s)<0$) gives the desired result.

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu